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FUNCTION

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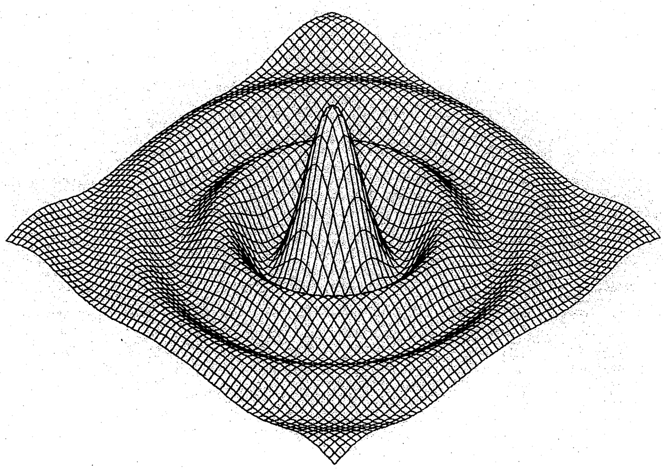

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting traffic lights, that do not involve mathematics. *Function* will contain articles describing some of these uses of mathematics. It will also have articles, for entertainment and instruction, about mathematics and its history. There will be a problem section with solutions invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168

The magazine will be published five times a year in February, April, June, August, October. Price for five issues: $3.50; single issues: 90 cents. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

Front cover: $y = \dfrac{\sin x}{x}$ rotated about the y-axis

Our main article in this issue is by John Stillwell and is an account, written specially for *Function*, explaining how one of the famous problems of mathematics, first posed in the 1850's, the so-called Four Colour Problem, was solved in 1976. Its solution is a great event and was unusual in that it involved over 1200 hours of computer time and so, perhaps, is suspect until the computing as well as the reasoning has been adequately checked.

We are pleased to include two articles by authors Peter Watterson and Christopher Stuart who were in sixth form in 1976, but who will be starting university courses this year. We call for further articles to be offered for publication from authors still at school.

Each issue of *Function* will contain problems for your enjoyment and interest. We ask you to send us any problems you think may interest others. We hope to cover a wide range of applications, and a wide range of mathematical techniques will be called upon to handle those applications. Some problems may yield up their solutions only after some numerical calculations have been carried through - calculators and computers are not barred. Some problems will be harder than others, of course; we hope none prove to be impossible!

We shall invite you to send us your solutions to some of the problems (these will be specially indicated). *Function* will publish some of the solutions received. Please tell us your name, your form, and the name of your school.

CONTENTS

BALBUS'S ESSAY

"When a solid is immersed in a liquid, it is well known that it displaces a portion of the liquid equal to itself in bulk, and that the level of the liquid rises just so much as it would rise if a quantity of liquid had been added to it, equal in bulk to the solid. ...

"Suppose a solid held above the surface of a liquid and partially immersed: a portion of the liquid is displaced, and the level of the liquid rises. But, by this rise of level, a little bit more of the solid is of course immersed, and so there is a new displacement of a second portion of the liquid, and a consequent rise of level. Again, this second rise of level causes a yet further immersion, and by consequence another displacement of liquid and another rise. It is self-evident that this process must continue till the entire solid is immersed, and that the liquid will then begin to immerse whatever holds the solid, which, being connected with it, must for the time be considered a part of it. If you hold a stick, six feet long, with its ends in a tumbler of water, and wait long enough, you must eventually be immersed."

Lewis Carroll, *A Tangled Tale*

LOUIS PÓSA

adapted from Chapter 2 of 'Mathematical Gems'* by Ross Honsberger

This is a story about the life and work of a remarkable young Hungarian named Louis Pósa (pronounced po-sha) who was born in the late 1940's. When quite young he attracted the attention of the eminent Hungarian mathematician Paul Erdös (pronounced air-dosh). Erdös is now 63 years old and is an internationally known mathematician. His three great interests are combinatorics, number theory and geometry and he has written more than 500 mathematical papers. For decades Erdös has travelled the world's universities, seldom visiting anywhere for more than a few months. During 1970 he visited the University of Waterloo in Ontario, Canada and spoke about a number of the child prodigies he has known. Except for a few minor changes the following is his story of Pósa.

In the course of the story Erdös mentions some mathematical terms with which you may not be familiar. At the end of this article there are some explanatory notes to help you follow what Erdös is talking about.

Erdös' Story

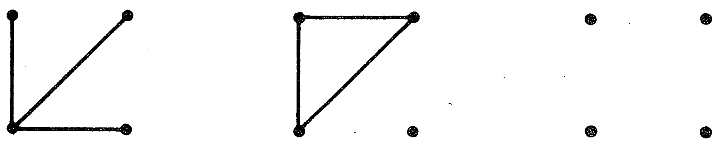
"I will talk about Pósa who is now 22 years old and the author of about 8 papers. I met him before he was 12 years old. When I returned from the United States in the summer of 1959 I was told about a little boy whose mother was a mathematician and who knew quite a bit about high school mathematics. I was very interested and the next day I had lunch with him. While Pósa was eating his soup I asked him the following question: Prove that if you have $n + 1$ positive integers less than or equal to $2n$, some pair of them are relatively prime (see Note 1). It is quite easy to see that the claim is not true of just n such numbers because no two of the n even numbers up to $2n$ are relatively prime. Actually I discovered this simple result some years ago but it took me about ten minutes to find the really simple proof. Pósa sat there eating his soup, and then after a half a minute or so he said "If you have $n + 1$ positive integers less than or equal to $2n$, some two of them will have to be consecutive and thus relatively prime" (Note 2). Needless to say, I was very much impressed, and I venture to class this on the same level as Gauss' summation of the positive integers up to 100 when he was just 7 years old" (Note 3).

At this point Erdös discussed a few problems in graph theory which he gave to Pósa. In order to avoid any misunderstandings, let us interject here a brief introduction to this material.

*'Mathematical Gems' was published by the Mathematical Association of America in 1976. We are most grateful to Professor Honsberger for permission to use his article as reproduced here. If you enjoy this article, you will enjoy reading the book.

By a *graph* we do not mean anything connected with axes and coordinates. A graph consists of a collection of *vertices* (points) and a set of *edges*, each joining a pair of vertices. How the graph is pictured on paper is not essential. The edges may be drawn as straight lines or curves, and it is immaterial whether they are drawn so as to intersect or not. Points of intersection obtained by edges which cross do not count as vertices. Only the given vertices are the vertices of the graph. Also a graph need not contain every possible edge which could be drawn, that is, there are in general many different graphs with the same set of vertices.

For example, here are three different graphs each with four vertices.



I II III

Graphs I and II each have three edges and graph III has no edges (Note 4).

A *loop* is an edge both of whose endpoints are the same vertex. When two or more edges join the same pair of vertices we call these *multiple edges*. In general a graph may contain loops and multiple edges, but throughout this article we use the word *graph* to mean a graph without loops and without multiple edges.

We continue now with the story.

"From that time onward I worked systematically with Pósa. I wrote to him of problems many times during my travels. While still 11 he proved the following theorem which I proposed to him: *A graph with* $2n$ *vertices and* $n^2 + 1$ *edges must contain a triangle* (Note 5). Actually this is a special case of a well-known theorem of Turán, which he worked out in 1940 in a Hungarian labor camp. I also gave him the following problem: Consider an infinite series whose nth term is the fraction with numerator 1 and denominator the lowest common multiple of the integers 1, 2, ..., n; prove that the sum is an irrational number. This is not very difficult, but it is certainly surprising that a 12 year old child could do it.

"When he was just 13, I explained to him Ramsey's theorem for the case $k = 2$: *Suppose you have a graph with an infinite number of vertices; then there is either an infinite set of vertices every two of which are joined by an edge, or there is an infinite set of vertices no two of which are joined by an edge* (Note 6). It took about 15 minutes for Pósa to understand it. Then he went home, thought about it all evening, and before going to sleep had found a proof.

"By the time Pósa was about 14 years old you could talk to him as a grown-up mathematician. It is interesting to note that he had some difficulty with calculus. He never liked geometry, and he never wanted to bother with anything that did not really excite him. At anything that did interest him, however, he was extremely good. Our first joint paper was written when he was $14\frac{1}{2}$. Pósa wrote several significant papers by himself, some of which still have a great deal of effect. His best known paper, on Hamiltonian circuits, for which he received international acclaim, he wrote when he was only 15! (Note 7).

"The first theorem that he discovered and proved by himself which was new mathematics was the following: *A graph with* n *vertices* ($n \geq 4$) *and* $2n - 3$ *edges must contain a circuit with a diagonal* (Note 8). This result is the best possible, because for every n, one can construct a graph with n vertices and $2n - 4$ edges which fails to have a circuit with a diagonal.

"A problem which I had previously solved is the following: *A graph with* n *vertices* ($n \geq 4$) *and* $2n - 3$ *edges must contain two circuits which have no vertices in common.* I told Pósa of the problem, and in a few days he had a very simple proof which was miles ahead of the complicated one I had come up with. A most remarkable thing for a child of 14.

"I would like to make a few conjectures why there are so many child prodigies in Hungary. First of all there has been for at least 80 years a mathematical periodical for high school students. Then there are many mathematical competitions. The Eötvös-Kurshák competition goes back 75 years. After the first world war a new competition was begun for students just completing high school, and after the second war several new ones were started.

"A few years ago a different kind of competition was started. It is held on television. Bright high school students compete in doing questions in a given period of time. The questions are usually very ingenious and the solutions are judged by a panel of leading mathematicians such as Alexits, Turán, and Hajos. It seems many people watch these competitions with great interest even though they do not understand the problems.

"In Hungary a few years ago a special high school, the Michael Fazekas High School, was opened in Budapest for children who are gifted in mathematics. The school started just when Pósa was due to go to high school. He liked the school very much, so much so, in fact, that he refused to leave it for entrance into university two years early."

This is the end of the story told by Erdös. In later issues of *Function* we will publish solutions to some of the problems mentioned here. You may like to find solutions for yourself. If you have a solution to any of the problems you are invited to submit them to the editors. The address is on the inside front cover of this issue.

Notes

(1) Two integers are called *relatively prime* if their highest common divisor is 1. For example 4 and 9 are relatively prime. However 4 and 10 are *not* relatively prime because both 4 and 10 are divisible by 2. Note that the property of being relatively prime is a property of two numbers considered *together*. We can say that 4 and 9 are relatively prime but we cannot say separately that 4 is relatively prime or that 9 is relatively prime.

(2) Pósa's explanation uses the fact that for any positive integer n no number greater than 1 divides both n and $n + 1$. We can show that this is true as follows. Whenever a number k divides both the numbers a and b then k also divides $(a - b)$. So any number which divides both n and $n + 1$ will also divide $(n + 1) - n = 1$. But the only divisor of 1 is 1 itself.

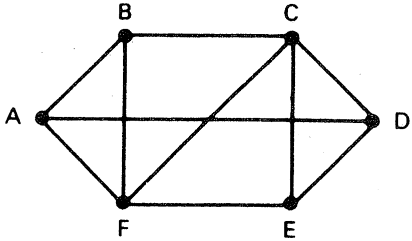
(3) Carl Friedrich Gauss was born in Göttingen, Germany, in 1777. He was a scientist of great note whose main interests were in the fields of mathematics, astronomy, and physics. There are many stories about Gauss's ability even in early childhood. One such episode occurred in his arithmetic class at school when he was 7 years old. The class was given the exercise of writing down all the numbers from 1 to 100 and adding them. Gauss found this total almost immediately. Instead of adding up a column of 100 numbers, as did the rest of the class, Gauss reached his answer by noticing that $100 + 1 = 101$, $99 + 2 = 101$, $98 + 3 = 101$, etc., and that there are 50 such 'pairs' thus the total is 50×101, i.e. 5050.

(4) There are eleven different graphs with four vertices. Try to draw them all.

(5) A *triangle* in a graph is a set of three vertices in the graph such that each pair of the vertices is joined by an edge.

(6) This theorem is due to F.P. Ramsay, an English mathematician who made important contributions to mathematics, mathematical logic, mathematical economics and philosophy. His death at age 26 in 1930 was a tragic loss. His father was a well-known applied mathematician. The Archbishop of Canterbury from 1961 to 1974 was his brother.

(7) A *circuit* in a graph is a path along edges of the graph which returns to its starting point. A *Hamiltonian circuit* is one which passes through every vertex of the graph exactly once. In the following graph which has 6 vertices A, B, C, D, E, F and 10 edges we see that $\langle AB, BF, FA \rangle$ and $\langle FB, BC, CD, DE, EF \rangle$ are circuits, but $\langle AD, DE, EC \rangle$ is not (because it does not return to its starting vertex A). The circuit $\langle AB, BC, CD, DE, EF, FA \rangle$ is a Hamiltonian circuit in the graph. Can you find any other Hamiltonian circuits in this graph? (Opposite page.)



(8) If S is a circuit in a graph and XY is an edge in the graph whose endpoints are on the circuit S call the edge XY a *diagonal* of S if XY is not itself an edge used in S. For example, in the graph drawn above the edge FC is a diagonal of the circuit $\langle FB, BC, CD, DE, EF \rangle$.

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ODDS SPOT

by G. A. Watterson, Monash University

On 18 October, 1976, the eight dogs in the last greyhound race at Olympic Park finished in the same order as they appeared in the race book. As there are $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\ 320$ different orders in which the dogs could have finished, a most unlikely event occurred. But then, a large number of races have been run at Olympic Park, Melbourne!

Examples of unlikely coincidences have been argued in the law courts. In one case*, the Australian Taxation Commissioner was trying to assess a bookmaker for taxation. The bookmaker possessed some Treasury Bonds, namely three bonds each for £1000, one bond for £500 and five bonds of £100. The case depended on how long the bookmaker had possessed the bonds. There was evidence that he had had £4000 in bonds at an earlier date, split up according to denominations in exactly the same way as the above. This was considered unlikely unless the bonds were identical. Moreover, there were eleven series of bonds available; we shall call them A, B, C, D, E, F, G, H, I, J, K. It was shown that on both the earlier and the later occasions, the bonds in the bookmaker's possession were

1 × £1000 series A

2 × £1000 series B

1 × £500 series A

5 × £100 series A.

*The author thanks Sir Richard Eggleston for information concerning this taxation case.

Since each bond could have come from any of the eleven series, a barrister argued that, even assuming that the 5 × £100 bonds would very likely be of the same series as each other, there were still 11 × 11 × 11 × 11 × 11 = 161 051 different orders in which the series *could* have been allocated to the bonds. Only three of these orders (depending on whether the A series was allocated to the first, second, or third of the £1000 bonds) were consistent with the actual series arrangement on both occasions. He thought that this was pretty conclusive evidence that the bookmaker had kept the *same* packet of £4000 worth of bonds throughout. I cannot vouch that the series were equally likely, though.

Another example of the use of probability in the law was in a case in Sweden. A policeman noticed the position of the valves of the front and rear tyres on one side of a parked car. There was a maximum period allowed for parking in that spot. Well after that period had expired, the policeman returned and found the car still there, with its tyre valves in the same positions as before. The car driver claimed he had driven away and returned later and that the tyre valves just happened to return to the same positions. The judge accepted his argument. If there were twelve different positions for a tyre valve to take (like 12 hours on a clock face) then the chance of *both* valves returning to their previous positions was claimed to be $(1/12)^2 = 1/144$. Although this is not very large, the judge agreed that the case was not proved beyond reasonable doubt.

Incidentally, would it be correct that the front and back tyres of a car move independently, as assumed in the above calculation? Observe what happens next time you go for a car ride.

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CRICKET TEAMS

On February 11, 1977, the Melbourne 'Age' announced the winners of its Centenary Test cricket competition. The Age stated:

"Readers were asked to pick from a list of 30 Test players, Mr Johnson's choice as the best post-war team in batting order, 12th man, captain and vice-captain.

The probability of only one reader filling the requirements was 455,730,000,000,000,000 to one."

[Twenty seven people chose the correct 12 players, of whom one person chose the correct order. The number of entries was not announced.]

What, if anything, does The Age statement mean, and is it correct?

THE FOUR COLOUR PROBLEM

by John Stillwell, Monash University

"A student of mine asked me today to give him a reason for a fact which I did not know was a fact, and do not yet. He says, that if a figure is anyhow divided, and the compartments differently coloured, so that figures with any portion of common boundary *line* are differently coloured - four colours may be wanted, but no more."

Letter from Augustus de Morgan to Sir William Rowan Hamilton

October 23, 1852.

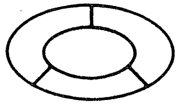
Introduction

Few mathematical problems are as easy for the layman to understand as the four colour problem, in fact the problem is so down-to-earth it does not even *sound* like mathematics to many. Nevertheless, since it was first proposed in 1852 most great mathematicians have tackled it without success, and a large number of "solutions" have been proposed only to have errors exposed shortly afterwards. False solutions eventually became so numerous that experts became extremely wary of claims that the problem had been solved. The first reaction was to disbelieve any such claim. And when the solution of Haken and Appel was announced in 1976 there was a strange sense of anticlimax, because the mathematical community still couldn't quite believe it.

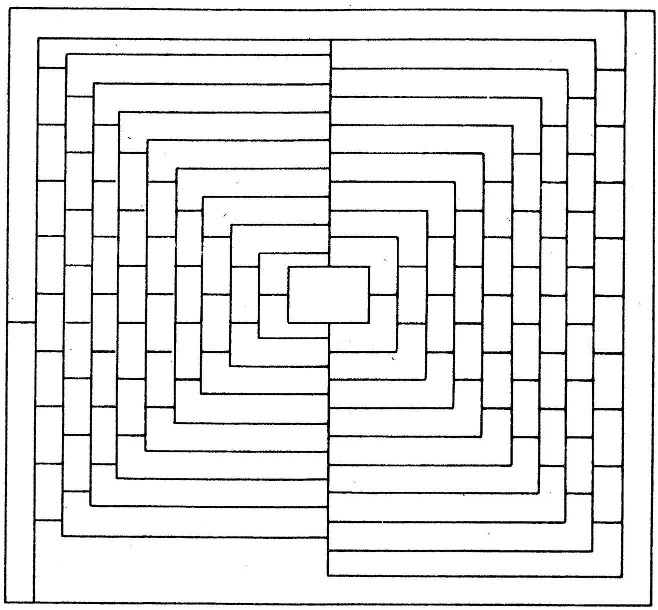
Before going on to outline their solution, a few more words about the history of the problem may be of interest. Although it is easy to imagine (as some popularizers of mathematics have done) that map makers had long been aware that four colours seem to suffice, there is actually no evidence that they were. Probably the fact was quite obscured by other requirements of map-making, such as reserving blue for lakes and seas, and colouring colonies the same as the mother country, which can force more than four colours to be used. At any rate, extensive research has failed to find any statement of the problem earlier than the letter of Augustus de Morgan quoted above (the student incidentally was Francis Guthrie, who later became professor of mathematics at Cape Town).

The problem first appeared in print in 1878 when Arthur Cayley asked, in the Proceedings of the London Mathematical Society, whether it had been solved. The following year A.B. Kempe published a "proof" that four colours suffice, in the American Journal of Mathematics. This solution evidently did not set the world on fire, for it was not until 1890 that P.J. Heawood discovered a simple mistake in it. However, Heawood was able to save enough of Kempe's argument to prove that *five* colours always suffice.

The simple map



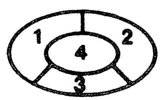
shows that four colours are sometimes necessary, so Heawood's theorem left an apparently "small" gap between necessary and sufficient. Despite this, the story of the next 80 years is one of painfully slow progress - by the end of the 1960's little was known except that four colours suffice for maps of not more than 40 regions - punctuated by the occasional false "solution". One of the most amusing of these was the deliberate hoax by Martin Gardner in the April (Fool's Day) issue of Scientific American in 1975. Gardner claimed that the following map cannot be coloured with four colours.



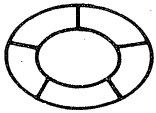
Can you prove him wrong?

Outline of the Haken - Appel solution

It might be best to begin by disposing of the false "solution" which everyone thinks of when they first see this problem. When we construct the map (alongside) which needs 4 colours it is easy to see that there is nowhere to put a 5th region so that it has common boundaries with each of the regions 1, 2, 3, 4. In fact it



is a theorem which goes back to Möbius in 1840 that 5 regions cannot be arranged so that each has common boundaries with all the others. Unfortunately, this is quite beside the point, because a map may need 4 colours even though no 4 of its regions touch each other, e.g.



Now let us look at a more serious way to attack the problem. There are of course infinitely many maps, so no amount of colouring individual maps will solve the problem. We could somehow *classify* maps into finitely many types, and try to prove each *type* is 4-colourable, but there is still the difficulty that some types will have infinitely many members. The way round this is to define each type by the presence of a particular part M, and make the argument depend on the properties of M alone. To classify all maps we therefore need a finite set of M's such that every map contains one of them. Such a set of M's is called an *unavoidable set*.

Then the property of M that we need is 4-*colour reducibility*, meaning that the 4-colourability of any map containing M can be made to depend on 4-colourability of a smaller map obtained by shrinking M to a map M' with fewer regions. Another way to express this property is by saying that M cannot be a part of the (hypothetical) smallest map which needs 5 colours (since shrinking M to M' would then give a smaller map which required 5 colours*). If every type of map is 4-colour reducible, then there cannot be a map which needs 5 colours.

A finite solution of the four colour problem is now conceivable - we have only to find a suitable unavoidable set of M's, then prove them all 4-colour reducible.

This argument is actually a fair summary of what Kempe was trying to do in 1879. What Kempe did not know was that the unavoidable set side of the argument does not mesh with the reducibility side until both have been elaborated to an incredible length, requiring computations that were out of the question before the age of computers.

It is against a mathematician's instincts to embark on long calculations without a clear picture of where they will lead, because the desire is always for a *beautiful* solution, particularly in the case of a famous problem like the four colour problem. So perhaps the decisive step towards the present solution occurred in 1969 when Heesch dared to produce ugly mathematics by enumerating thousands and thousands of

*For let K be a map with the smallest possible number of regions among those requiring 5 colours and suppose that K contains M. Then shrinking M to M' replaces K by a map K' containing fewer regions than K. So K' can be coloured by 4 colours. Hence, since M is by assumption 4-colour reducible, K can also be coloured by 4 colours, a contradiction.

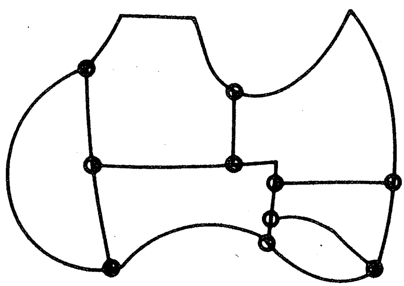
reducible map types in the hope that eventually all maps would be included.

Heesch's computer tests for reducibility were refined by Appel and Haken at the University of Illinois, and complemented by an equally ugly theory of unavoidable sets. They divided all maps into 1936 types which they had reason to believe were reducible, then used their tests (and 1200 hours of computer time) to verify that this was in fact the case. The maps M used by Haken and Appel to define their types have as many as 14 rings of regions and they claim that this cannot be improved upon, so a proof of the four colour theorem by this method is bound to be complex.

I shall say some more about ultra-complex proofs in the concluding remarks below; in the meantime, readers who want to get their teeth deeper into the mathematics are invited to study the next two sections, where the basic results on unavoidable sets and reducibility are proved.

Unavoidable Sets

At this stage we should make it clear that we are talking about maps on the plane with finitely many regions, and every part of the plane is counted as belonging to some region. E.g. the sea surrounding the figure of mainland Australia below counts as the 7th region.



Also, we omit any region which is completely surrounded by another (as the A.C.T. is surrounded by New South Wales) since there is no problem colouring such regions if they are put back later on. Our maps will then satisfy *Euler's formula*:

$$V - E + F = 2$$

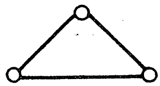
where F = number of faces ("regions")
E = number of edges ("borders" between regions)
V = number of vertices (points where two or more borders meet).

In the example the vertices are marked by small circles and the edges are the pieces of line (not necessarily straight) between these circles. It can then be checked that

$$V - E + F = 10 - 15 + 7 = 2$$

so the formula is true for this example.

A detailed proof of Euler's formula would take up too much space, but it is easy to be convinced it is true in general by cutting all faces into triangles by new edges (where a "triangle" now means any region with 3 edges, such as Victoria) and then removing triangles successively from the outer boundary. If this is done carefully, $V - E + F$ will be unchanged at each step and one can end with a triangle



for which $V - E + F = 3 - 3 + 2 = 2$.

Euler's formula implies that all maps are subject to a surprising restriction:

Kempe's Theorem: In any map at least one region has not more than 5 neighbours.

Proof: Suppose there are n regions (so $F = n$) and A is the average number of edges which each region has.

Then $E = \frac{1}{2}nA$ (because the average region has A edges, but each is shared by 2 regions),

and $V \leqq \frac{1}{3}nA$ (because the average region has A vertices, but each is shared by 3 or more regions).

(Here we are using the assumption that no region completely surrounds another. This means each region has vertices, and their number equals the number of edges of the region.)

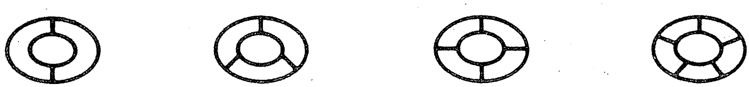
So

$$V - E + F \leqq \frac{1}{3}nA - \frac{1}{2}nA + n = n(1 - \frac{A}{6})$$

But $V - E + F$ must be greater than 0 (namely, 2) which is only possible if A, the average, is less than 6. If the average is less than 6 then at least one region must have not more than 5 edges and so not more than five neighbours.

Q.E.D.

This theorem gives a particularly simple example of an unavoidable set, for it says every map must contain one of the following configurations:



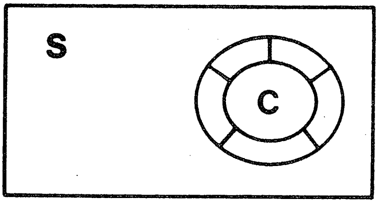
which we shall call M_2, M_3, M_4, M_5.

M_2, M_3, M_4 are 4-colour reducible (this can be proved using the ideas in the next section), and Kempe thought that M_5 was also. Unfortunately, this is where he was wrong, and the damage can only be repaired by a far larger unavoidable set (1936 configurations in the Haken and Appel proof, which they believe could not be lessened by more than about 25%).

Reducibility

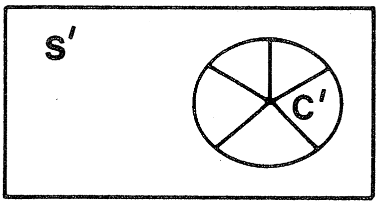
Reducibility can be defined with respect to any number of colours, and we shall warm up for the main exercise by using 6-*colour reducibility* to prove that 6 colours suffice for any map.

Suppose there are maps which need 7 colours, and let S be one which has the smallest possible number of regions. S must contain one of M_2, ..., M_5 and we shall assume for the sake of argument that S contains M_5:



If we shrink the centre region C of M_5 down to a point C' the result is a map S' with fewer regions, so by hypothesis S' can be coloured with no more than 6 colours.

But at most 5 colours are used in the regions surrounding the centre point C', so the colouring of S' can be transferred back to S by using the 6th colour for C.

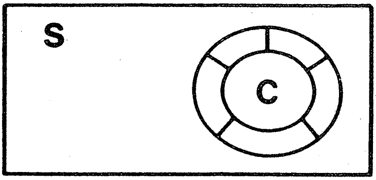


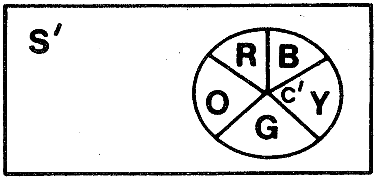
Transfer of the 6-colouring from S' to S says that M_5 is 6-*colour reducible*, and it means that M_5 cannot be part of the hypothetical smallest map S which requires 7 colours. By a similar argument, neither can M_2, M_3, M_4; so S *does not exist*.

Using a clever idea of Kempe, we can improve this result to 5-colour reducibility, and hence prove Heawood's theorem that 5 colours suffice for every map.

Heawood's Theorem: No map requires 6 colours.

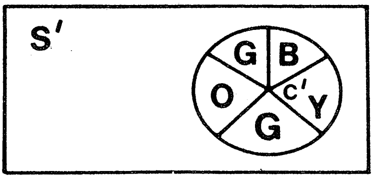
Proof: If there are maps which need 6 colours, let S be one which has the smallest possible number of regions, and look at the case where S contains M_5. We shrink C down to a point C', so that the new map S' has fewer regions, and hence can be coloured with no more than 5 colours.





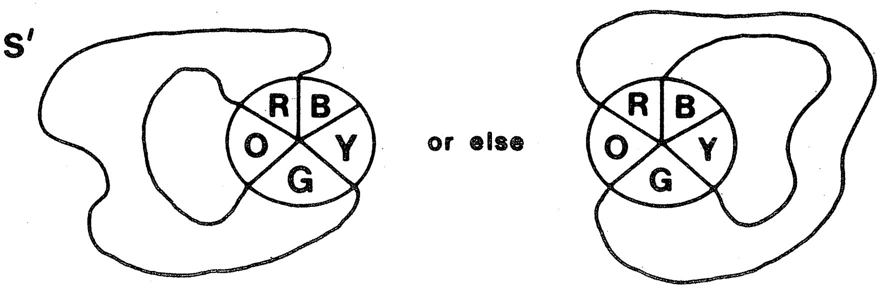
This time it is not enough simply to transfer the colouring of S' back to S, since all the available colours may have been used in the 5 regions surrounding C'. We shall have to revise the colouring of S' until only 4 colours surround C'.

Suppose that the colours initially surrounding C', are red, blue, yellow, green, orange (in the regions marked R, B, Y, G, O in the diagram). Starting at R we construct what is called the *red-green Kempe chain* containing R, by combining R with all its green neighbours, then combining these with all their red neighbours, and so on, until no more red or green neighbours are found. A Kempe chain can be constructed for any pair of colours X, Y, with any starting point, and since any regions bordering the chain are *not* coloured X or Y, X and Y can be interchanged throughout the chain without spoiling the colouring of the map.



If the red-green Kempe chain containing R does not contain G then a red-green interchange will produce a validly coloured S' (shown at left) where only 4 colours now surround C'. This revised colouring can now be transferred back to S, using red for C, and we have a 5-colouring of S, contrary to assumption.

If the red-green Kempe chain containing R *does* contain G we must have a situation like this:



In either case the red-green chain separates O from Y, so the yellow-orange Kempe chain containing Y will not contain O and we can similarly use yellow-orange interchange to obtain a 5-colouring of S.

We have thus proved that M_5 is 5-colour reducible, and it is obvious, that M_2, M_3, M_4 are also, so the hypothetical map S which required 6 colours does not exist. Q.E.D.

Heawood's theorem is the end of the line for the unavoidable set $\{M_2, M_3, M_4, M_5\}$. It is of no use in proving 4 colours suffice because (short of proving the 4 colour theorem itself) *we cannot prove that* M_5 *is 4-colour reducible.* M_2 and M_3 obviously are, and we can prove that M_4 is 4-colour reducible by the Kempe chain method, but not M_5, so we are forced to consider larger unavoidable sets.

This is where the going gets rough and enormous calculations have to be made. Haken and Appel had to experiment with 10 000 different configurations before they found their set of 1936 which is both unavoidable and reducible, so a brief description of their work is impossible. Nevertheless, the basic tools remain Euler's formula for finding unavoidable sets and Kempe chains for proving reducibility.

Concluding Remarks

In 1971 there was an interesting "proof" of the four colour theorem based on a computer check of reducibility by Heesch's test. Shimamoto, assuming that the theorem was false, showed that there must be a non-4-colourable map containing a configuration H which had already passed Heesch's test. This contradiction showed that the theorem was in fact true, Q.E.D. Shimamoto's argument was checked, and circulated in a detailed form by Haken, only to have its support shot away when it was found that the computation had been wrong - it did *not* pass the test!

The moral is that computer programs are just as subject to error as mathematical arguments, and we shall therefore have to wait until Haken and Appel's 1976 computations have been independently checked before we can be sure that the four colour theorem is really proved. This is a barbarous way to do mathematics, and most mathematicians will continue to hope for a solution which uses powerful ideas rather than massive amounts of brute computation.

On the other hand, there may be no other way. The four colour theorem may be a new breed of animal - a question whose answer can be known but not understood. To give another example of this type of question, suppose one looks at a table of prime numbers and picks one at random, say 9 925 387. Then the question: "Is 9 925 387 a prime?" has a known answer, yes, but we do not know *why* unless we actually divide 9 925 387 by hundreds of different numbers, which can hardly be called "understanding". It is one thing to concoct questions of this type using random processes, but something else to find a *natural* question whose answer is known but not understood. The four colour problem will not really be settled until we know whether or not it has a more understandable answer.

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MATHEMATICS AS IT WAS

by Peter A. Watterson, 6th form, Haileybury College

There are many mathematical textbooks on the market today, but one less commonly found is "A Treatise on Practical Mensuration" by A. Nesbit. If it is of little relevance to today's students, it is certainly of great interest, for it was written in 1841. (Note: Darwin's revolutionary book, "Origin of Species" was not published until 1859.) Here are the opening lines of the preface.

"Various have been the conjectures concerning the origin of Geometry or Mensuration; but as it is a science of general utility, there can, I think, be little doubt that its existence is nearly coeval with the Creation of Man. Indeed I can see no reasonable objection why we may not attribute its invention to our first parent Adam; especially as we are informed in Holy Writ, that his son Cain built a city; to do which, it is evident, would require some knowledge of a measuring unit which is the first principle of Mensuration. By the same infallible testimony, we find that the Arts and Sciences were cultivated to a considerable extent long before the Flood."

I will use some of the Preface notes to give you an idea of the book's contents. After geometrical definitions and theorems, the mensuration of superficies, and land-surveying, "Part the Fourth" includes "Rules and Directions for measuring and valuing standing Timber; many of which were never before published. I have given a description of Timber Trees, and pointed out the purposes for which their wood is best adapted; for it is impossible to become a valuer of timber without being made acquainted with the properties of trees." Mathematics was certainly "Applied" in those days.

"Part the Fifth treats of the Method of measuring the Works of Artificers viz. Bricklayers, Masons, Carpenters and

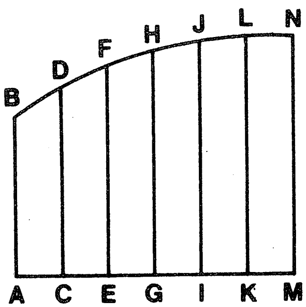
Joiners, Slaters and Tilers, Plasterers ... and Pavers. Part the Sixth treats of the Mensuration of Haystacks, Drains, Canals, Marlpits ... Clay-heaps, and irregular figures, by means of equidistant, parallel sections, founded upon the method of equidistant ordinates." I will return to this method later.

The antiquity of the book itself, the faded brown pages, the old language and the type it's written in, are a source of pleasure, but the exactness of the worked examples (7 or 8 significant figures), is equally fascinating. Here's a technique on arched roofs - "To find the solid content of the vacuity formed by a groin arch, either circular or elliptical.

RULE. Multiply the area of the base by the height and the product by ·904."

I won't ask for an interpretation of this rule, but an interesting exercise could be to derive "the invaluable Rule finding the areas of curvilineal figures, by means of equidistant ordinates." The author says it was first demonstrated by "the illustrious Newton" and later given on p. 109 of "Simpson's Dissertations" - hence, today, we attribute it to him as Simpson's Rule.

In the curvilinear figure $AMNB$ below, take the equidistant ordinates AB, CD, EF etc., where $AC = CE = \cdots$. AB and MN are the first and last ordinates; EF, IJ are the odd ordinates; and CD, GH, KL are the even ordinates.



"RULE. To the sum of the first and last ordinates, add four times the sum of all the even ordinates, and twice the sum of all the odd ordinates, not including the first and last; multiply this sum by the common distance of the ordinates (e.g. AC), divide the product by 3 and the quotient will be the area ($AMNB$)."

"Note: By this Rule, the contents of all solids, whether regular or irregular, may be found, by using the areas of the sections perpendicular to the axe, instead of the ordinates; and it is evident that the greater the number of ordinates or sections are used, the more accurately will the area or solidity [volume] be determined."

This rule appears to have occupied an important place in mathematics of the past (certainly in this treatise.) Hopefully its derivation will appear in a later issue of *Function*.

From the above examples you will probably notice the literary style of all "rules" etc.[†] In fact virtually no algebraic symbolism appears in this practical manual, because "algebraic form is seldom perfectly comprehended by learners".

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A PERPETUAL CALENDAR

by Liz Sonenberg, Monash University

We use a calendar to tell us on which day of the week a given date falls. In this article we shall describe a formula which enables you to work this out for any date.

Our calendar has its origin in Roman times. It is based on a calendar introduced in 45 B.C. by Julius Caesar. Before this the Romans used a calendar of 12 months based on the lunar cycle. The first month of their year was Martius and the eleventh and twelfth months were Januarius and Februarius. Adjustments were made to keep the lunar months in line with the seasons. The process of inserting additional days in a standard calendar is called *intercalation*. The early Romans made intercalations at the end of the year, i.e. after Februarius. It is for this historical reason that now the intercalation in a leap year is made at the end of February.

The new Roman calendar was a modified version of a calendar developed in Egypt about 3000 B.C. Astronomical evidence in Caesar's time suggested that the length of the solar year (i.e. the time of one revolution of the earth around the sun) was $365\frac{1}{4}$ days. To accommodate this odd quarter day it was decided to adopt a four year cycle in which the first three years had 365 days and the fourth had 366 days. The new calendar also had 12 months and the previous names of the months were retained but each year was counted to start in Januarius. Each month had 30 or 31 days except Februarius, which had 29 days (and 30 days every fourth year). A later change took one day away from Februarius and added it to the month Augustus, giving this month 31 days.

Since Roman times, more accurate astronomical observations showed that by reckoning the solar year as $365\frac{1}{4}$ days, its length was overestimated by about 11 minutes 14 seconds; an error amounting to approximately one day in every 128 years. In 1582 Pope Gregory III ordered that henceforth of the years which are a multiple of 100, not all should be leap years but only those which are a multiple of 400 should be retained as leap years, i.e. the year 1600 was to be a leap year but 1700, 1800, and 1900 were not.

This Gregorian calendar was adopted forthwith by the principal states of the Holy Roman Empire but was not adopted in Protestant Great Britain or her American colonies until 1752. This calendar is still in error by about one day in 3323 years. It has been proposed to correct this by omitting the Gregorian intercalation in the year 4000 and all its multiples!

There are many interesting stories surrounding the development of the calendar and you should be able to find books in your local library describing some of the history.

The week-day problem

We begin by looking at dates in the twentieth century. Since January 1, 1900 was a Monday, "January 0 1900" would be a Sunday and we shall use this as our 'standard' date. If we are given a date after our standard date and we know how many days (x days, say) have passed from "January 0 1900" to the given date then we can find the given day quite easily. For divide x by 7 and look at the remainder. If the remainder is 0 then a number of complete weeks has passed since the standard day, which was a Sunday, so the given day must also be a Sunday; if the remainder is 1 then the given day must be a Monday, and so on.

But how are we to work out the number x in the first place? Actually we are not really interested in the number x itself but rather the remainder after dividing x by 7 and so we'll show how to work out what this remainder is without working out x.

We can get the general idea by looking at dates in 1900. What day was April 12, 1900? From our standard date there are 31 + 28 + 31 = 90 days before April 1st 1900, so April 12 is day 90 + 12. Now $90 = (12 \times 7) + 6$ and $12 = (1 \times 7) + 5$ so $(90 + 12) = (13 \times 7) + 11 = (14 \times 7) + 4$, i.e. April 12, 1900 is 4 days after a Sunday, i.e. a Thursday.

Notice something about this last calculation. We didn't add 90 and 12. All we did was divide each by 7, find the remainders, and *add the remainders*. This observation will be the key to our calculations.

The numbers we deal with are non-negative integers 0, 1, 2, ... and we call the remainder after dividing the number x by 7 the *residue* (*modulo* 7) *of* x. So 6 is the residue (modulo 7) of 90, 5 is the residue (modulo 7) of 12, and 4 is the residue (modulo 7) of 90 + 12. It is a nuisance to write (modulo 7) all the time, so we'll leave it out and just refer to the *residue of* x. We write $\overline{x}$ for the residue of x, e.g. $\overline{90} = 6$, $\overline{12} = 5$, and $\overline{90 + 12} = 4$.

Now $6 + 5 = 11 = (1 \times 7) + 4$, i.e. $\overline{6 + 5} = 4$. So we have $\overline{90} + \overline{12} = 6 + 5$ and $\overline{6 + 5} = 4$ and $4 = \overline{90 + 12}$, i.e. the residue of $\overline{90} + \overline{12}$ equals the residue of 90 + 12. Is there something special about the numbers 90 and 12 or is this just part of a general rule?

We say two numbers are *congruent* if they have the same residue. If a and b are congruent we write $a \equiv b$. So $a \equiv b$ if and only if $\overline{a} = \overline{b}$. We can prove the following rule:

Rule: If $a \equiv b$ and $c \equiv d$ then $a + c \equiv b + d$.

This rule explains the properties of the numbers 90 and 12 that we pointed out in the last paragraph, so let us prove the rule.

Suppose $a \equiv b$ and $c \equiv d$. Then a and b have the same residue, let us call it r, i.e. $a = 7m_1 + r$ and $b = 7m_2 + r$ for some numbers m_1 and m_2. Similarly there is a number s such that $\overline{c} = \overline{d} = s$, i.e. $c = 7n_1 + s$ and $d = 7n_2 + s$ for some numbers n_1 and n_2. Then $a + c = (7m_1 + r) + (7n_1 + s) = 7(m_1 + n_1) + (r + s)$. And $b + d = (7m_2 + r) + (7n_2 + s) = 7(m_2 + n_2) + (r + s)$. So $\overline{a + c} = \overline{r + s}$. Notice that $r + s$ may be greater than 6 so we cannot say $\overline{a + c} = r + s$. Similarly $\overline{b + d} = \overline{r + s}$. So we have $\overline{a + c} = \overline{b + d}$. This says $a + c \equiv b + d$ so we have proved the rule.

We have already calculated that the number of days up to and including March 31, 1900, is 90 and $\overline{90} = 6$. We can use our rule to work out the day of *any* date in April 1900. For example, April 23, 1900, is 23 days after March 31, so there are 90 + 23 days from the standard date to April 23. But using the rule, $90 + 23 \equiv 6 + 23 = 29 \equiv 1$. So April 23, 1900 was a Monday.

Similarly, if for each month we know the residue of the number of days up to the first of that month, then we can calculate the day for dates in that month. Here is a table of these residues.

January... 0	April.. 6	July...... 6	October... 0
February.. 3	May.... 1	August.... 2	November.. 3
March..... 3	June... 4	September. 5	December.. 5

For example let us look at November 15, 1900. From the table the residue of the number of days before November 1 is 3. Now 3 + 15 = 18 and $18 \equiv 4$ so November 15, 1900, was a Thursday.

We can also use this table to extend our calculations to cover all dates in the twentieth century. Notice that $365 = (52 \times 7) + 1$ so the residue of 365 is 1, and the residue of 366 is 2. This means that by knowing that November 15, 1901 was a Friday, November 15, 1902, was a Saturday and November 15, 1903, was a Sunday. But as 1904 was a leap year, November 15, 1904, was a Tuesday (not a Monday). Can we write down a formula to take account of all this? We need to take into account the number of years that have passed since 1900 (this is just the last two digits of the year) and we need to know the number of leap years that have passed.

For any real number c we write $[c]$ for the greatest integer less than or equal to c. For example,

$[3\frac{1}{4}] = 3$, $[3\frac{3}{4}] = 3$, and $[4] = 4$. Using this we can count the number of leap years since 1900. For example 1922 is 22 years after 1900 and $[\frac{22}{4}] = [5\cdot5] = 5$. This says that 5 leap years occurred between 1900 and 1922. Similarly $[\frac{24}{4}] = [6] = 6$, so 1924 is the sixth leap year after 1900.

Now we can give a formula which, with the aid of the table of residues, enables us to calculate the day of any date in the 20th century. The formula is

$$x \equiv M + D + Z + [\tfrac{Z}{4}] - L$$

where x is the day number (Sunday = 0, etc.), M is the residue for the month (from the table), D is the day of the month, Z is the last two digits of the year, and $L = 0$ except when the given date is in January or February of a leap year and then $L = 1$.

Since x is the only unknown it can readily be calculated by working out the residue of the right-hand side of the formula. The term L is in the formula because in a leap year it is only days which occur *after* February 29 that are advanced because of the extra day in the year. Dates *before* February 29 should be treated as though they were in an ordinary year, but in our calculation $[\frac{Z}{4}]$ of the number of leap years since 1900 we also count the year itself if it is a leap year. For example only 5 leap years should be counted when considering a date in January 1924, but $[\frac{24}{4}] = 6$. The term L subtracts this extra 1 where necessary.

Example: What day of the week was November 11, 1918?

$L = 0$, $M = 3$, $D = 11$, $Z = 18$, $[\frac{Z}{4}] = [4\cdot5] = 4$,

$M + D + Z + [\frac{Z}{4}] - L = 3 + 11 + 18 + 4 - 0 = 36 \equiv 1$,

so this was a Monday.

Example: What day of the week was February 8, 1976?

$L = 1$, $M = 3$, $D = 8$, $Z = 76$, $[\frac{Z}{4}] = [19] = 19$,

$x \equiv 3 + 8 + 76 + 19 - 1 = 105 \equiv 0$,

so this was a Sunday.

To use the formula you need to have the table of residues for each month (or to remember them). Other formulae have been worked out which don't require such a table. Here is one which was worked out in 1882. Notice that this formula works not only for dates in the twentieth century but for all dates in the Gregorian calendar.

$$x \equiv D + [\frac{26(N+1)}{10}] + Z + [\tfrac{Z}{4}] + [\tfrac{J}{4}] - 2J - 1$$

where the given date is day D of month N of year $J \times 100 + Z$ (i.e. D and Z are just as before). On applying this formula the convention is made that January and February of any year are to be regarded as the 13th and 14th months of the previous year. (This is for the same reason as the term L was included in the first formula.)

N.B. Work out the residues of the numbers $[\frac{26(N+1)}{10}]$ for $N = 3, \ldots, 14$ and compare these with the given tables of residues. You should then be able to see that for dates in the twentieth century (i.e. when $J = 19$) the two formulae are exactly the same.

Example: January 26, 1788.

$D = 26$, $N = 13$, $J = 17$, $Z = 87$, so $26(N + 1) = 364$ and

$x \equiv 26 + [36\cdot4] + 87 + [21\cdot75] + [4\cdot25] - 34 - 1 = 139 \equiv 6.$

So January 26, 1788, was a Saturday.

Using the formulae given above we can answer questions of the form "On which day of the week does a given date occur?". With a knowledge of some properties of residues and congruences one can use the formulae to answer different sorts of questions, for example:

"In which years of this century does February have 5 Sundays?", and

"In which years of this century does my birthday fall on a Saturday?".

Unfortunately we do not have the space here to go into these things. You might like to try to solve the following problems yourself.

1. Show that if $a \equiv b$ and $c \equiv d$ then $ac \equiv bd$. (Hint: Look at the way we proved the *RULE* on page 4 of this article.)

2. Find the residue of 6^{40}. (Hint: Use question 1.)

3. Show that if $ab \equiv ac$ then $b \equiv c$, or $a \equiv 0$. (Hint: Recall that we write $x \equiv y$ to mean that x and y have the same remainder after dividing by 7. To solve this problem you will need to use the fact that 7 is a prime number.)

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PROBLEM 1.1 (i.e. problem number 1 in Part 1 of *Function*).

Show that the thirteenth day of the month is more likely to fall on a Friday than on any one other day of the week. (Solutions invited.)

FIBONACCI SEQUENCES

by Christopher Stuart, 6th form, Kingswood College

The Fibonacci Sequence is a sequence of numbers beginning 1, 1 ... in which each term is equal to the sum of the previous two. The sequence is thus:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 ...

It has many fascinating properties, and a detailed study opens many exciting avenues for research, some of which will be explored here.

The first notable point is that the sequence is defined if two terms are specified. For example, the first two terms 1, 1 determine the Fibonacci Sequence. If different first two terms are used, and the same rule is used to generate subsequent terms, a similar sequence is formed. For example, when the first two are 3, 4, the following sequence results:

3, 4, 7, 11, 18, 29, 47, 76, 123 ...

The sequence beginning 1, 1, ... will be referred to as the Fibonacci Sequence, and any others as fibonacci sequences.

Denote the n^{th} term of a fibonacci sequence by t_n. Thus $t_1 = 1$, $t_2 = 1$, $t_3 = 2$, $t_4 = 3$, etc., for the Fibonacci Sequence, and for any fibonacci sequence $t_3 = t_1 + t_2$, $t_4 = t_2 + t_3$, and so on; in general the $(n + 1)^{th}$ term is the sum of the n^{th} and $(n - 1)^{th}$ terms, i.e. $t_{n+1} = t_n + t_{n-1}$. There is no reason why a fibonacci sequence cannot be extended to the left by rewriting this rule as, $t_{n-1} = t_{n+1} - t_n$. Allowing n now also to run through the negative integers the extended Fibonacci Sequence becomes:

... -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21 ....

Note the symmetry, except for the sign of the terms, of the sequence about zero. There is basically only one other symmetrical fibonacci sequence, the Lucas Sequence, which is:

... -29, 18, -11, 7, -4, 3, -1, 2, 1, 3, 4, 7, 11, 18, 29 ....

All other symmetrical fibonacci sequences are multiples of one of these. A multiple of a fibonacci sequence is one which has had every term multiplied by a constant. For example, the

Lucas Sequence multiplied by four is:

... 72, -44, 28, -16, 12, -4, 8, 4, 12, 16, 28, 44, 72 ....

A multiple of a fibonacci sequence is another fibonacci sequence. If two fibonacci sequences are added term by term, then the sum is another fibonacci sequence.

Using these results it is possible to form a general fibonacci sequence in which the first two terms are x, y, as follows:

...	-x,	x	0,	x,	x,	2x,	3x,	5x	8x,	13x	...
...	y,	0,	y,	y,	2y,	3y,	5y,	8y,	13y,	21y	...
...	y-x,	x,	y,	x+y,	x+2y,	2x+3y,	3x+5y,	5x+8y,	8x+13y,	13x+21y	...

This general sequence gives us a method of calculating any term of any fibonacci sequence, given the Fibonacci Sequence.

Let f_n be the n^{th} term of the Fibonacci Sequence, where $f_1 = 1$ and $f_2 = 1$; and let t_n be the n^{th} term of a fibonacci sequence where $t_1 = x$ and $t_2 = y$. Looking at the general fibonacci sequence above, it is seen that:

$$t_n = xf_{n-2} + yf_{n-1} \qquad (1)$$

This formula enables us to calculate terms for a fibonacci sequence.

For example, if $t_1 = 3$ and $t_2 = 4$, find t_7. Here $t_n = xf_{n-2} + yf_{n-1}$, where $x = 3$ and $y = 4$. Hence $t_7 = xf_5 + yf_6$ $= 3f_5 + 4f_6 = 3 \times 5 + 4 \times 8$, since $f_5 = 5$ and $f_6 = 8$, so that $t_7 = 47$.

The sequence is:

3, 4, 7, 11, 18, 29, 47 ...,

so the answer is correct.

An important find is made if each term in the Fibonacci Sequence is divided by the previous one. A sequence is formed which is as follows:

$$\ldots, \frac{5}{-8}, \frac{-3}{5}, \frac{2}{-3}, \frac{-1}{2}, \frac{1}{-1}, \frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$$

This sequence approaches a limit as the following argument shows. Put $d_n = f_{n+1}/f_n$. Then it is not too difficult to check that $d_{n+1} - d_n = (-1)^{n+1}/f_n f_{n+1}$. A standard argument, using the theory of alternating series†, then shows that d_n tends to a limit as n tends to infinity.

Let us now find this limit. By the symmetry of the Fibonacci Sequence,

$$d_{-n} = f_{-n+1}/f_{-n} = -f_{n-1}/f_n$$
$$= -1/d_{n-1}.$$

Put $\lim_{n\to-\infty} d_n = d_{-\infty}$ and $\lim_{n\to\infty} d_n = d_\infty$. Thus $d_{-\infty} = -1/d_\infty$,

i.e. $$d_\infty \cdot d_{-\infty} = -1. \qquad (2)$$

Also $$d_n + d_{-n} = f_{n+1}/f_n - f_{n-1}/f_n$$
$$= f_{n+1}/f_n - (f_{n+1} - f_n)/f_n$$
$$= f_n/f_n = 1.$$

Thus, taking the limit,

$$d_\infty + d_{-\infty} = 1. \qquad (3)$$

From (2) and (3) it follows that d_∞ and $d_{-\infty}$ are the solutions of the quadratic equation

$$x^2 - x - 1 = 0 \qquad (4)$$

Thus $d_\infty = (\sqrt{5} + 1)/2$ and $d_{-\infty} = -(\sqrt{5} - 1)/2$. Note that these values satisfy $d_{-\infty} = -1/d_\infty$.

We shall denote $(\sqrt{5} + 1)/2$ by ϕ. It is called the *golden ratio*. Hence $d_\infty = \phi$, $d_{-\infty} = -1/\phi$.

In fact any fibonacci sequence displays this same limit property. Let $t_1 = x$, $t_2 = y$ be the first two terms of a fibonacci sequence. Then, by equation (1), $t_n = xf_{n-2} + yf_{n-1}$, so

$$t_{n+1}/t_n = (xf_{n-1} + yf_n)/(xf_{n-2} + yf_{n-1})$$
$$= \frac{f_{n-1}}{f_{n-2}} \cdot \frac{x + y(f_n/f_{n-1})}{x + y(f_{n-1}/f_{n-2})}.$$

Now $\lim_{n\to\infty} (x + y(f_n/f_{n-1})) = \lim_{n\to\infty} (x + y(f_{n-1}/f_{n-2}))$. Hence, if this common limit is non-zero,

$$\lim_{n\to\infty} t_{n+1}/t_n = \lim_{n\to\infty} f_{n-1}/f_{n-2} = \phi.$$

†See a later issue of *Function* for a discussion of alternating series.

Similarly, $\lim_{n\to-\infty} t_{n+1}/t_n = -1/\phi$

except when $\lim_{n\to\infty} (x + y(f_{-n}/f_{-n-1})) = 0.$

I now deal with the exceptional cases.

When the terms of a fibonacci sequence are not necessarily integers we meet in particular the special cases where the arguments used to show that $\lim_{n\to\infty} t_{n+1}/t_n = \phi$ and $\lim_{n\to-\infty} t_{n+1}/t_n = -1/\phi$ break down. As noted, the argument broke down when

$$\lim_{n\to\infty} (x + y(f_n/f_{n-1})) = 0$$

i.e. when $$x + y\phi = 0, \qquad (5)$$

and when $$\lim_{n\to\infty} (x + y(f_{-n}/f_{-n-1})) = 0,$$

i.e. when $$x + y(-1/\phi) = 0. \qquad (6)$$

Let us consider the case of equation (6). In particular (6) is satisfied when $x = 1$, $y = \phi$. Since, by equation (4) $\phi^2 - \phi - 1 = 0$, we have

$$1 + \phi = \phi^2,$$

$$\phi + \phi^2 = \phi(1 + \phi) = \phi \,.\, \phi^2 = \phi^3,$$

$$\phi^2 + \phi^3 = \phi^2(1 + \phi) = \phi^2 \,.\, \phi^2 = \phi^4,$$

and so on. Hence the sequence is

$$\ldots\ 1,\ \phi,\ \phi^2,\ \phi^3,\ \ldots,\ \phi^n,\ \ldots$$

This fibonacci sequence is a geometric sequence as well. In fact the continuation to the left,

$$\ldots\ 5\phi - 8,\ 5 - 3\phi,\ 2\phi - 3,\ 2 - \phi,\ \phi - 1,\ 1\ \ldots$$

may be shown to have the same constant ratio between terms. For this sequence, therefore

$$\lim_{n\to\infty} t_{n+1}/t_n = \phi = \lim_{n\to-\infty} t_{n+1}/t_n.$$

This sequence is called the *perfect* fibonacci sequence.

Similarly, considering the case of equation (5), we get a fibonacci sequence which is also a geometric sequence, with common ratio now $-1/\phi$. This sequence is called the *imperfect* fibonacci sequence.

Any fibonacci sequence can be expressed as the sum of multiples of these two sequences. I shall do this for the general fibonacci sequence.

$$k\phi^{-3}, \quad k\phi^{-2}, \quad k\phi^{-1}, \; k, \quad k\phi, \quad k\phi^{2}, \quad k\phi^{3} \; \ldots$$

$$\ldots \; c(\tfrac{-1}{\phi})^{-3}, \; c(\tfrac{-1}{\phi})^{-2}, \; c(\tfrac{-1}{\phi})^{-1}, \; c, \; c(\tfrac{-1}{\phi}), \; c(\tfrac{-1}{\phi})^{2}, \; c(\tfrac{-1}{\phi})^{3} \; \ldots$$

$$\ldots \; 2y - 3x, \quad 2x - y, \quad y - x, \; x, \quad y, \quad x + y, \; x + 2y \; \ldots$$

We find the required multiples k and c in terms of x and y from the equations

$$k + c = x$$

$$k\phi - c/\phi = y.$$

Solving these gives

$$k = (x/\phi + y)/(\phi + 1/\phi)$$

$$c = (x\phi - y)/(\phi + 1/\phi).$$

But, since $\phi = (\sqrt{5} + 1)/2$, it follows that $\phi + 1/\phi = \sqrt{5}$. Hence

$$k = (x/\phi + y)/\sqrt{5},$$

$$c = (x\phi - y)/\sqrt{5}.$$

We now know a new method of calculating the n^{th} term in a fibonacci sequence:

$$t_n = k\phi^n + c(-1/\phi)^n \qquad (7)$$

where $t_0 = x, \quad t_1 = y,$

and k and c are given above.

This new formula may be used to prove many results. For example, that $\lim_{n\to\infty} t_{n+1}/t_n = \phi.$

Using these new formulae, it is possible to give a meaning to t_n for non-integral n; $t_{1/2}$, for example, but the answer is a complex number. In general, for n not an integer, several values of t_n will satisfy (7). There is room for more research here.

One final aspect of ϕ is its value as a number base, and was found quite accidentally.

Our number system is to base 10, computers use base 2, and other integer bases may easily be used. However, what is the most simple non-integer base?

To find this, consider the simple problem, 1 + 1 = 2, and express 2 in numbers to a certain base involving ones and zeros. In base 2, the answer is 10. Any solution with only one one in it will be basically binary, and will not do. Any solution involving only two ones will just use the ones for counting. Thus the solution must have three ones. The most simple solutions are 111, 11·1 and 1·11.

Finding bases for which these results hold yields the following bases:

For 1 + 1 = 111, bases $1/\phi$ or $-\phi$.

For 1 + 1 = 11·1, no solution, if we exclude complex bases.

For 1 + 1 = 1·11, bases $-1/\phi$ or ϕ.

Since the simplest base is one which is greater than one, ϕ is the simplest non-integer base. The interesting aspect of this number as a base is that integers may be expressed so easily, in spite of the fact that the base is not rational.

Here are a few 'phinary' numbers:

2 = 1·11

3 = 11·01 = 100·01

5 = 1000·1001 = 1000·0111 = 110·1001 = 101·1111

10 = 10011·0101.

Note that there are always infinitely many ways of expressing a number.

Every new discovery opens new avenues for exploration, but an end must be made, leaving further exploration into the wonders of mathematics to those who have the same fascination as I of the pattern that lies behind all things.

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LEONARDO FIBONACCI (1175-1250), though born in Pisa, was brought up and educated at Bugia in Barbary where his father was a merchant and customs controller. His book *Liber Abaci* (1202) is generally credited with having introduced the arabic numeral symbols into Europe. By the end of the 13th century they had largely superseded Roman numerals.

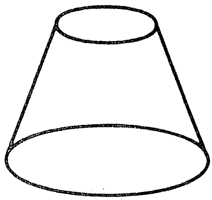
He had a great reputation as a mathematician and Emperor Frederick II stopped at Pisa in 1225 to hold a mathematical tournament to test Leonardo's skill. Leonardo solved all the questions set and his competitors solved none of them.

PROBLEM 1.2. (This was one of the problems solved by Leonardo.)

Find a number of which the square when increased or decreased by 5 remains a square. (All numbers are to be taken to be rational numbers.)

PROBLEM 1.3.

Prove that the volume of a frustum of a cone is obtained by either of the rules: (1) To the areas of the two ends of the frustum add the square root of their product; multiply the result by $\frac{1}{3}$ of the perpendicular height. (2) To the product of the diameters of the two ends, add the sum of their squares; multiply this sum by the height, and again by ·2618.



(From A. Nesbit, A Treatise on Practical Mensuration, 1841.)

PROBLEM 1.4.

The left hand digit of a natural number is removed and replaced at the right hand end, and this results in increasing the original number by fifty percent. Find such a natural number. (Solutions invited.)

PROBLEM 1.5.

A newspaper report stated that the combined effect of Australia's 17·5% devaluation and New Zealand's 7% devaluation was to revalue the New Zealand dollar by 12·7% in comparison with the Australian dollar. Where does this figure come from. Is it correct?

PROBLEM 1.6.

The blackboard has been filled with 100 statements, as follows:

"Exactly one of these statements is incorrect.
Exactly two of these statements are incorrect.

.
.
.

Exactly one hundred of these statements are incorrect".

Which (if any) of the 100 statements is correct?

PROBLEM 1.7. (Supplied by Christopher Stuart.)

Show that if t_n is the n^{th} term of any fibonacci sequence then

$$t_n^2 - t_{n-2}^2 = t_{n-1}(2t_{n-1} + t_{n-4}).$$

PROBLEM 1.8.

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \text{ and } \log_e n$$

are not equal. But for large natural numbers n, the difference between them is quite small. Use a calculator or computer to investigate how the difference between them varies as n increases.

In fact, $\lim\limits_{n\to\infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log_e n)$ exists.
Roughly what is the limiting value?

PROBLEM 1.9.

Fibonacci (1202 A.D.) considered the following problem. A pair of rabbits is put into an enclosure. They produce one pair of offspring in the first month and they reproduce just once more, namely a second pair of offspring in the second month.

Similarly each pair of offspring follows exactly the same pattern of reproduction, beginning to reproduce one month after birth. There is no other breeding between other pairs of rabbits.

Show that the number of pairs produced in a certain month is equal to the numbers produced during the preceding two months. Relate this to the Fibonacci Sequence as discussed in Christopher Stuart's article.

PROBLEM 1.10.

Consider the following

$$(1 + \sqrt{2})^2 = 3 + 2\sqrt{2} \simeq 3 + 2{\cdot}828$$

$$(1 + \sqrt{2})^3 = 7 + 5\sqrt{2} \simeq 7 + 7{\cdot}070$$

$$(1 + \sqrt{2})^4 = 17 + 12\sqrt{2} \simeq 17 + 16{\cdot}968.$$

From the above we might guess that if

$(1 + \sqrt{2})^n = a + b\sqrt{2}$ where a, b, n are positive integers

then a is the integer closest to $b\sqrt{2}$. Prove this.

Use a computer to print a, an approximation to $b\sqrt{2}$ and the difference between a and $b\sqrt{2}$ as n increases.

Can you generalize the above problem in any way? If so prove your generalization. (Solutions invited.)

"Let us sit on this log at the roadside," says I, "and forget the inhumanity and ribaldry of the poets. It is in the glorious columns of ascertained facts and legalized measures that beauty is to be found. In this very log we sit upon, Mrs Sampson," says I, "is statistics more wonderful than any poem. The rings show it was sixty years old. At the depth of two thousand feet it would become coal in three thousand years. The deepest coal mine in the world is at Killingworth, near Newcastle. A box four feet long, three feet wide, and two feet eight inches deep will hold one ton of coal. If an artery is cut, compress it above the wound. A man's leg contains thirty bones. The Tower of London was burned in 1841."

"Go on, Mr Pratt," says Mrs Sampson. "Them ideas is so original and soothing. I think statistics are just as lovely as they can be."

O. Henry, *The Handbook of Hymen*

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"Would you tell me, please, which way I ought to go from here?"

"That depends a good deal on where you want to get to," said the Cat.

"I don't much care where ——" said Alice.

"Then it doesn't matter which way you go," said the Cat.

Lewis Carroll, *Alice's Adventures in Wonderland*

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M A T H E M A T I C S   L E C T U R E S

A series of lectures will be held at Monash University for 5th and 6th form mathematics students. At present, it is planned to have one lecture per fortnight, on Friday evenings, 7 p.m. to 8 p.m. (approximately). A detailed programme will be circulated later, but the first lecture will be given on

FRIDAY, MARCH 25, 1977

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MR NEIL CAMERON WILL SPEAK ON:

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