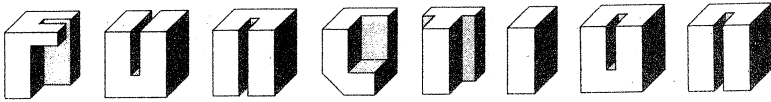
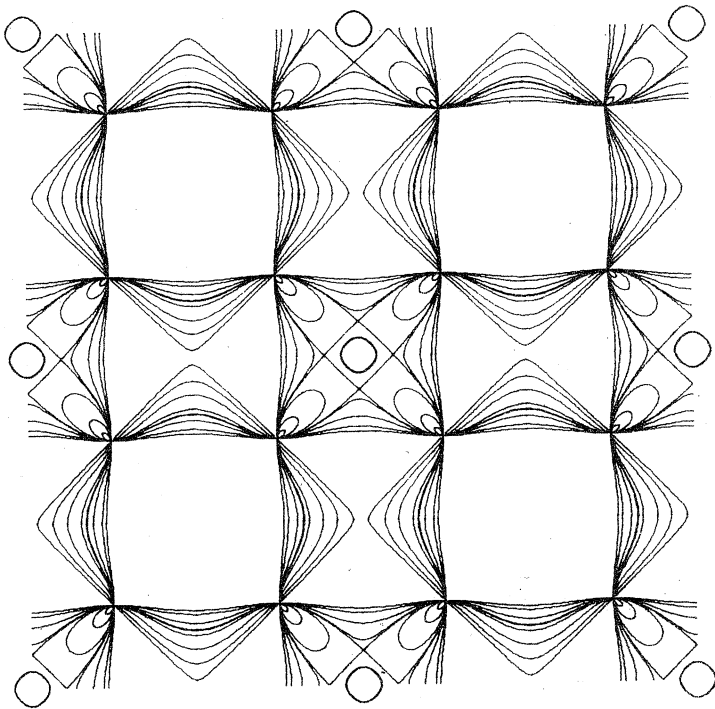


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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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What happens, from generation to generation, to the size of a population of insects? Our leading article this issue, by Peter Kloeden, discusses some possible mathematical models that have been used to try to find answers to this question. The mathematics used includes some that has been discovered only recently and has helped to unravel and give an explanation for what at first sight seemed bizarre and almost random fluctuations in population sizes.

This year has seen an increase in the number of schools taking *Function*. We have tried to put into effect the suggestions we have received in letters from readers. Continue please to make comments. We would like more articles from school students please. They do not have to be on difficult subjects. Anything that interests you will probably interest your friends.

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THE FRONT COVER

J. O. Murphy, Monash University

"- the trigonometrical functions acting in fact as a kind of kaleidoscope" is a description, extracted from a section on repeating curves in a book on Curve Tracing, by P. Frost, first published in 1872, of the form of curves whose equations involve trigonometrical functions of the coordinates in place of the coordinates themselves. If $f(x,y) = 0$ is the equation of any curve, then by replacing x by a trigonometrical function of y and y by a trigonometrical function of x , a possible new curve, using sines for example, is given by

$$f(\sin y, \sin x) = 0.$$

There appear to be no specific applications for this approach in curve tracing other than the development of elegant figures which are obtained from symmetrical arrangements. The total effect is made up of a basic pattern repeated periodically across the plane.

The front cover diagram is based on the equation

$$\sin^3 y + \sin^3 x - 3a \sin y \sin x = 0,$$

derived from the auxiliary equation $x^3 + y^3 - 3axy = 0$, where a is a parameter. For $a < \sqrt[3]{2}/2$ loops are obtained which all pass through the origin (0,0) and when $a = \sqrt[3]{2}/2$ the loop, which is the largest one possible, touches the lines $y = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$ and has a discontinuity in its gradient at the point of contact with these lines. For $\sqrt[3]{2}/2 < a < \frac{2}{3}$ we have closed curves which are centered on the point $(\frac{\pi}{2}, \frac{\pi}{2})$ and for values of $a > \frac{2}{3}$ the sine type curves result.

Our computer drawn version of the curve compares very well with the diagram published in Frost's book mentioned above and in view of the total computing effort involved, running time and programming, we now have considerable regard for the accuracy of the techniques applied by Frost over one hundred years ago. The numerical procedures applied for plotting the curves used the Runge-Kutta method for solving systems of simultaneous differential equations. This now well known numerical method was initially developed some years after the first publication of Frost's book by Runge (about 1894) and modified a little later on by Kutta.

Frost also exhibited drawings of the interesting curves

$$\tan^3 y + \sin^3 x - 3a \tan y \sin x = 0$$

and
$$\tan^3 y + \tan^3 x - 3a \tan y \tan x = 0$$

which are clearly other trigonometrical variations of the above auxiliary equation.

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STABILITY AND CHAOS IN INSECT POPULATION DYNAMICS

Peter Kloeden, Murdoch University

For many species of insects, such as cicadas, the insects of a given generation lay eggs and die before any insects of the next generation appear. Biologists say that such species have *non-overlapping generations*. Compared with human beings this is a relatively uncomplicated situation, with the population of the next generation depending only on the population of the current generation, being related to it by what mathematicians call a difference equation.

A *difference equation* is an equation of the form

$$x_{n+1} = f(x_n) \quad (1)$$

where $f(x)$ is a known function and $n = 0, 1, 2, \dots$. In the biological context being considered here, the function $f(x)$ depends on the particular species of insects, for which x_n and x_{n+1} are the populations of the n^{th} and $(n + 1)^{\text{th}}$ generations, respectively. Equation (1) says that for this species the population x_{n+1} of the $(n + 1)^{\text{th}}$ generation is equal to $f(x_n)$ where x_n is the population of the n^{th} generation. Moreover this relationship holds for all generations $n = 0, 1, 2, \dots$. Consequently if the population x_0 of the initial generation is known, then the populations $x_1, x_2, x_3, \dots, x_n, \dots$ of all successive generations can be calculated by repeatedly using equation (1):

$$x_1 = f(x_0), \quad x_2 = f(x_1), \quad x_3 = f(x_2), \quad \dots \quad (2)$$

The sequence $x_0, x_1, x_2, \dots, x_n, \dots$ thus describes the population dynamics for this species of insects and initial population. Usually a different initial population will give a different sequence of populations.

The simplest kind of difference equation has the form

$$x_{n+1} = \alpha x_n \quad (3)$$

where α is a constant. Here the function $f(x) = \alpha x$ is a linear function and equation (3) is called a *linear difference equation*. For insect populations the constant α is called the *growth coefficient* of the species under consideration. It is the ratio of populations of any two successive generations and is of course a positive number. For such an equation the simple formula

$$x_n = \alpha^n x_0 \quad (4)$$

expresses the population x_n of the n^{th} generation in terms of n and the initial population x_0 . (To see this, note by (2) that $x_1 = \alpha x_0$, $x_2 = \alpha x_1 = \alpha(\alpha x_0) = \alpha^2 x_0$, and so on.) Such a formula

is called a *solution* of the difference equation. Knowing it is very useful because the population for any initial population and generation can then be quickly calculated without having to do the tedious iterative calculations in (2). More importantly, it allows a complete classification to be made of all possible behaviour of the population dynamics. For difference equation (3) there are three different possibilities depending on the size of the growth coefficient α :

Case 1. When $\alpha > 1$, then $\alpha^n \rightarrow \infty$ as $n \rightarrow \infty$ and so $x_n \rightarrow \infty$ as $n \rightarrow \infty$ for all positive initial populations. Here the population grows larger and larger with successive generations. This is called *exponential growth* and was discovered by the 18th century English clergyman Malthus, who made dire predictions of a future world of standing room only!

Case 2. When $\alpha = 1$, then $\alpha^n = 1$ and so $x_n = x_0$ for all n . Here the population remains constant for all future generations, which is called *zero population growth* by the American biologist Paul Ehrlich. It offers a more optimistic view of the future than the unbounded exponential growth of case 1.

Case 3. When $0 < \alpha < 1$, then $\alpha^n \rightarrow 0$ as $n \rightarrow \infty$ and so $x_n \rightarrow 0$ as $n \rightarrow \infty$ no matter what was the initial population. Here the population tends to extinction. This is called *exponential decay* and is well known to nuclear physicists with the radioactive decay of elements such as plutonium.

In the past biologists often used linear difference equations like (3) as models of the dynamics of insect populations. A major reason for this seems to have been the simplicity and completeness of the above analysis. Such models are however rarely realistic and it is not too hard to see why: a larger population requires more food and living space, the availability of which are not unlimited. Also pollution and other environmental stresses increase as the population increases. These factors not only limit the population that can be supported, but also tend to decrease the rate of population increase as the population approaches its maximum saturation value. Hence the growth coefficient α in equation (3) is not a constant, but decreases as the population increases to its saturation value, where there is no growth. The simplest way of describing this mathematically is

$$\alpha = \beta(S - x) \quad (5)$$

where β is a positive constant and S is the saturation population. With this varying growth coefficient the difference equation becomes

$$x_{n+1} = \beta(S - x_n)x_n \quad (6)$$

with $f(x) = \beta(S - x)x$, which is a quadratic function, so equation (6) is a *nonlinear difference equation*.

For equation (6) to be biologically realistic some restrictions must be placed on the values of β and x . First, the population x cannot be negative and cannot exceed the saturation population S , so must be restricted to the interval $0 \leq x \leq S$. Second, the constant β must be restricted to the interval

$0 < \beta \leq 4/S$. To see why it cannot exceed $4/S$ note that the quadratic function $f(x) = \beta(S - x)x$ has the maximum value $\beta S^2/4$ when $x = S/2$. As all population sequences are to lie in the interval $0 \leq x \leq S$, this maximum value $\beta S^2/4$ cannot exceed S . See figure 1.

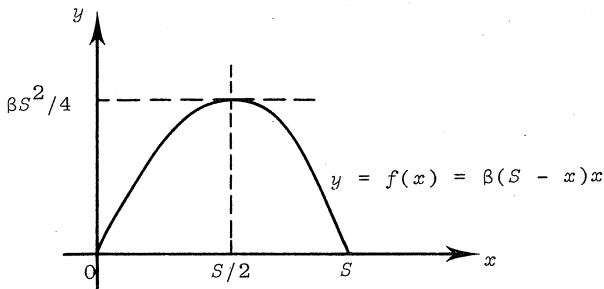


Figure 1.

Unfortunately nobody has been able to discover a simple formula like (4) for the solutions of the nonlinear difference equation (6). This makes an analysis of the behaviour of its population sequences much more difficult than for the linear difference equation (3), but such an analysis is well worth doing because, as will be seen below, some very interesting and unusual things can happen. To make it simpler, a new variable $z = x/S$, the ratio of population x to saturation population S , will be used. Substitution into equation (6) and using a new constant $\gamma = \beta S$ gives a new difference equation

$$z_{n+1} = \gamma(1 - z_n)z_n \quad (7)$$

which has the same quadratic form as equation (6), but with β replaced by γ and S set equal to 1. Consequently the restrictions on β and x to make equation (6) biologically meaningful become $0 \leq z \leq 1$ and $0 < \gamma \leq 4$ in terms of γ and z for equation (7).

The behaviour of the sequences of population ratios z_0, z_1, z_2, \dots depends crucially on the magnitude of γ , with more and more interesting and unusual possibilities as γ gets closer to its upper bound 4. Unlike the linear difference equation (3), the behaviour here in many cases also depends very much on the initial population ratio z_0 .

The simplest case is one of exponential decay which occurs when $0 < \gamma < 1$. To see this note for any $0 \leq z \leq 1$ that $0 \leq 1 - z \leq 1$ and so $\gamma(1 - z)z \leq \gamma z$, which gives

$$z_{n+1} \leq \gamma z_n \quad (8)$$

for $n = 0, 1, 2, \dots$ when used in equation (7). Repeated application of inequality (8) and the fact that population ratios are non-negative leads to the bounds

$$0 \leq z_n \leq \gamma^n z_0 \quad (9)$$

for $n = 1, 2, 3, \dots$. (To see this note by (8) that $z_1 \leq \gamma z_0$, $z_2 \leq \gamma z_1 \leq \gamma(\gamma z_0) = \gamma^2 z_0$, and so on.) These bounds do not give an exact formula for the solution of difference equation (7) as (4) does for the linear difference equation (3), but they are just as useful in the case under consideration in investigating the behaviour of the population ratio sequences. Here $0 < \gamma < 1$, so $\gamma^n \rightarrow 0$ as $n \rightarrow \infty$ and hence from the bounds in (9) $z_n \rightarrow 0$ as $n \rightarrow \infty$ for every initial population ratio z_0 . Thus for $0 < \gamma < 1$, every population ratio sequence decays exponentially to zero, which corresponds to the extinction of the insect species.

Note here that if the population ratio is zero then it remains zero for all future generations, which is biologically expected for a closed environment without immigration from elsewhere. Hence the population ratio sequence z_0, z_1, z_2, \dots corresponding to initial population ratio $z_0 = 0$ consists only of the zero population ratio, that is $z_n = 0$ for $n = 0, 1, 2, \dots$. This zero population ratio is called an *equilibrium* population ratio of difference equation (7) and biologically corresponds to zero population growth. Mathematically an equilibrium population ratio is a *fixed point* of the function $f(z) = \gamma(1 - z)$ in terms of which difference equation (7) is defined, that is $z = 0$ satisfies the equation

$$z = f(z). \quad (10)$$

When $0 < \gamma < 1$ the zero population ratio $z = 0$ is the only equilibrium or zero population growth population ratio. The above analysis shows that every other population ratio sequence converges towards this equilibrium population ratio, which is then said by mathematicians to be *stable*. (The terms equilibrium and stability here are borrowed from mechanics.) Mathematically this means for the case $0 < \gamma < 1$ that the behaviour of the population ratio sequences of difference equation (7) is very regular and predictable. Biologically it means that the insect species is doomed to extinction, which is not necessarily a disaster if the insects are a pest.

The next case is when $\gamma = 1$, where again the only equilibrium population ratio is $z = 0$. But do all population ratio sequences converge to this zero growth population ratio? As $\gamma = 1$ the second inequality in (9) gives no useful information here. A few numerical calculations illustrated in figure 2 for different initial population ratios seems to indicate that the sequences do converge to zero.

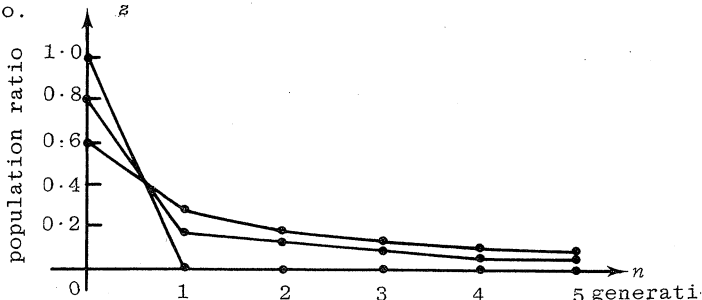


Figure 2.

To prove this mathematically note that $0 \leq 1 - z \leq 1$ and so $(1 - z)z \leq z$ for all $0 \leq z \leq 1$, which gives

$$0 \leq z_{n+1} \leq z_n$$

for $n = 0, 1, 2, \dots$ when used in equation (7). Hence the successive population ratios in a sequence are decreasing. As $z = 0$ is the only equilibrium ratio this means every sequence of population ratios must converge to it. (They must converge because they are nonnegative and decreasing. Also they cannot converge to a non-zero population ratio, because that population ratio would be equilibrium population ratio for difference equation (7), which would contradict the fact that $z = 0$ is the only equilibrium population when $\gamma = 1$.) Consequently when $\gamma = 1$ the zero population ratio is also stable. There is however a slight difference from the first case in that the population ratio sequences all converge to zero, but do not converge in the rapid exponential fashion of the first case indicated by the second inequality in (9).

The next case is when $1 < \gamma < 3$. Here there are two equilibrium population ratios for difference equation (7), namely $z = 0$ and $z = 1 - \gamma^{-1}$. To see this solve equation (10), that is, $z = \gamma(1 - z)z$, for z . This can be done either algebraically or graphically, where the equilibrium points are just the intersection points of the graphs of $y = z$ and $y = \gamma(1 - z)z$. See figure 3.

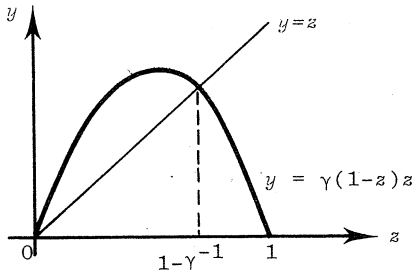


Figure 3.

There are now two zero growth population ratios. The first is the same extinction ratio $z = 0$ as in the earlier cases and the second is the positive ratio $z = 1 - \gamma^{-1}$. What happens now to the population ratio sequences for initial population ratios different from these two equilibrium ratios? Do they converge to one or the other equilibrium ratio? Numerical calculations for $\gamma = 2$ are illustrated in figure 4 for several different population ratios. These suggest that the population ratio sequences with initial ratio z_0 satisfying $0 < z_0 < 1$ converge to the nonzero equilibrium ratio $z = 1 - \gamma^{-1}$, with only those sequences starting at $z_0 = 0$ or 1 converging to the zero equilibrium ratio.

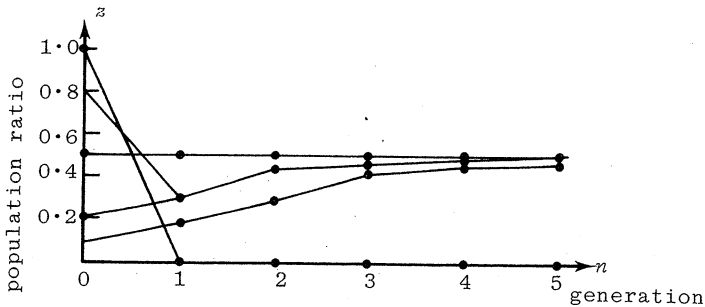


Figure 4.

To prove this mathematically is not so simple. On changing variables to $d = z - 1 + \gamma^{-1}$ difference equation (7) becomes

$$\begin{aligned}
 d_{n+1} &= z_n - 1 + \gamma^{-1} \\
 &= \gamma(\gamma^{-1} - d_n)(d_n - \gamma^{-1} + 1) - 1 + \gamma^{-1} \\
 &= (2 - \gamma)d_n - \gamma d_n^2.
 \end{aligned} \tag{11}$$

When z_n is close to $1 - \gamma^{-1}$, then d_n is close to 0 and d_n^2 is even closer to 0. Neglecting this quadratic term in (11) then gives the linear approximation

$$d_{n+1} \approx (2 - \gamma)d_n \tag{12}$$

and so

$$d_n \approx (2 - \gamma)^n d_0 \tag{13}$$

for $n = 0, 1, 2, \dots$. As $1 < \gamma < 3$, then $-1 < 2 - \gamma < 1$ and so $(2 - \gamma)^n \rightarrow 0$ as $n \rightarrow \infty$. Hence $d_n \rightarrow 0$ as $n \rightarrow \infty$ or equivalently $z_n \rightarrow 1 - \gamma^{-1}$ as $n \rightarrow \infty$ when d_0 is sufficiently close to 0 or z_0 sufficiently close to $1 - \gamma^{-1}$, that is the population ratio sequences all converge to the nonzero equilibrium ratio $1 - \gamma^{-1}$ whenever they start sufficiently close to it. Actually every population ratio sequence with initial z_0 satisfying $0 < z_0 < 1$ converges to the nonzero equilibrium ratio $1 - \gamma^{-1}$. To see this note from figure 3 that for any such z_0 the successive ratios z_n eventually get close enough to $1 - \gamma^{-1}$ for the linear approximation (12) to become valid. The only population ratio sequences which do not converge to $1 - \gamma^{-1}$ are those starting at $z_0 = 0$ or 1. Disregarding these two extreme cases, the nonzero equilibrium population ratio $1 - \gamma^{-1}$ is thus stable. Biologically this means there is a stable nonzero population with zero population growth.

Interestingly when $2 < \gamma < 3$ the population ratio sequences oscillate about the equilibrium ratio $1 - \gamma^{-1}$ as they converge towards it. This is because $2 - \gamma$ is then negative, so $(2 - \gamma)^n$, and hence by (13) d_n too, oscillates in sign. Such dying or

damped oscillations are actually observed in many laboratory experiments, which suggests that difference equation (7) is a fairly realistic model. See figure 5 for $\gamma = 2.5$.

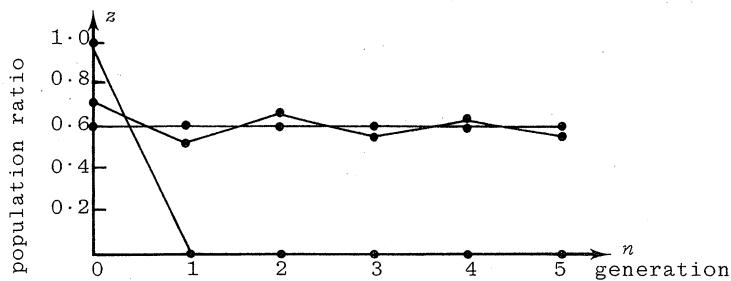


Figure 5.

When $\gamma = 3$ numerical calculations, not illustrated here, indicate that the behaviour of the population ratio sequences is much the same as in the previous case. The interested reader is invited to think of how to verify this mathematically. Note that the linear approximation (12) does not give any useful information in this case.

What happens when $\gamma > 3$? As in the two previous cases both 0 and $1 - \gamma^{-1}$ are still equilibrium points, but neither is stable as there are population ratio sequences which do not converge to either of them. In fact there are population ratio sequences which are *cyclic* or *periodic*, that is which oscillate between several different population ratios. The number of different population ratios in such a cyclic sequence is called its *period*. Sequences consisting of just one of the equilibrium ratios 0 or $1 - \gamma^{-1}$ can be considered cyclic with period 1. The simplest non-trivial cyclic sequence has period 2, that is consists of just two different population ratios, between which it alternates. If these two population ratios are a and b respectively then such a cyclic sequence has the form

$$z_0 = a, z_1 = b, z_2 = a, z_3 = b, z_4 = a, \dots$$

Hence $z_{n+2} = z_n$ for $n = 0, 1, 2, \dots$. Also such a sequence satisfies difference equation (7) so for $n = 0, 1, 2, \dots$

$$z_{n+2} = f(z_{n+1}) = f(f(z_n))$$

where $f(z) = \gamma(1 - z)z$. Hence a and b satisfy the equation

$$\begin{aligned} z &= f(f(z)) \\ &= \gamma(1 - f(z))f(z) \\ &= \gamma(1 - \gamma(1 - z)z)\gamma(1 - z)z. \end{aligned} \tag{14}$$

Note that both of the equilibrium ratios 0 and $1 - \gamma^{-1}$ also satisfy equation (14) when $\gamma > 1$ since they both satisfy equation (10). When $\gamma \leq 3$ they are the only values of z which satisfy equation (14), but when $\gamma > 3$ there are another two values

$$\frac{\gamma(1 + \gamma) \pm \sqrt{(1 + \gamma)(\gamma - 3)}}{2\gamma}$$

These two values form a cyclic sequence of period 2. For instance when $\gamma = 3.1$ they are approximately equal to 0.5580 and 0.7646, respectively. The cyclic sequence starting at the smaller of them is illustrated in figure 6.

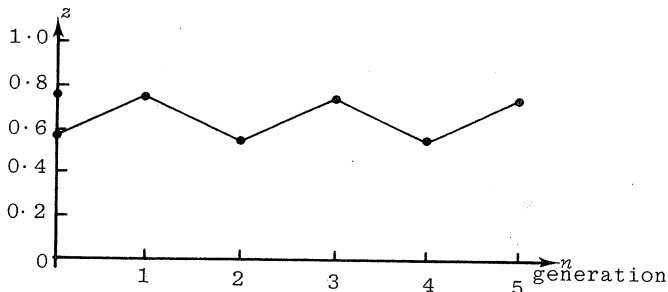


Figure 6.

When γ lies between about 3 and 3.4 it can be shown that the only cyclic sequences present are the cyclic sequence of period 2 and the two cyclic sequences of period 1. It can also be shown that every other sequence of population ratios converges to one of these cyclic sequences. Hence the behaviour of the populations is still quite regular and predictable, although slightly more complicated than in the earlier cases. The mathematical details are however quite complicated and will thus not be given here. The interested reader may wish to do some numerical calculations to check that this is really what does happen.

When γ is slightly bigger than 3.4 there is also a cyclic sequence of period 4. A slight increase in γ will cause a cycle of period 8 to appear, a further increase a cycle of period 16, and so on with cycles of periods of the form 2^k appearing as γ is increased more and more. In each case the increase in γ needed to cause a new cycle of higher period to appear gets smaller and smaller. However in all cases every sequence of population ratios converges to one of the cyclic sequences, so their behaviour is still quite regular and predictable, although becoming increasingly more complicated. To see this it is well worth doing some numerical calculations with different initial population ratios and different values of γ , for example with γ equal to 3.4, 3.45, 3.5 and 3.55.

So far all of the cyclic sequences have had periods of the form 2^k and the behaviour of all population ratio sequences has been fairly regular, though somewhat complicated. When γ is increased above about 3.6 cyclic sequences appear with periods equal

to an odd number or divisible by an odd number other than 1 and the behaviour of some sequences of population ratios becomes very erratic or chaotic. In fact many of the sequences do not converge to cyclic behaviour at all. Also many sequences starting very close to each other do completely different things. Biologically it is a very disturbing discovery, which has become known as *chaos*. It means that virtually nothing can be predicted in advance about the behaviour of any given population ratio sequence, even though this sequence obeys a fairly simple difference equation. Spurred on by the concern this was giving to biologists, two American mathematicians Li and Yorke proved a remarkable theorem in 1975 using very sophisticated arguments from the branch of mathematics called *topological dynamics*. They showed that chaos occurs whenever a difference equation (1) has a cyclic sequence of period 3, in which case it also has cyclic sequences of every other period and infinitely many different sequences which do not converge to cyclic behaviour. Difference equation (7) has a cyclic sequence of period 3 when γ is greater than about 3.8. This can be seen by solving the equation

$$z = f(f(f(z))),$$

which is not an easy thing to do. Some numerical calculations for $\gamma = 4$ in figure 7 show that the behaviour of the population ratio sequences is indeed chaotic.

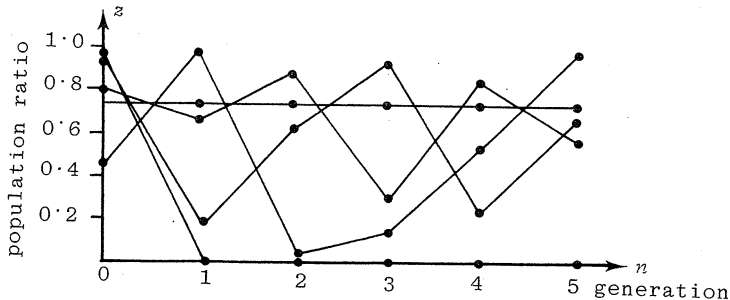


Figure 7.

Actually chaotic behaviour occurs whenever there is a cyclic sequence of period divisible by an odd number other than 1, for example 3, 5, 6, 15. For difference equation (7) this occurs when γ is just greater than 3.57. This was proved by the mathematician Sherkovsky in the early 1960's, using mathematical arguments familiar to every first year mathematics student. His work seems however to have been motivated purely by mathematical curiosity, not the need to explain a disturbing biological phenomenon. Consequently it attracted very little interest, even from other mathematicians, for about 15 years. The passage of time has however made the world more appreciative of his labours.

Exercise. Investigate the behaviour of the population sequences of the nonlinear difference equation

$$x_{n+1} = x_n e^{r(1-x_n)}$$

for $r > 0$. (Chaos occurs here when r is greater than about 3.7.)

LASERS

Gordon Troup, Monash University

LASER stands for "Light Amplification by Stimulated Emission of Radiation", and denotes a machine which does this. The radiation is electromagnetic radiation. 'Amplification' and 'amplify' are technical terms used in the following sense (see Figure 1). A device accepts a small signal, and by using some external source of power, converts this to a larger signal of the same kind. Power is drawn from the external source.

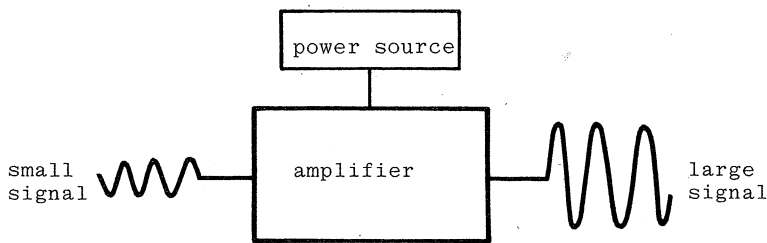


Figure 1.

If a tuning fork is struck and held only in the hand, it produces a very weak sound. However, if its base is placed on a large box, the sound is much louder. This is often termed amplification, but the energy comes from the tuning fork itself. 'Matching', or 'transducing' is a better term for this phenomenon.

The majority of lasers in teaching laboratories are actually oscillators rather than amplifiers - they generate their own signal, taking power from an external source. How does an amplifier become an oscillator?

Consider a loudspeaker system in a hall. We have all heard the 'squeal' at some time when the amplifier has too high a gain (ratio of output signal to input signal). What happens is that some of the output signal from the loudspeakers reaches the microphone and is fed back into the amplifier again. When the gain of the amplifier is sufficiently high, and the signal that is 'fed back' is in phase with any other input signal at a particular frequency, the system becomes an oscillator - it emits a signal without, eventually, the necessity for any outside 'input' signal. (See Figure 2.)

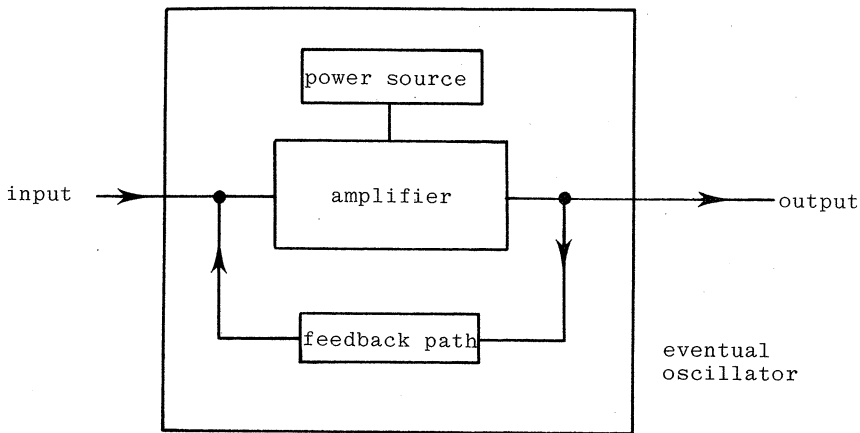
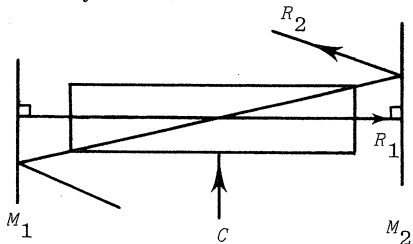


Figure 2.

The build-up of these oscillations, whether at a frequency of a few hertz or $\sim 10^{14}$ hertz (optical frequencies) can be described by a cubic equation - the simplest equation of 'catastrophe theory' (see *Function*, Volume 1, Part 2, pp.3-11).

What happens in a laser oscillator (we can see why 'loser' is *not* used!) is usually the following. Some material (an ionised gas [plasma], or a liquid, or a solid) is made amplifying at some particular frequency. The amplifying material is usually rod-shaped (long and thin), and mirrors are placed at either end of the rod. If for the moment we forget the spread of light by diffraction (the length and width of the rod are usually very large indeed compared with the wavelength of the radiation concerned), then reference to Figure 3 will show that, for plane parallel mirrors for example, only that radiation travelling along the axis of the amplifying rod will remain in the system for any appreciable length of time. The 'feedback' mechanism is obvious: the radiation emitted from the end of the rod is reflected directly back into the rod.



M_1, M_2 - mirrors

C - amplifying material

R_1 - path confining radiation within the system

R_2 - path not confining radiation within the system

Figure 3.

When the mirrors are spaced by an integral number of half-wavelengths at the frequency at which the rod amplifies (remember wavelength \times frequency = wave speed), and the power gain per single pass through the system is just slightly greater than the power losses due to diffraction and imperfect reflection, the system becomes an oscillator, generating an output signal by itself, but of course drawing power from some external source.

To understand why the laser oscillator needs no external signal to trigger it off, we need now to consider the interaction of atoms or molecules with electromagnetic radiation. First we need to know that atoms and molecules can only possess discrete, and not continuous values of energy. Consider, for example, the hydrogen atom. From the point of view of electromagnetic theory, the negatively charged electron can occupy any position with respect to the positively charged proton (nucleus). The stable orbits for the electron round the nucleus in the square-law Coulomb force field will be circles or ellipses, and the values of the total energy (potential plus kinetic) open to the whole system are continuous. But from the point of view of quantum theory, only certain values of the total energy are allowed. A particle on an inclined plane has a continuum of potential energies open to it; a particle on the rungs of a ladder, say, has only discrete (stable) potential energies open to it.

Quantum theory also tells us that the atom or molecule can only exchange energy with the electromagnetic radiation field under the following conditions. If the frequency of the electromagnetic field is f , then

$$hf = E_2 - E_1 \quad (1)$$

(Bohr frequency condition) where E_2, E_1 are two discrete energy states of the molecule, $E_2 > E_1$, and h is Planck's constant. If the energy of the atom increases, we have absorption of radiation: if the energy of the electromagnetic field increases, we have emission of radiation.

In 1917, Einstein considered the absorption and emission processes, and put forward the following theory. Firstly, atoms in an excited state (E_2 , say) would drop *spontaneously* to the ground state (E_1 , say), with the emission of radiation of energy and frequency given by equation (1). This is known as *spontaneous* emission. The elementary probability $\delta P_{(s)}$ that this should occur in a small time interval δt , Einstein wrote as

$$\delta P_{(s)} = N_2 A_{21} \delta t \quad (2)$$

where N_2 is the number of atoms in state E_2 and A_{21} is a constant depending on the atom. Atoms in the ground state E_1 will absorb energy from the radiation field at the frequency determined by equation (1); the elementary probability $\delta P_{(a)}$ for this is

$$\delta P_{(a)} = N_1 B_{12} U_f \delta t \quad (3)$$

where N_1 is the number of atoms in state E_1 , and U_f is the energy-

density per unit frequency range of the incident radiation. B_{12} is a constant depending on the atom.

In the presence of radiation of the appropriate frequency (Equation (1)), atoms in the excited state will be *stimulated* to give up their energy by the radiation. This is called *stimulated* emission, and can only occur in the presence of radiation of the right frequency, whereas spontaneous emission may occur whether such radiation is present or not. Further, stimulated emission bears a definite phase relationship to the incoming radiation whereas spontaneous emission does not. In general, stimulated emission adds to the incoming radiation (interferes constructively with it) whereas spontaneous emission can add to it or subtract from it (interfere constructively or destructively with it). Hence spontaneous emission is "noise".

The elementary probability $\delta P_{(st)}$ for stimulated emission is

$$\delta P_{(st)} = N_2 B_{21} U_f \delta t \quad (4)$$

where B_{21} is a constant depending on the atom. Einstein then postulated that

$$B_{12} = B_{21} \quad (5)$$

i.e. that the absorption and stimulated emission coefficients, as they are called, are equal. He further showed that, for an enclosure large compared with the wavelength of the interacting radiation,

$$A_{21} = B_{12} \times hf \times 8\pi f^2 c^{-3} \quad (6)$$

where c is the speed of light. It turns out that in the microwave region, where the wavelength of the radiation is of the order of a centimetre the factor $8\pi f^2 c^{-3}$ is so small that A_{21} is negligible. Consequently, we need only consider equations (3) and (4). We see that if $N_2 > N_1$ (more atoms in the upper energy state) the probability of stimulated emission is greater than that for absorption, so that the net result is an increase in energy of the radiation at the appropriate frequency: i.e., amplification.

In thermal equilibrium, the number of atoms in their various energy states is distributed so that

$$\frac{N_2}{N_1} = e^{[-(E_2 - E_1)/kT]}$$

where k is a fundamental constant called Boltzmann's constant: this is called the Boltzmann distribution over the energy states. So in thermal equilibrium $N_2 < N_1$, and we shall always get a net *absorption* of energy. We must find some way of making $N_2 > N_1$ to obtain amplification; i.e., we must put energy into the system to destroy the thermal equilibrium. Before going on to consider how we may do this, it is worth considering equation (6) again for the optical situation, where $8\pi f^2 c^{-3}$ is large, and it

would therefore appear that (Equation (2) spontaneous emission (noise) dominates. It turns out that $8\pi f^2 c^{-3}$ is the number of *modes* per unit volume per unit frequency range for the radiation at the interaction frequency. A mode is a solution of Maxwell's electromagnetic equations - which for this case reduce to a wave equation - satisfying the boundary conditions. So if we can reduce the number of modes available we can reduce the spontaneous emission power. How do we do this? We can make the container for the atoms long and thin, as discussed at the beginning of the article, and so define only a single preferred direction by the careful use of mirrors.

Let us further suppose that in our system, we *have* achieved the situation $N_2 > N_1$. An atom may *spontaneously* emit along the preferred direction. This radiation will be amplified by the stimulated emission process, pass to one of the mirrors, be reflected (fed) back, and so on, until eventually if the gain is sufficiently high, oscillation is set up. So it is the spontaneous emission in the laser which triggers the onset of the oscillations.

Finally, let us consider a possible way of getting more atoms into the upper state E_2 than the lower state E_1 . Basically, we need at least one extra energy level - let us call this E_3 , and let $E_3 > E_2$ (see Figure 4). In thermal equilibrium, $N_3 < N_2 < N_1$. But suppose by some means (e.g. an electric discharge; electron bombardment; even electromagnetic radiation at the appropriate frequency) we make $N_3 \approx N_1$, while leaving N_2 relatively undisturbed. Since we cannot, in this situation, create or destroy electrons, $N_1 + N_2 + N_3$ is a constant; the dashes denote non-thermal equilibrium. Then it can occur that $N_2 > N_1$, and the condition for amplification is fulfilled. Note that the energy to 'pump' the system, as it is called, comes from outside the atoms; and since $(E_3 - E_1) > (E_2 - E_1)$ the laser is not generally a particularly efficient system.

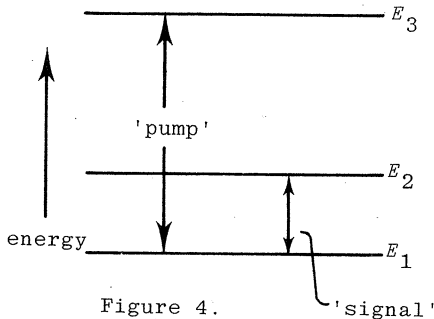


Figure 4.

Of course, actual energy-level diagrams and situations may be much more complex than have been shown here. And there is a great deal of very beautiful, if complicated, theory that has been omitted. If you get the impression that there is much more physics to the laser than mathematics, you are correct. It is perhaps the most wonderful and elegant invention of the twentieth century, and requires a great deal of physics (naturally with the appropriate concomitant mathematics) for complete understanding.

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DESIGNING A PRACTICAL CALENDAR FOR MARS

George Strugwell, 106 Bell Street, Coburg

The planet Mars may be regarded as the Earth's little brother: their axial tilts relative to their respective orbital planes are about the same, so that each displays north and south polar caps which wax and wane with the seasons; their days are approximately the same length; and the product of the sidereal year and equatorial diameter of the one precisely equals the other. Of all the planets and satellites of the solar system, apart from the Earth, Mars alone presents the possibility of colonization or, at any rate, of the establishment of an observation station and, where there are men, the need will arise for a calendar.

The reference books provide the following particulars:

Martian sidereal day \approx 24 hours 37 minutes 22.678 seconds (in terrestrial solar units)
 \approx 1.025 96 terrestrial solar days (in terrestrial solar units)

Martian year \approx 686 days 23 hours 30 minutes 53 seconds (in terrestrial solar units)
 \approx 686.979 78 terrestrial solar days.

Therefore, a Martial year \approx $686.979\ 78 \div 1.025\ 95$
 \approx 669.599 Martial sidereal days.

In its orbit around the sun the Earth revolves once more in regard to the stars than to the sun: i.e., a year of 366.2422 terrestrial sidereal days equals one of 365.2422 terrestrial solar days. Likewise for Mars, a year equals 668.599 Martian solar days. Thus in a thousand Martian years there will be 668 599 solar days, which to preserve reasonable uniformity of length and division into whole days will need to consist of 599 years of 669 days and 401 years of 668 days. A proposed arrangement appears hereunder.

A year of 669 or 668 days will need subdivisions and the ones suggested are 24 months of 28 or 27 days each: they have nothing to do with the moon, but are so-called merely because they are approximately a terrestrial month long. The names proposed are

combinations of those of the 24 letters of the Greek alphabet and of the Greek cardinal numbers one to six repeated four times, i.e. once each quarter-year. Since Martian geographical nomenclature is mostly, if not entirely, Greek, names derived from that language seem most apt for the purposes of a Martian calendar.

For practical workaday purposes and for religious reasons the seven-day week will need to be kept up as in the Gregorian calendar, but with one important exception. A year of 669 days plus one of 668 days equals 1337 days, i.e. exactly 191 weeks. However, 99 times in a thousand years a day - here called "Decaday" - will appear quite outside the seven-day week and on those occasions give rise to a three-day weekend. Thus, the Martian calendar will be maintained as a perpetual biennial repetition, always commencing on a Sunday and ending on a Saturday (or Decaday) as follows:

MARTIAN PERPETUAL CALENDAR

ALPHEN							BEDU							GANTRIA							DELTETRA						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7
8	9	10	11	12	13	14	8	9	10	11	12	13	14	8	9	10	11	12	13	14	8	9	10	11	12	13	14
15	16	17	18	19	20	21	15	16	17	18	19	20	21	15	16	17	18	19	20	21	15	16	17	18	19	20	21
22	23	24	25	26	27	28	22	23	24	25	26	27	28	22	23	24	25	26	27	28	22	23	24	25	26	27	28
...		
EPPENT							ZETHEX							ETHEN							THEDU						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	
8	9	10	11	12	13	14	8	9	10	11	12	13	14	2	3	4	5	6	7	8	2	3	4	5	6	7	8
15	16	17	18	19	20	21	15	16	17	18	19	20	21	9	10	11	12	13	14	15	9	10	11	12	13	14	15
22	23	24	25	26	27	28	22	23	24	25	26	27	..	16	17	18	19	20	21	22	16	17	18	19	20	21	22
...	23	24	25	26	27	28	..	23	24	25	26	27	28	..	
IOTRIA							KAPTETRA							LAMPENT							MUHEX						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
...	1	1	1	1		
2	3	4	5	6	7	8	2	3	4	5	6	7	8	2	3	4	5	6	7	8	2	3	4	5	6	7	8
9	10	11	12	13	14	15	9	10	11	12	13	14	15	9	10	11	12	13	14	15	9	10	11	12	13	14	15
16	17	18	19	20	21	22	16	17	18	19	20	21	22	16	17	18	19	20	21	22	16	17	18	19	20	21	22
23	24	25	26	27	28	..	23	24	25	26	27	28	..	23	24	25	26	27	28	..	23	24	25	26	27
NUHEN							XIDU							OMICTRIA							PITETRA						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
...	1	2	1	2	1	2	1	2
3	4	5	6	7	8	9	3	4	5	6	7	8	9	3	4	5	6	7	8	9	3	4	5	6	7	8	9
10	11	12	13	14	15	16	10	11	12	13	14	15	16	10	11	12	13	14	15	16	10	11	12	13	14	15	16
17	18	19	20	21	22	23	17	18	19	20	21	22	23	17	18	19	20	21	22	23	17	18	19	20	21	22	23
24	25	26	27	28	24	25	26	27	28	24	25	26	27	28	24	25	26	27	28
RHOPENT							SIGMEX							TAUHEN							UPSILDU						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
...	1	2	1	2	1	2	3	1	2	3		
3	4	5	6	7	8	9	3	4	5	6	7	8	9	4	5	6	7	8	9	10	4	5	6	7	8	9	10
10	11	12	13	14	15	16	10	11	12	13	14	15	16	11	12	13	14	15	16	17	11	12	13	14	15	16	17
17	18	19	20	21	22	23	17	18	19	20	21	22	23	18	19	20	21	22	23	24	18	19	20	21	22	23	24
24	25	26	27	28	24	25	26	27	25	26	27	28	25	26	27	28		
PHITRIA							CHITETRA							PSIPENT							OMEGHEX						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
...	1	2	3	1	2	3	1	2	3	1	2	3				
4	5	6	7	8	9	10	4	5	6	7	8	9	10	4	5	6	7	8	9	10	4	5	6	7	8	9	10
11	12	13	14	15	16	17	11	12	13	14	15	16	17	11	12	13	14	15	16	17	11	12	13	14	15	16	17
18	19	20	21	22	23	24	18	19	20	21	22	23	24	18	19	20	21	22	23	24	18	19	20	21	22	23	24
25	26	27	28	25	26	27	28	25	26	27	28	25	26	27	28

EVEN-NUMBERED YEARS

ALPHEN							BEDU							GANTRIA							DELTETRA										
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S				
..	1	2	3	1	2	3	1	2	3	1	2	3
4	5	6	7	8	9	10	4	5	6	7	8	9	10	4	5	6	7	8	9	10	4	5	6	7	8	9	10				
11	12	13	14	15	16	17	11	12	13	14	15	16	17	11	12	13	14	15	16	17	11	12	13	14	15	16	17				
18	19	20	21	22	23	24	18	19	20	21	22	23	24	18	19	20	21	22	23	24	18	19	20	21	22	23	24				
25	26	27	28	25	26	27	28	25	26	27	28	25	26	27	28				

EPPENT							ZETHEX							ETHEN							THEDU								
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S		
..	1	2	3	1	2	3	1	2	3	4	1	2	3	4
4	5	6	7	8	9	10	4	5	6	7	8	9	10	5	6	7	8	9	10	11	5	6	7	8	9	10	11		
11	12	13	14	15	16	17	11	12	13	14	15	16	17	12	13	14	15	16	17	18	12	13	14	15	16	17	18		
18	19	20	21	22	23	24	18	19	20	21	22	23	24	19	20	21	22	23	24	25	19	20	21	22	23	24	25		
25	26	27	28	25	26	27	28	26	27	28	26	27	28				

IOTRIA							KAPTETRA							LAMPENT							MUHEX							
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	
..	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	6	7	8	9	10	11	5	6	7	8	9	10	11	5	6	7	8	9	10	11	5	6	7	8	9	10	11	
12	13	14	15	16	17	18	12	13	14	15	16	17	18	12	13	14	15	16	17	18	12	13	14	15	16	17	18	
19	20	21	22	23	24	25	19	20	21	22	23	24	25	19	20	21	22	23	24	25	19	20	21	22	23	24	25	
26	27	28	26	27	28	26	27	28	26	27	28				

NUHEN							XIDU							OMICTRIA							PITETRA						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
..	..	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
6	7	8	9	10	11	12	6	7	8	9	10	11	12	6	7	8	9	10	11	12	6	7	8	9	10	11	12
13	14	15	16	17	18	19	13	14	15	16	17	18	19	13	14	15	16	17	18	19	13	14	15	16	17	18	19
20	21	22	23	24	25	26	20	21	22	23	24	25	26	20	21	22	23	24	25	26	20	21	22	23	24	25	26
27	28	27	28	27	28	27	28			

RHOWENT							SIGMEX							TAUHEN							UPSILDU						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
..	..	1	2	3	4	5	1	2	3	4	5	..	1	2	3	4	5	6	..	1	2	3	4	5	6
6	7	8	9	10	11	12	6	7	8	9	10	11	12	7	8	9	10	11	12	13	7	8	9	10	11	12	13
13	14	15	16	17	18	19	13	14	15	16	17	18	19	14	15	16	17	18	19	20	14	15	16	17	18	19	20
20	21	22	23	24	25	26	20	21	22	23	24	25	26	21	22	23	24	25	26	27	21	22	23	24	25	26	27
27	28	27	28	28	28			

PHITRIA							CHITETRA							PSIPENT							OMEGHEX						
S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S	S	M	T	W	T	F	S
..	1	2	3	4	5	6	..	1	2	3	4	5	6	..	1	2	3	4	5	6	..	1	2	3	4	5	6
7	8	9	10	11	12	13	7	8	9	10	11	12	13	7	8	9	10	11	12	13	7	8	9	10	11	12	13
14	15	16	17	18	19	20	14	15	16	17	18	19	20	14	15	16	17	18	19	20	14	15	16	17	18	19	20
21	22	23	24	25	26	27	21	22	23	24	25	26	27	21	22	23	24	25	26	27	21	22	23	24	25	26	27
28	28	28	28			

There are always 669 days in odd-numbered years and 668 in most of the even-numbered years: of the latter those ending in -0 (other than -000) also have 669 days, the additional day, Decaday 28^R Omeghex being outside the seven-day week and occurring between Saturday 27^R Omeghex and Sunday 1st Alphen. Thus, in a cycle of a thousand years there are:

Odd-numbered years	500	of 669 days =	334 500
Decadal other than millennial years	99	of 669 days =	66 291
Millennial year	1	of 668 days =	668
Other even-numbered years	400	of 668 days =	267 200
	1000	years equals	668 599 days

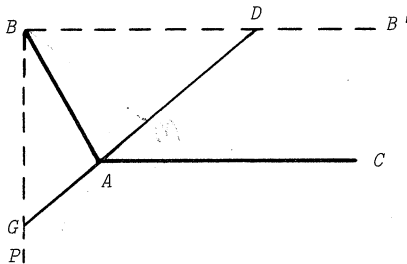
27 days has ZETHEX
 As well does MUHEX and SIGMEX,
 All the rest have 28,
 But to keep the record straight,
 OMEGHEX in even-numbered years
 Loses one; but shed no tears,
 A holiday is on the way
 With the 28th a Decaday.

Derek A. Holton

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HOW TO TRISECT AN ANGLE

Let $\angle BAC$ be any angle; an angle whose measure



is $\frac{1}{3}$ of that of $\angle BAC$ can be constructed as follows: construct BB' parallel to AC , and BP perpendicular to BB' (see the figure). Mark a length equal to twice BA on a ruler then by placing your ruler on the point A turn it and slide it until the marked length has its ends on BP and BB' - as GD in the figure. $\angle DAC$ is $\frac{1}{3}$ of $\angle BAC$.

PROBLEM 5.1

Prove this construction: it may help you to join B to the mid-point of GD .

Addendum. This construction appears in a manuscript of Isaac Newton which was written about 1672, but whether it is Newton's own idea, or originates earlier I do not know.

Gordon C. Smith

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At every stage in business, a disaster can occur. These kinds of considerations can make business a little less satisfying than proving a good theorem. For instance, once Pythagoras had his theorem down, he didn't have to worry about people finding a better one, or producing a cheaper one, or some kid swallowing a triangle and gagging on the hypotenuse.

Sam Savage - inventor of *Shmuzzles*.

TO RUN OR NOT TO RUN?

P. L. Galbraith, University of Queensland

An interesting and much discussed question is whether it is better to run or walk to the nearest shelter when caught in the rain with no protection.

It has been rumoured that this problem was first formulated by English cricketers during intervals in which the sun interrupted their adjournments for rain. The development of the umbrella field was seen as an attempt to find a comprehensive solution. Recently however Weiner (1) has assured us through his reference (2) that the problem has exercised the minds of a wider audience.

The problem can be modelled in a number of ways: one (two dimensional) version forms the basis of this article.

Assumptions

- (1) A Test Match is due to start at the M.C.G. (This is an existence condition: it ensures the rain.)
- (2) The rain has speed u , density ρ (drops/unit area) and falls in a direction making an angle α with the vertical.
- (3) A person caught in the rain runs at speed U on horizontal ground towards a shelter at distance L . (U can have any constant value up to the maximum of which the individual is capable.)

While running the body is angled at θ to the direction of motion where $0 < \theta \leq \pi/2$.

- (4) We consider the situation in which the direction of the rain is with or against the motion of the victim. This makes the problem a two dimensional one. In these circumstances no rain will strike the sides of the person who may thus be modelled as a plane figure with a front and back each of area A . (This involves the further assumption that rain falling on the head top is negligible in comparison to that falling on the front and/or back.)

Case 1: Running with following rain.

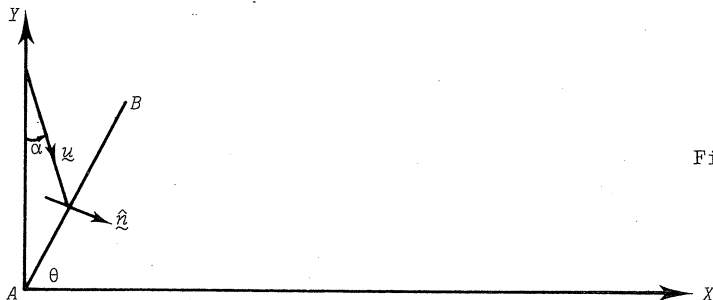


Figure 1.

AB represents the cross-section of the person described in the assumptions who is moving parallel to the X -axis with speed U . \hat{n} is the inward normal unit vector to AB from the 'weather side'. Taking the unit vectors \hat{i} and \hat{j} along the X and Y axes respectively we have

$$\begin{aligned}\underline{U} &= U\hat{i} \\ \hat{n} &= \sin\theta \hat{i} - \cos\theta \hat{j}, \\ \underline{u} &= u(\sin\alpha \hat{i} - \cos\alpha \hat{j}).\end{aligned}$$

The velocity of the rain relative to AB is

$$\underline{u} - \underline{U} = (u \sin \alpha - U)\hat{i} - u \cos \alpha \hat{j}$$

so the volume of water striking AB per unit time is given by

$$\begin{aligned}\rho A(\underline{u} - \underline{U}) \cdot \hat{n} &= \rho A(u \sin \alpha - U)\sin \theta + \rho A U \cos \alpha \cos \theta, \\ &= \rho A[u(\cos \alpha \cos \theta + \sin \alpha \sin \theta) - U \sin \theta], \\ &= \rho A[u \cos(\theta - \alpha) - U \sin \theta].\end{aligned}$$

Since the time taken to reach shelter is given by L/U the total volume of water received by AB is

$$\begin{aligned}V &= \rho A[u \cos(\theta - \alpha) - U \sin \theta] \cdot L/U \\ &= \rho AL \left[\frac{u}{U} \cos(\theta - \alpha) - \sin \theta \right] \\ &= k \left[\frac{u}{U} \cos(\theta - \alpha) - \sin \theta \right] \text{ where } k = \rho AL \text{ is constant.}\end{aligned}$$

Now for a uniform rainstorm u, α are fixed. If the 'running angle' θ is also held constant then V becomes a function of U .

The quantity of water received is thus determined by the behaviour of the function defined by

$$V(U) = \left[k \frac{u}{U} \cos(\theta - \alpha) - \sin \theta \right] \text{ provided } V \geq 0.$$

A sketch graph of this function is shown in Figure 2.

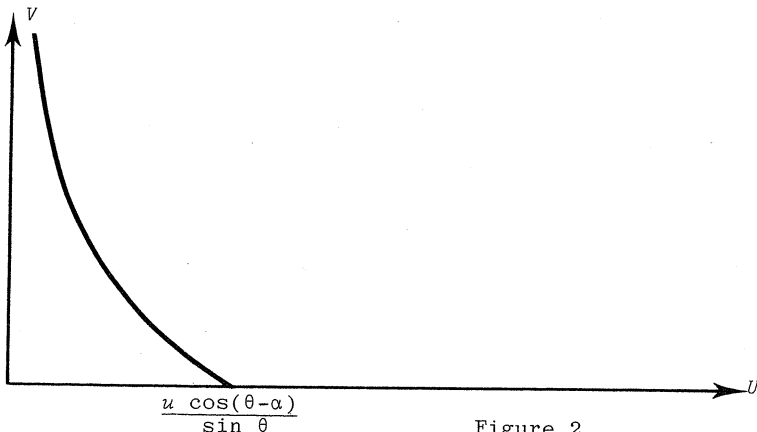


Figure 2.

Since $V = 0 \iff U = \frac{u \cos(\theta - \alpha)}{\sin \theta}$ and the function (whose graph is part of an hyperbola) is strictly decreasing the minimum value is attained for this value of U .

Maintaining this speed ensures that the velocity of the rain relative to AB is kept tangential to AB .

If U is increased beyond this value then the rain has a normal component inwards to the underside of AB i.e. the person is effectively running into the rain by going too fast. Such a situation is not satisfactorily described by a function which takes negative values (which would happen here) so a re-examination of the solution is suggested.

This is achieved by adjusting the expression for the volume of water striking AB per unit time to $\rho A |(\underline{u} - \underline{U}) \cdot \hat{n}|$ which can never be negative.

Recalling that $|x - a| = x - a$ if $x \geq 0$,
 $a - x$ if $x < 0$,

we see that the volume function becomes

$$\begin{aligned} V(U) &= k \left| \frac{u}{U} \cos(\theta - \alpha) - \sin \theta \right|, \\ &= k \left[\frac{u}{U} \cos(\theta - \alpha) - \sin \theta \right] \text{ if } 0 < U \leq \frac{u \cos(\theta - \alpha)}{\sin \theta}, \\ &= k \left[\sin \theta - \frac{u}{U} \cos(\theta - \alpha) \right] \text{ if } U > \frac{u \cos(\theta - \alpha)}{\sin \theta}. \end{aligned}$$

The graph of this function is sketched in Figure 3.

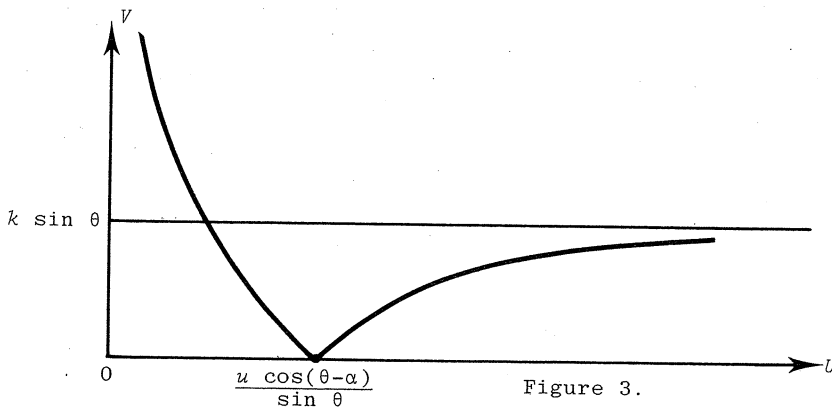


Figure 3.

As U becomes very large the volume of water received approaches a value $k \sin \theta$ which is independent of U . There are human limits on U but it seems that running too fast,

$\left(U > \frac{u \cos(\theta - \alpha)}{\sin \theta} \right)$ increases the wetting properties of the rain.

This might seem a bit surprising so let us check the solution by an alternative method.

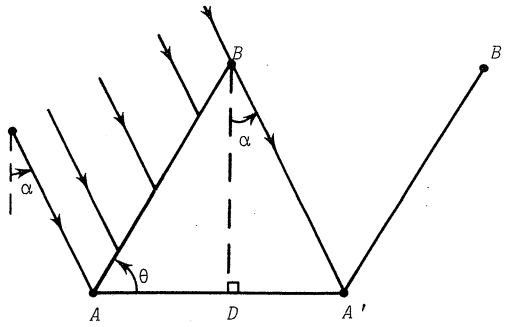


Figure 4.

In order to avoid wetting from the 'underside' the time taken for rain to fall the vertical distance BD must be less than or equal to the time taken for A to reach A' when moving with speed U .

$$\text{Hence it is necessary that } \frac{BD}{u \cos \alpha} \leq \frac{(AD + DA')}{U},$$

$$\text{i.e. } \frac{AB \sin \theta}{u \cos \alpha} \leq \frac{(AB \cos \theta + AB \sin \theta \tan \alpha)}{U},$$

$$\text{i.e. } \frac{\sin \theta}{u \cos \alpha} \leq \frac{(\cos \theta \cos \alpha + \sin \theta \sin \alpha)}{U \cos \alpha},$$

$$\text{i.e. } U \leq \frac{u \cos(\theta - \alpha)}{\sin \theta} \text{ which checks with the above.}$$

Case 2: Approaching rain.

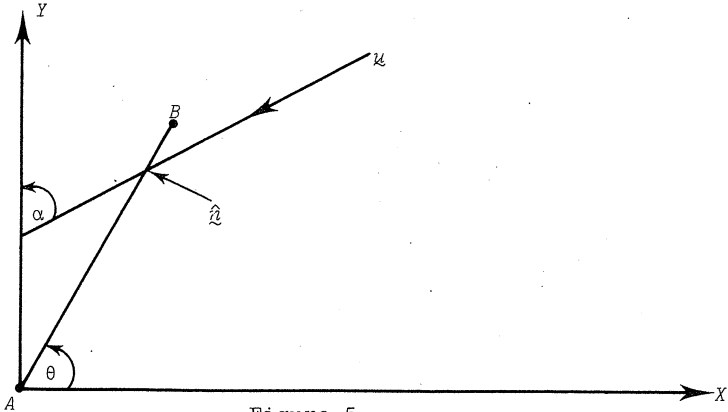


Figure 5.

In this case we have $\vec{U} = U\hat{i}$,

$$\hat{n} = -\sin\theta \hat{i} + \cos\theta \hat{j},$$

$$\vec{u} = u(-\sin\alpha \hat{i} - \cos\alpha \hat{j}),$$

so that the velocity of rain relative to AB is given by $\underline{v} - \underline{U} = -(u \sin \alpha + U)\hat{i} - u \cos \alpha \hat{j}$.

The volume of rain falling on AB while traversing distance L at speed U is as before

$$\begin{aligned} V &= \rho A |(\underline{v} - \underline{U}) \cdot \hat{n}| \cdot \frac{L}{U} \\ &= k \left| \sin \theta - \frac{u}{U} \cos(\theta + \alpha) \right| \end{aligned}$$

after substitution and simplification.

$$\begin{aligned} \text{Hence } V &= k \left[\sin \theta - \frac{u}{U} \cos(\theta + \alpha) \right] \text{ if } U \geq \frac{u \cos(\theta + \alpha)}{\sin \theta}, \\ &= k \left[\frac{u}{U} \cos(\theta + \alpha) - \sin \theta \right] \text{ if } U < \frac{u \cos(\theta + \alpha)}{\sin \theta}. \end{aligned}$$

Firstly we note that the real possibility exists that $\theta + \alpha > \frac{\pi}{2}$.

If this is so then $V(U) = k \left[\sin \theta - \frac{u}{U} \cos(\theta + \alpha) \right]$ for all U since $U > 0$.

In this case the graph of $V(U)$ for $U > 0$ is as shown in Figure 6.

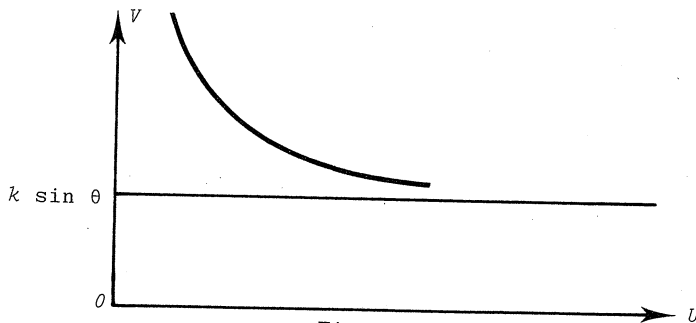


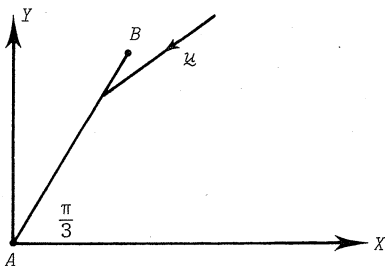
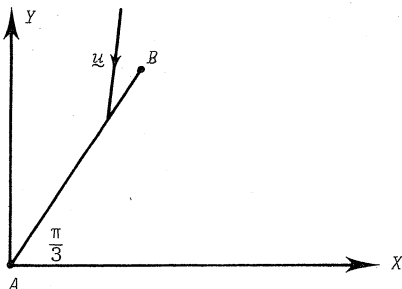
Figure 6

The function is strictly decreasing on $U > 0$ so the volume of water received is minimized by running as fast as possible.

If $0 < \theta + \alpha < \frac{\pi}{2}$ then the graph of $V(U)$ has precisely the same form as the graph of Figure 3 except that the minimum point is now given by $U = \frac{u \cos(\theta + \alpha)}{\sin \theta}$.

By moving with this speed the velocity of the rain relative to AB is directed along \overrightarrow{BA} .

The two different circumstances are illustrated in Figures 7(a) and (b).

Figure 7(a) $\theta + \alpha > \frac{\pi}{2}$,e.g., $\theta = \frac{\pi}{3}$, $\alpha = \frac{\pi}{4}$ Figure 7(b) $0 < \theta + \alpha < \frac{\pi}{2}$,e.g., $\theta = \frac{\pi}{3}$, $\alpha = \frac{\pi}{12}$

For the experimentally minded, weather bureau (approximate) figures for various conditions have been given as

Light shower (15 k.p.h.), sharp shower (45 k.p.h.),
thunderstorm with downdraft (180 k.p.h.).

Interpretation

The model presented here has the following messages for flat people caught in rainstorms.

- (1) If caught in heavy following rain run for it.
- (2) If the following rain is light do not run as fast as possible. (The most comfortable course of action is to maintain an upright position and a speed equal to the horizontal component of the velocity of the rain. Why?)
- (3) If running into the weather take due note of the angle of the rain. If the body can be inclined so that the (steeply falling rain) falls on the back then provided that the shower is not too sharp minimum wetting will be achieved by maintaining the speed $U = \frac{u \cos(\theta + \alpha)}{\sin \theta}$.
- (4) If the oncoming shower is sharp or if it is light but despite all contortions directed into the face then run for it.

Discussion

- (1) To evaluate the above findings we need to examine our assumptions in relation to the real world.

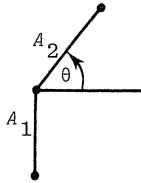
Assumption (4) suggests that the results may be of relevance to humans (particularly advanced weight watchers) but are of little or no value to plum puddings. On the other hand gingerbread men should find the theoretical results highly reliable. Indeed confidence in the theoretical results has been increased by recent research into the origins of the famous gingerbread poem³.

'Run, run as fast as you can
You can't catch me, I'm the gingerbread man.'

It has been established that these words were first penned during a *heavy* rainstorm and are seen to be consistent with the findings above.

(2) Returning to the human approximation some subjects find it difficult to incline the whole body (from top to toe) at a uniform angle θ to the direction of motion.

A modification of the model is shown below in which the lower half (the running part) of the body is held vertical and the upper half inclined.



With respective areas A_1 and A_2 theory can be developed in an exactly parallel way to the foregoing. Try it.

(3) A further practical problem has arisen in cases where the optimum value of U has to be calculated. Mathematicians concerned about precision have traditionally carried with them a protractor, theodolite, or some other means of measuring the angle of the rain, and a set of logarithm tables. Unfortunately the time taken to calculate U has invariably defeated the purpose of knowing it. Such people by virtue of the combination of water and log tables are said to become waterlogged. It is hoped that the development of the electronic calculator will revolutionize this whole area. Work is continuing.

References

- (1) B. Weiner: "A Theoretical Solution to the Great Raindrop Problem", *Journal of Irreproducible Results*, 1977, pp.17-19.
- (2) Archimedes, several monks in the middle ages, and Ben Franklin, to name a few.
- (3) Little Golden Books.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

A physicist, biologist and a mathematician were in an aircraft. They saw below a large flock of sheep all of which were white, except for one which was black. The physicist concluded that black sheep occur only rarely on the earth's surface. The biologist concluded that other sheep, namely the parents of the one he saw, were probably black. The mathematician concluded that there is at least one sheep, which, at any rate when viewed from the top, is black.

SOLUTION TO PROBLEM 1.1

The problem is:

We have a pack of cards, an even number c of them. By a "shuffle" we shall mean that we divide the pack into a top half and a bottom half, then put the pack back together again by alternately taking one card from each half starting with the bottom half. For instance, if the cards were initially 1, 2, 3, 4, after one shuffle they would become 3, 1, 4, 2. How many shuffles does it take for the cards to return to their original position?

Solution

The answer is that n shuffles are required where n is the smallest integer such that 2^n is divisible by $c + 1$. For example, for an ordinary pack of 52 cards, $n = 52$; for a pack of 12 cards, $n = 12$; for a pack of 14 cards, $n = 4$; for a pack of 16 cards, $n = 8$.

To prove that this is the answer consider what happens when one shuffle takes place. After the shuffle the 1st card has moved into 2nd place, the 2nd card has moved into 4th place, ..., the $\frac{1}{2}c$ -th card (remember c is even) has moved into c -th place; the $(\frac{c}{2}+1)$ -th card has moved into 1st place, the $(\frac{c}{2}+2)$ -th card has moved into 3rd place, ..., and finally the c -th card has moved into $(c-1)$ -th place. Consider again the last $\frac{c}{2}$ cards. The $(\frac{c}{2}+1)$ -th card has moved to the $(2(\frac{c}{2}+1) - (c+1))$ -th place; the $(\frac{c}{2}+2)$ -th card has moved to the $(2(\frac{c}{2}+2) - (c+1))$ -th place; ...; and finally the c -th card has moved into the $(2c - (c+1))$ -th place.

So, in summary, we can say for each r , $r = 1, 2, 3, \dots, c$, the r -th card has moved into the position k where $1 \leq k \leq c$ and k is the remainder obtained when $2r$ is divided by $c + 1$.

Consequently, after 2 shuffles the card initially in the r -th place has moved to the place given by the remainder obtained when $2^2 r$ is divided by $c + 1$. Similarly, after n shuffles the r -th card has moved to the place given by the remainder when $2^n r$ is divided by $c + 1$.

Thus the cards first return to their original position when n is the smallest integer such that, for each r , $r = 1, 2, \dots, c$, the remainder when $2^n r$ is divided by $c + 1$ equals r . This will hold for all r if and only if it holds for $r = 1$.

Note: When c is even it can be shown that there always exists an integer n such that 2^n gives remainder 1 when divided by $c + 1$. Let $E(d)$ denote the number of positive integers less than d and prime to d : for example $E(2) = 1$, $E(8) = 4$, $E(15) = 8$, $E(53) = 52$. Then it may be shown that the smallest such integer n is a divisor of $E(c + 1)$.

SOLUTION TO PROBLEM 1.2

The problem is:

I have the following options of depositing \$100 for one year. The bank will give me \$4 interest at the end of the year. A Housing Co-operative will give me interest at the rate of 2% per half-year (compound, so it pays interest in the second half also on the first half's interest). A Credit Union will give me interest at the rate 1/3% per month, compound. A friend says he will give me interest equivalent to the 4% per annum rate, but compounding *every instant!* Which should I choose, and how much interest do I get?

Solution

This problem has a trick in it: the final alternative, which is described as being "equivalent to the 4% per annum rate" is hence, by definition, the same as the first alternative. Each gives \$4 interest at the end of the year. The second alternative gives \$2 interest at the half-year and then 2% on \$102 (the \$100 deposit together with the interest already earned), i.e. \$2.04 at the end of the year. The total interest earned from the Housing Co-operative is therefore \$4.04.

With the third alternative, the Credit Union, if I have \$P invested at the beginning of any month, then at the end of the month I get $\frac{1}{3}\%$ interest, i.e. I have $\$(P + \frac{1}{300}P) = \$P(1 + \frac{1}{300})$, *in toto*. Since I start with \$100, at the end of the first month I therefore have $\$100(1 + \frac{1}{300})$; hence at the end of the second month I have $\$(100(1 + \frac{1}{300}))(1 + \frac{1}{300})$; and so on, until finally, at the end of the twelfth month I have $\$100(1 + \frac{1}{300})^{12}$. This gives the best return: it may be calculated that, to the nearest cent, the interest at the end of the year is 4 dollars 7 cents.

SOLUTION TO PROBLEM 1.3

The problem is:

Is 22/2/2022 a Twosday (Tuesday)? How about 2/2/2202?
(See *Function* (1977), Volume 1, pp 19-23.)

Solution

In the article of *Function* referred to, at the foot of page 22, there is the formula

$$x \equiv D + \left[\frac{26(N + 1)}{10} \right] + Z + \left[\frac{Z}{4} \right] + \left[\frac{J}{4} \right] - 2J - 1,$$

where x is the day of the week ($x = 0, 1, 2, 3, 4, 5$ or 6 , corresponding to Sunday, Monday, ..., Saturday, respectively), where D is the number giving the date of the month, N the number of the month of the year $100J + Z$, and where, in applying the formula, January and February in any year are regarded as the 13th and 14th months, respectively, of the previous year.

Applying the formula to 22/2/2022 gives

$$\begin{aligned} x &\equiv 22 + \left[\frac{26(14 + 1)}{10} \right] + 21 + \left[\frac{21}{4} \right] + \left[\frac{20}{4} \right] - 40 - 1 \\ &= 22 + 39 + 21 + 5 + 5 - 40 - 1 \\ &\equiv 1 + 4 + 0 + 3 + 2 - 1 \\ &\equiv 2. \end{aligned}$$

So 22/2/2022 is a Tuesday.

Similarly, using the formula for 2/2/2202, we find that this is also a Tuesday.

SOLUTION TO PROBLEM 3.5

The problem is:

If a side of a triangle is of length less than the average length of the other two sides, show that its opposite angle is less, in magnitude, than the average of the other two angle magnitudes.

Solution

Let the sides of the triangle be of lengths a , b , and c with opposite angles of size A , B , and C , degrees, respectively. We have to show that if $a < \frac{1}{2}(b + c)$, then $A < \frac{1}{2}(B + C)$.

Suppose that a and A are fixed. Then, as we now show, $b + c$ takes its maximum value when $B = C$, i.e. when $b = c$.

From

$$\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a}{\sin A} = k, \text{ say, a constant,}$$

we get $b + c = k(\sin B + \sin C)$.

Thus $b + c = k(\sin B + \sin(180 - A - B))$,

since $A + B + C = 180$. Since A is constant therefore $b + c$ is a function of B , $b + c = f(B)$, say. Then

$$f'(B) = k(\cos B - \cos(180 - A - B))$$

and $f''(B) = -k(\sin B + \sin(180 - A - B))$;

and $b + c$ attains a (local) maximum where $f'(B) = 0$ and $f''(B) < 0$. Now $f'(B) = 0$ when

$$\cos B = \cos(180 - A - B),$$

and the only solution to this with $0 < B < 180$, is

$$B = 180 - A - B,$$

i.e. $B = C = 90 - \frac{A}{2}$.

For this value of B , $f''(B) < 0$. Hence $b + c$ takes a maximum value when $b = c$. Since f has no (local) minima for the domain of values

of B , viz. $0 < B < 180 - A$, being considered, $B = C$ gives the absolute maximum value of f .

An alternative geometrical proof of the above result, which makes it intuitively clear, can be found as follows. Consider the side of length a as fixed; if A is constant then the remaining vertex of the triangle is restricted to lie on an arc of a circle of which the fixed side is a chord; if $b + c$ is fixed then the remaining vertex lies on an ellipse for which the fixed side has the foci of the ellipse as its end points. For fixed A and a , $b + c$ takes its maximum value when the circle and ellipse touch and this happens only when $b = c$.

Now let us return to solving the problem. So we assume that $a < \frac{1}{2}(b + c)$. We shall show that $A \geq \frac{1}{2}(B + C)$ leads to a contradiction, and so deduce that $A < \frac{1}{2}(B + C)$.

Since $\frac{1}{2}(A + B + C) = 90$, $A \geq \frac{1}{2}(B + C)$ is the same as $A \geq 90 - \frac{1}{2}A$, i.e. $A \geq 60$. If $A = 60$, then the largest value of $b + c$, as has just been proved, is when $B = C$. Thus the triangle is equilateral and $a = b = c$, whence $a = \frac{1}{2}(b + c)$; which contradicts the assumption $a < \frac{1}{2}(b + c)$. If $A > 60$, then considering again the case $B = C$ in which the triangle is isosceles, which gives the maximum value of $b + c$, we easily see that $b + c$ is smaller than in the equilateral triangle with the same base of length a (the isosceles triangle on the same base has its apex inside the equilateral triangle). Hence $\frac{1}{2}(b + c) < a$, which is again a contradiction.

The proof of the result required is complete.

SOLUTION TO PROBLEM 3.1

The problem is:

As a classroom project, two students keep a calendar of the weather, according to the following scheme: Days on which the weather is good are marked with the sign +, while days on which the weather is bad are marked with the sign -. The first student makes three observations daily, one in the morning, one in the afternoon and one in the evening. If it rains at the time of any of these observations, he writes -, but otherwise he writes +. The second student makes observations at the same times as the first student, writing + if the weather is fair at any of these times and - otherwise. Thus it would seem that the weather on any given day might be described as ++, +-, -+ or -- (the first symbol made by the first student, the second symbol by the second student). Are these four cases all actually possible?

Solution

The answer to this problem depends on the meaning the second student gives to "fair weather". If by "fair" he means "not raining" then ++ and -- are the only pairs possible. On the other hand, with normal usage of fair to indicate, say, that it is not raining and that (this would depend on personal judgement) the sky is not overcast, then +- would be possible but -+ would not.

SOLUTION TO PROBLEM 3.4

The problem is:

Spot the fallacy:

$$\text{Since } \cos^2 x = 1 - \sin^2 x,$$

$$\text{it follows that } 1 + \cos x = 1 + (1 - \sin^2 x)^{\frac{1}{2}},$$

$$\text{that is } (1 + \cos x)^2 = \{1 + (1 - \sin^2 x)^{\frac{1}{2}}\}^2.$$

In particular, when $x = \pi$, we have

$$(1 - 1)^2 = \{1 + (1 - 0)^{\frac{1}{2}}\}^2,$$

$$\begin{aligned} \text{or } 0 &= (1 + 1)^2 \\ &= 4. \end{aligned}$$

Solution

From $\cos^2 x = 1 - \sin^2 x$ it follows that

$$\cos x = \pm \sqrt{1 - \sin^2 x},$$

the choice of sign depending on the value of x . When $\cos x$ is negative, the negative sign must be chosen. So when $x = \pi$, so that $\cos x = -1$, $\cos x$ may be replaced by $-\sqrt{1 - \sin^2 x}$ but not by $+\sqrt{1 - \sin^2 x}$ as was done in the above argument.

PROBLEM 5.2

Observe that the value of

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!}$$

is $\frac{1}{2}$, $\frac{5}{6}$, $\frac{23}{24}$, for $n = 1, 2, 3$, respectively. Guess the general law and prove your guess.

PROBLEM 5.3

A right angled triangle has area A and hypotenuse of length c . On each side of the triangle draw a square, exterior to the triangle. Now imagine a tight rubber band placed around your figure. What area would it enclose?

PROBLEM 5.4

Let P be a non-constant polynomial with integer coefficients. If $n(P)$ is the number of distinct integers k such that $[P(k)]^2 = 1$, prove that $n(P) - \deg(P) \leq 2$, where $\deg(P)$ denotes the degree of the polynomial P .

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"A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician. If this description were accurate, mathematics would not attract any intelligent person. It would be a game with definitions, rules, and syllogisms, without motive or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only guided by intrinsic necessity, can the free mind achieve results of scientific value."

What is Mathematics? R. Courant and H. Robbins, 1941.

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"There are two kinds of reasoning, as we said: demonstrative reasoning and plausible reasoning. Let me observe that they do not contradict each other; on the contrary, they complete each other. In strict reasoning the principal thing is to distinguish a proof from a guess, a valid demonstration from an invalid attempt. In plausible reasoning the principal thing is to distinguish a guess from a guess, a more reasonable guess from a less reasonable guess. If you direct your attention to both distinctions, both may become clearer.

A serious student of mathematics, intending to make it his life's work, must learn demonstrative reasoning; it is his profession and the distinctive mark of his science. Yet for real success he must also learn plausible reasoning; this is the kind of reasoning on which his creative work will depend."

Mathematics and Plausible Reasoning, Volume 1,
G. Pólya, 1954.

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