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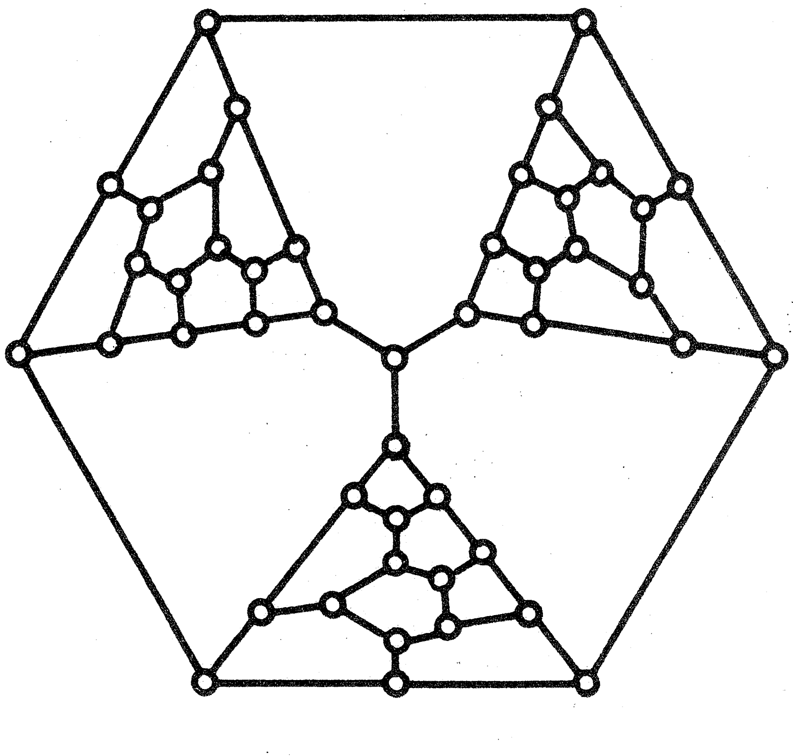

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Our main article this issue concerns the mathematical art of M.C. Escher. This fascinating artist produced so many works of interest to mathematicians that the author, John Stillwell, has concentrated on a single group of pictures. There are, however, many others which will intrigue you. Many of these are available as posters (see the advertisement on the back cover), and can lead to hours of fascinated enjoyment.

We also print a further paper on Einstein and his work and another of the Monash Schools' lectures, Professor Ewens' popular talk on Probability Theory and Statistics. As to the cover, it touches on what may be the best mathematical contribution by an amateur this century: Wayne Watts' discovery of a connection between the Tutte Graph (cover) and the Kozyrev-Grinberg theory of graphs. To complete the list, we have a true story of Statistics in a real life situation.

CONTENTS

THE FRONT COVER – THE TUTTE GRAPH

D.A. Holton, University of Melbourne

The path to the solution of the Four Colour Map Theorem is paved with a large number of false "proofs". The present cover shows a graph which arose in the course of one of these attempts.

For a background to the problem, see John Stillwell's article in *Function*, Vol.1, Part 1. The idea, of course, is to prove that the regions of any map can be coloured with four or fewer colours, so that no two adjacent countries have the same colour. This can be converted into a graph theoretical problem by taking a dot to represent the capital of each country and joining two dots by a line if the corresponding countries have a common boundary. Each dot of the resulting graph can then be coloured in four or fewer colours so that no two joined dots have the same colour, if and only if the original map can be coloured in four or fewer colours. For convenience, this graph theoretical formulation was the one used by Appel and Haken in their proof of the Four Colour Theorem.

In 1880, Tait observed that in order to prove this Four Colour Theorem, it would be enough to show that the lines of every cubic, 3-connected planar graph† can be coloured in 3 colours so that no two lines with the same colour meet at a given dot.

Tait claimed that, clearly, every cubic 3-connected planar graph had a *hamiltonian cycle*, that is that there is a route along the lines of the graph which starts and ends at one dot and which passes through all the other dots once and only once.

Now if a graph is cubic it is not difficult to prove that it must have an even number of dots. Should the hamiltonian cycle exist, its lines could then be coloured alternatively in two colours. The remaining lines could then use the third colour. So if Tait was right about these cubic 3-connected graphs having hamiltonian cycles, the Four Colour Theorem would surely follow.

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†A cubic graph is one in which 3 lines come in to every dot and a graph is 3-connected if it takes the removal of at least 3 dots, and their incident lines, to disconnect the graph. A planar graph is one that can be drawn in the plane so that no two lines cross.

For a long while people puzzled over Tait's hamiltonian claim. It waited, however, till 1946 before Tutte produced the graph on the cover which finally showed that not all cubic 3-connected planar graphs were hamiltonian.

But how do we prove that the Tutte graph is not hamiltonian? One way, of course, would be to illustrate, by exhausting all possibilities, that there is no cycle through every point of the graph. This is enormously tedious.

Is there a characterisation of hamiltonian graphs? If there is, then there is a theorem that says "*G* is hamiltonian if and only if 'blaa'". All we need to do then is to test Tutte's graph. If it didn't have the property 'blaa' then we would know that it was not hamiltonian.

Unfortunately, despite a great deal of effort, no one has yet come up with this magic property 'blaa'. If and when they do they will be accorded instant mathematical fame. In the meantime we have a graph at hand. What to do?

Well, fortunately we have some partial 'blaa' results. For instance, Tutte himself has proved that every 4-connected planar graph is hamiltonian. Unfortunately, Tutte's graph is not 4-connected. We can find three vertices whose removal will disconnect the graph. So that test is out.

Another partial 'blaa' comes from two Russians - Kozyrev and Grinberg. Their result says that if G is planar and hamiltonian with n dots, then

$$\sum_{i=2}^{n} (i - 2)(f_i - f_i') = 0.$$

i.e. $0(f_2-f_2') + 1(f_3-f_3') + 2(f_4-f_4') + \ldots + (n-2)(f_n-f_n') = 0.$

To explain the f_i and the f_i' look at the planar graph of Figure 1, which has the hamiltonian cycle 1, 2, 3, 4, ..., 15, 16, 1.

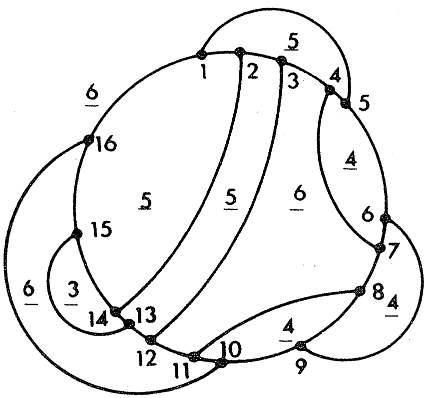

Figure 1.

This cycle divides the plane in two. Various lines of the graph divide the inside of the cycle into regions. One region is formed by 1, 2, 14, 15, 16, another by 3, 4, 7, 8, 11, 12. The term f_i is the number of such regions inside the hamiltonian cycle formed by i lines. Hence, for the graph in Figure 1 we have, $f_2 = 0$, $f_3 = 0$, $f_4 = 2$, $f_5 = 2$ and $f_6 = 1$. The f' term tells us the number of regions formed in a similar way on the outside of the cycle. So we see that $f'_3 = 1$, $f'_4 = 1$, $f'_5 = 1$; $f'_6 = 2$.

Substituting these values in the Kozyrev-Grinberg expression

$$\sum_{i=2}^{n} (i - 2)(f_i - f'_i)$$

we get

$$(3-2)(0-1) + (4-2)(2-1) + (5-2)(2-1) + (6-2)(1-2)$$

which is indeed zero.

Now Tutte proved his graph was not hamiltonian, but a nicer proof was given by an amateur mathematician, Watts, in 1972. This proof uses the Kozyrev-Grinberg result. It is a little too long to include here but the basic idea is to assume that Tutte's graph has a hamiltonian cycle and use the Kozyrev-Grinberg equation to obtain a contradiction. The complete proof can be found in the book "Mathematical Gems", by R. Honsberger (published by the American Mathematical Society, 1974). In the same book the Kozyrev-Grinberg theorem is also proved.

For more results in the area of hamiltonian graphs you should read L. Lesniak-Foster: "Some recent results in hamiltonian graphs", *Journal of Graph Theory*, Vol.1, 1977, 27 - 36. If you have any difficulty in getting hold of this paper then write to me care of the Department of Mathematics, University of Melbourne.

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CONJUGATING SENTENCES

I am a statistician,
You are unreliable,
He lies.

I am a systems analyst,
You programme,
He codes,
She punches.

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PROBABILITY IN THEORY AND PRACTICE†

W.J. Ewens, Monash University

My aims in this discussion are fourfold: first, to describe what probability and statistics are; second, to discuss why probability and statistics enter into modern science; third, to illustrate this by describing a specific probabilistic process and various of its scientific applications, and finally to discuss what changes in scientific procedure have been brought about by a statistical way of thinking.

Although in some quarters statistics is seen as being tied up with batting averages, records, lists, computers and so on, I wish to concentrate on the more specifically scientific meaning of the word "statistics". In a scientific context, statistics is in a sense the converse of probability theory, or, better, statistics and probability theory form the two sides of the same discipline. More precisely, probability theory and statistics are respectively the deductive and inductive sides of arguments involving random unpredictable phenomena. In probability theory we assume some state of the real world and deduce the probabilities of various possible outcomes, whereas in statistics we start with the observation of an outcome and try to make an induction about the state of the real world which generated this outcome. As a simple example, consider tossing a coin ten times. A typical statement of probability theory would be "If the coin is fair, the probability of getting ten heads is 1/1024", while a typical statement of statistics might roughly be "I have observed this coin to give ten heads in ten tosses. I therefore doubt that it is fair". This last statement is a rough and ready one, but it does nevertheless indicate the inferential or inductive nature of a statistical statement. The probability argument assumes a state of the real world (that the coin is fair) and makes a deductive statement based on this assumption, and the corresponding statistical induction is in large part based on this statement. The structure of probability theory is like that of Euclidean geometry, whereby one makes certain assumptions and by logical argument deduces various conclusions, whereas the nature of statistics is like that of an experimental science where, from certain observations, one tries to understand the state and rules of the real world.

My next aim is to ask why statistics, as just defined, enters into modern science. The main reason for this is that modern science (in contrast to the science of past centuries) deals very much with random phenomena. There are two main reasons why this is so. The first is that developments within even an

† Text of a Schools' Lecture, delivered on April 20, 1979.

"exact" science such as physics show that at a fundamental level an observer must accept some degree of uncertainty in his measurements of physical phenomena. Perhaps more important, however, is the second reason, which is that science itself now embraces a far wider area of activity than previously and in most of the new areas, particularly agriculture, biology, economics and sociology, it is quite impossible to avoid unpredictable random phenomena entering in. A student of evolutionary theory, for example, has to accept that there will be a random transmission of genes from parent to offspring which cannot be predicted in detail and he must gear his theory to encompass and allow for this random behaviour. In other words, evolution must be seen as a statistical process and an analysis of evolution must be a statistical one (rather than a deterministic one). Similarly an agricultural scientist must accept that unpredictable rainfall, unknown soil fertility differences, and so on, influence his experiments, and so also must treat his observations on a statistical basis as the outcome of an experiment necessarily involving chance phenomena. The same is true of the economist, the sociologist and indeed of almost all contemporary scientists.

It goes without saying that the unavoidability of some degree of uncertainty leads to a rather different philosophical view of the nature of science and scientific laws than was held previously. Such a change in outlook was resisted to some extent, particularly in physics, but the great majority of scientists nowadays accept the philosophical implications of the uncertain nature of science.

It is perhaps worth noting, at this point, that much of our mathematical and scientific training, deriving often from the work of Newton, is deterministic in nature and that perhaps because of this our intuition is often all at sea in a non-deterministic, random, context. Consider for example the following very simple question. A fair coin is tossed twice and we are told that at least one head appeared. What is the probability that in fact two heads appeared? Try to give a quick answer to this, and then check with the correct answer, which I give at the end of this paragraph. This very question is very close to the following question in genetic counselling. Individuals can be classed as AA, AB or BB. Those who are AA or AB are normal, those who are BB have some genetic defect or disorder. If we know someone is normal (by observing him) we know he is AA or AB, and for purposes of genetic counselling we might wish to know the probability that he is AA. If A and B are equally frequent in the population the required probability is 1/3, and this also is the probability in the penny-tossing experiment referred to above. Very few students get this question correct when asked for a quick intuitive answer.

I now turn to the third aim, namely to describe a certain probabilistic process and to show how it can be used in various scientific contexts. The process I will consider is the so-called random walk (or drunkard's walk or gambler's ruin), the mathematics of which was largely established by mathematicians several centuries ago in connection with gambling. Imagine a gambler with initially x dollars and an adversary with initially $k - x$ dollars. A sequence of games is now conducted in the following way. A coin (with probability p for heads) is tossed and if it comes down heads the adversary gives our gambler a dollar, whereas if it comes down tails the gambler gives his adversary a dollar. The sequence of games continues until one or other gambler has all k dollars in the system, when it stops. We seek the probability that ultimately our gambler has all the money in the system and also the mean duration of play. The same mathematics can clearly be used to describe the behaviour of a drunkard's walk where steps to the right and left occur with probabilities p and $1 - p$ respectively, and the drunkard starts at the point x and continues walking until reaching 0 or k. In some scientific contexts it is convenient to think in terms of the gambler's ruin and in others in terms of the drunkard's walk, when analysing a specific scientific problem.

There are various scientific areas where the mathematics of this process can be used. One of them is in the estimation of Avogadro's number, namely the number of molecules per mole for any gas at standard temperature and pressure. It was observed by the botanist Robert Brown in 1827 that small particles immersed in a liquid undergo ceaseless irregular motion, and this is now known to be caused by the ceaseless bombardment of the particles by the molecules in the surrounding fluid. If we now think in terms of the drunkard's walk and imagine that the drunkard is staggering about not because he is drunk, but because he is being pushed to left and right by a platoon of policemen, then clearly the rate at which he staggers about will be related to the number of policemen in the platoon. Similarly the rate at which the particle moves is related to the density of molecules in the surrounding liquid, which can in turn be related to Avogadro's number. The mathematics of the drunkard's walk now allows an estimation of Avogadro's number from the rate at which the particle moves, and estimation of Avogrado's number through this method agrees very well with estimation from entirely different approaches.

A second application for the drunkard's walk arises in evolutionary theory, and here it is more convenient to think in terms of the gambler's ruin. The fundamental process in evolution is the replacement of an inferior gene in a population by a superior gene, brought about by natural selection over the course of a number of generations. Now this replacement process is by no means certain and deterministic. Random events in each generation, together with the random transmission of genes from one generation to another, ensure that random changes occur (both up and down) in the number of superior genes from one generation to another, just as the fortune of the gambler goes up and down depending on the results of the tosses of the coin. In the evolutionary context we may interpret the probability that the gambler eventually wins all the money in the system as the probability that eventually the superior gene takes over in the population, and similarly calculations on the mean duration of the gambling game give us some idea of the rate of evolutionary genetic processes.

Another application of the gambler's ruin problem concerns testing for ESP. One way of testing whether a certain person who claims to have ESP really does have some such ability is to allocate him, say, 50 points, and give him a series of tests where we deduct one point if he fails the test and add one point if he passes the test. Testing continues until he has no points left or has (say) 100 points. The theory of the gambler's ruin can clearly be used to test whether the outcome of this experiment gives evidence of whether the person in the experiment does have some ESP ability or not.

A final application of the gambler's ruin arises in pure mathematics. Suppose we decide to represent heads by 1 and tails by 0. Then the sequence heads, heads, tails, heads, ... in a penny-tossing experiment become 1101 ... . We can put a decimal point in front of this sequence to produce the number .1101 ... , which can be thought of as a number between 0 and 1 represented in binary notation. It follows that there is a one-to-one correspondence between the binary representation of any number between 0 and 1 and the complete history of an (infinite) gambler's ruin game. Here the theory of the gambler's ruin can be used to suggest and prove results in number theory. For example, it is perhaps expected that with a fair coin the fraction of tosses giving heads should approach one half as the number of tosses increases, and this suggests that the fraction of 1's in the binary expansion of most numbers should also approach one half. A proper analysis of the gambler's ruin problem indicates the extent to which this number theory result is true, and the insight into these problems gained by viewing them in the gambler's ruin context is often most valuable.

My final aim is to outline the changes in scientific procedure brought about by the statistical, or probabilistic, approach to science. Consider for example the problem of weighing two objects on a beam balance. There are at least two ways we can do this. The first way is to weigh the objects separately and the second is to weigh both objects together and then put them in separate pans to find the difference of their weights. If the weights of the two objects are x_1 and x_2, and if the weighing can be done without error, these two procedures give the equations

$$x_1 = w_1, \quad x_2 = w_2$$

and

$$x_1 + x_2 = w_3, \quad x_1 - x_2 = w_4,$$

where w_1, w_2, w_3 and w_4 are the weights put in the pans to achieve a balance. From the last two equations we can calculate x_1 and x_2 and we will get the same values as through the first approach. Thus the two procedures are equivalent, although in practice we would probably prefer the first approach because of its greater simplicity.

The situation is changed when we recognize the existence of errors in the weighing procedures. We now have equations

$$x_1 = w_1 + e_1, \qquad x_2 = w_2 + e_2$$

and

$$x_1 + x_2 = w_3 + e_3, \quad x_1 - x_2 = w_4 + e_4$$

(i.e.

$$x_1 = \frac{w_3 + w_4}{2} + \frac{e_3 + e_4}{2}$$

$$x_2 = \frac{w_3 - w_4}{2} + \frac{e_3 - e_4}{2}\Big) \quad ,$$

where e_1, e_2, e_3 and e_4 are small error terms associated with errors in the weighing procedure. Curiously the two weighing procedures are now no longer equivalent, the second procedure being more accurate than the first. The variances of the estimates of the weights of the objects, under the second procedure, are half the values of those under the first procedure. This indicates that when the existence of uncertainty in scientific measurements is admitted, a whole new problem, namely the best way to design experiments so as to minimize the errors in one's estimation of various quantities, arises. Strangely (and this is already apparent in the above example), the best way to achieve this is usually through a complex, "many at a time" experiment, instead of the previously favoured simple "one at a time" experiment.

In this brief outline I have tried to sketch the general nature of probability and statistics and to indicate how they extend into, and indeed determine, the nature of current-day science and scientific experimentation. Few subjects today can have such a far-reaching effect, or lead to such a varied and interesting scientific life for its practitioners, as do probability theory and statistics.

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FLYING BLIND WITH STATISTICS

The Skylab project director, Mr Richard Smith, said it was statistically unlikely that any one of the billions of people along Skylab's path would be injured.

This was because bits of Skylab debris travelling at up to 500 kmh will be spread out along a track 6500 kilometres long.

The Age, 9.7.79.

THE VALUE OF PARITY CHECKS

It can't be 112 points - not with an odd number of behinds.

Jack Dyer, 3KZ, 16.6.79,
(detecting an error in the
South Melbourne scoreboard).

TO TURN OR NOT TO TURN -THAT IS THE QUESTION

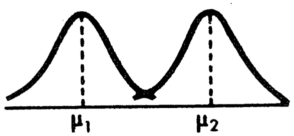
N. Barnett,

Footscray Institute of Technology

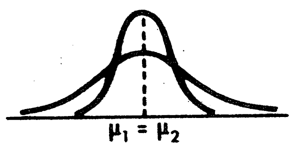
It is common to have someone approach a statistician with a problem for which none of the usual statistical analyses at his disposal is quite appropriate, in which case he has to improvise. Recently, however, a doctor from the Dandenong General Hospital presented me with some figures for which some simple meaningful calculations were possible. This turned out to be fortunate because he wanted a statistical analysis of his experimental results available for the following day!

Before proceeding to share one of the calculations with you, I would like to draw your attention to a feature of Dr Watterson's article on "Trapping Animals" (*Function*, Vol.3, No.3). The aim is to contrast the method adopted there with the procedure to be used here to analyse the doctor's data.

Dr Watterson's method was to assume the existence of conditions which permit use of the Poisson distribution and then use the experimental results (or observations) to estimate μ, the mean of the Poisson distribution. (μ is otherwise called the parameter of the distribution.) In this respect his procedure could be termed *parametric* - he used a parametric technique. Most distributions have one or more parameters, these are merely constants which need to be known in order to define the distribution uniquely. Some distributions possess parameters which affect the location of the distribution, while others affect the shape. For example, the normal distribution has two parameters, μ and σ called, respectively, the mean and standard deviation. Whatever the values of μ and σ, the distribution is always bell-shaped but the point of symmetry and the spread change with different values of μ and σ. This is illustrated in the figures below.



Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$.



Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$.

In the analysis of the doctor's data the technique I use isn't centred around estimating the value of a parameter of an assumed distribution. In this respect it is typical of statistical procedures called *non-parametric* or distribution-free. You will notice that a well-known distribution is used but not in the context of attempting to estimate parameters.

The doctor's problem was as follows. In performing a certain treatment on patients, the standard procedure is to keep the patient's head in the normal straight position. Unfortunately, in the ensuing treatment there is a possibility of neck injury which the doctor believed would be reduced if the head were first turned prior to treatment. The doctor subsequently treated some patients in the standard way and others by first turning their heads. Over a period of time he accumulated the following data pertaining to the incidence of neck injury in the two procedures.

	No. with neck injuries	No. without neck injuries	Total
Head turned	2	43	45
Head straight	5	50	55
Total	7	93	100

At first glance, the table indicates that turning the head prior to treatment is advantageous in reducing injury but the question remains, "Is this just chance or is a real effect evident?" In order to answer this question we calculate the probability of getting these tabulated values purely by chance.

We have 100 patients and suppose we randomly assign them to undergo either the head turned or head straight procedure. The number of ways we can choose the 45 patients who undergo the head-turned technique is

$$^{100}C_{45} = \binom{100}{45} = \frac{100.99.98...56}{1.2.3...45}.$$

The number of injuries in this 45, if 7 occur in the 100, we regard as a random variable capable of taking the values $0,1,2,\ldots,7$. In actual fact, we obtained only two injuries and the probability of two or fewer injuries is

$$\frac{\binom{7}{2}\binom{93}{43}}{\binom{100}{45}} + \frac{\binom{7}{1}\binom{93}{44}}{\binom{100}{45}} + \frac{\binom{7}{0}\binom{93}{45}}{\binom{100}{45}},$$

since, under the assumption that there is no real difference between the two treatments, the number of injuries has a hyper-geometric distribution. Calculation of the above probability yields a value of about 0·3. Our finding then is that there is a probability of approximately 0·3 that the tabulated values (those more favourable to the head-turning treatment) could have occurred purely by chance.

If this probability had been very small, say 0·05, then we would be inclined to disbelieve that this event occurred by chance. We would believe there to be a real difference between the effects of the two treatments rather than accept the occurrence of a highly unlikely event. We all know that unlikely events do occur, but in statistics we usually conclude that *if our assumptions give the observed events only a small chance of occurring, the assumptions are probably incorrect;* we can often calculate the probability of wrongly so rejecting the assumptions. So from the doctor's data we cannot, with any degree of conviction, state that the head-turning treatment is less hazardous.

If the same data had arisen in, say, the operation of two machines manufacturing small, fairly inexpensive mechanical parts (the two treatments corresponding to a new and old machine and injuries corresponding to the incidence of the production of faulty parts), my advice would have been to take larger samples and apply the analysis again to see if a more convincing result could be obtained. With large samples, we would be reducing the chance of getting 'untypical' results. However, in the case in question, we are dealing with injuries to people. We might well consider the result 0·3 sufficient ground for using head turning rather than risk subjecting patients to the standard and possibly more hazardous procedure. If the doctor does proceed to treat all patients by first turning the head, he can, of course, combine his observations of injury with those previously obtained and do his statistical analysis again.

Since the calculation of the figure 0·3 is based on the assumption that injuries due to the treatments do not differ significantly, you might like to play around with the table of values a little in the following manner. Through all changes, keep the totals (45 and 55) the same but alter the number of injuries in the first column, maintaining the total of 7. With these new data, make an intuitive comment about what you think is the effect of head turning (or not), and then make the calculation of this configuration occurring by pure chance, as was done above.

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TEETHING TROUBLES WITH METRICATION

The *Age Weekender's* Handy Hint for 15.6.79 gives a method of bracing fenceposts, taken from the Paul Hamlyn *Handyman's Encyclopedia*. Unfortunately, they used the metric conversion formula 2' 6" = 700 cm, and so wrote: "Allow that the fence post will go about 700 cm into the ground".

As a reader, Gary Donovan, pointed out (*Access Age*, 18.6.79), this needs 29' fence posts - a tall order indeed.

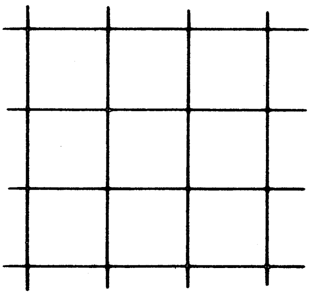
On the ABC, an agricultural journalist, Trevor Johnson, in News Commentary (4.7.79), also showed himself a little unhappy with the new units. He spoke of "a record acreage of 10.8 million hectares".

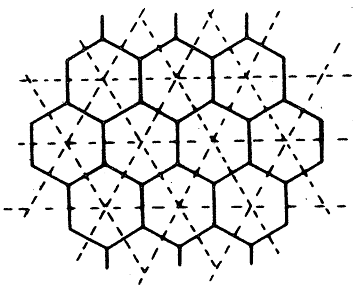
THE TESSELATION ART OF M.C. ESCHER

John Stillwell, Monash University

The Dutch artist Maurits Escher (1898 - 1972) is probably known to many readers for his mathematically inspired graphics, which have appeared as textbook illustrations, posters and record covers. A number of books have been written on his work, the most informative from the mathematical point of view being perhaps *The Magic Mirror of M.C. Escher* by Bruno Ernst (Ballantine Books, 1976). Nevertheless, the richness of Escher's work is such that much more could be written on its mathematical implications, some of which Escher himself did not realize. In this article I shall discuss just the *tessellation* designs.

A tessellation is any pattern produced by filling the plane with non-overlapping tiles. Floors have been made this way since ancient times, so it was well-known that the only regular polygons which could be used for tiles were squares, regular hexagons or equilateral triangles. The second figure



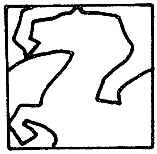


shows the "dual" relationship between the hexagonal and triangular tessellations. The symmetry of these designs is so blatant that artists through the ages have invented refinements of the basic patterns to make their symmetry more subtle. These refinements involve either

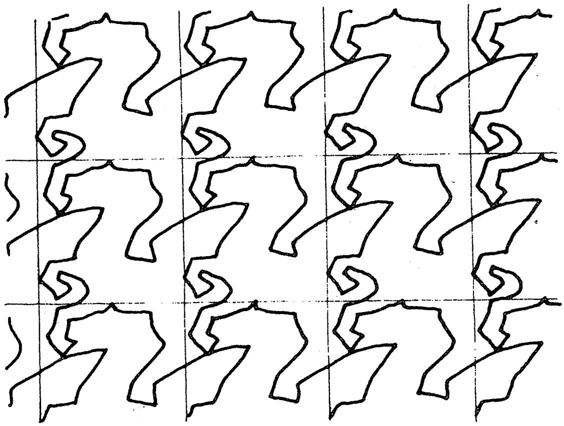
(1) changing the shape of the basic tile
or (2) imprinting a design on the basic tile

so as to eliminate certain lines of symmetry or to make them less obvious.

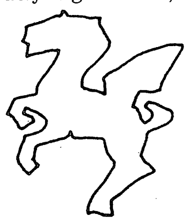
For example, to kill the diagonal lines of symmetry in the square tessellation one can imprint the basic tile with an asymmetric pattern, such as



which gives a more interesting tessellation, due to Escher:



Exactly the same result can be achieved by changing the shape of the basic tile to the flying horse;



The fact that the flying horse tessellation is symmetrical, but less so than the plain square tessellation, is reflected by an analysis of its *symmetry group*. We may define a *symmetry* to be any motion of the plane which results in the tessellation being superimposed on itself. Then it is clear that any symmetry of the flying horse tessellation is a combination of horizontal and vertical translations of unit length (= side of the square). Such a translation can be concisely specified by a pair (m,n) of integers, where m,n are respectively the horizontal and vertical distances covered in the motion.

It is clear that the result of symmetries (m_1, n_1) and (m_2, n_2) in succession is the symmetry $(m_1 + m_2, m_1 + m_2)$, and that the symmetry (m,n) has the *inverse* $(-m,-n)$ which moves every point back where it started. These properties imply that the symmetries form a *group* in the mathematical sense, and what we have just given is a precise description of the symmetry group F of the flying horse tessellation.

Another way to describe F is by letting X, Y denote the unit translations to the right and upwards, and X^{-1}, Y^{-1} their inverses. Then if we write combinations of the unit translations as "products" of the corresponding letters, any motion reduces to the form

$$X^m \ Y^n$$

which is what we previously denoted (m,n). For example

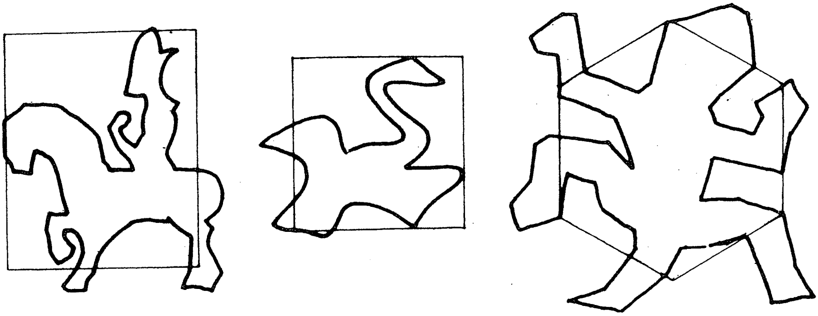
$$\begin{aligned} X^{-1}XYXXY &= YXXY \\ &= XXYY \\ &= X^2Y^2 \end{aligned}$$

This calculation uses the relation $XY = YX$, which is valid of course, since the result of translations is the same whatever their order.

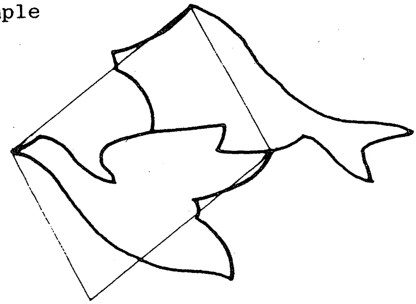
The symmetry group S of the plain square tessellation obviously includes X and Y, but also some additional motions, such as a rotation R through $\frac{\pi}{2}$ about a vertex. Without finding them and their relations explicitly, this confirms that the plain square tessellation is "more symmetrical" than the flying horse tessellation. The increase in symmetry is due to the greater symmetry of the basic tile - the square has a number of symmetries by reflection or rotation, while the flying horse has none. But one can use the fact that the square has only a finite number of symmetries to show that there are only finitely many symmetry groups based on square tiles. Similar reasoning for the hexagonal and triangular tessellations leads to the conclusion that *there are only finitely many plane symmetry groups.*

This means that artists designing floors, wallpapers etc. have only finitely many (17 in fact) types of symmetry to choose from. All of them were exhibited in the wall decorations of the Alhambra in Spain, made by the Moors in the 15th century, so Escher came on the scene far too late to make any innovation in this area! His contribution was to find interesting shapes for the basic tile, which is what group theorists call a *fundamental domain* for the symmetry groups.

He took advantage of the fact that fundamental domains could be recognizable human or animal shapes. Here are some of his best known ones:

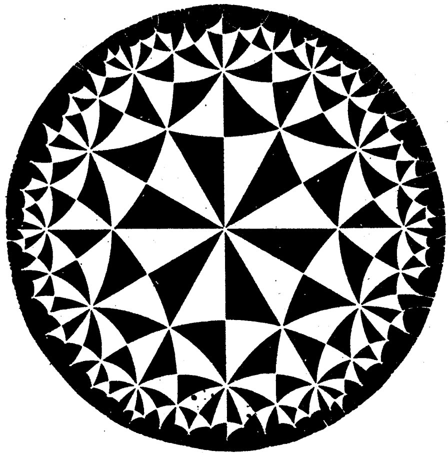


He also produced designs in which tiles of different types alternate, for example



in which case the fundamental domain is a union of tiles, one of each type. This can also happen when all the tiles are of the same shape, if they are coloured differently.

In the 1950's Escher became interested in depicting the infinite in terms of tessellations which packed (in theory) an infinite number of tiles into a finite space. Naturally the tiles cannot all be congruent in this case, and the artist faces the problem of making them become arbitrarily small in a natural and symmetrical way. Escher, who always freely admitted that he did not know much mathematics, was unaware that the ideal tessellations for his purpose had been found by mathematicians in the 19th century, but he was lucky enough to stumble on the example opposite in the book *Introduction to Geometry* by H.S.M. Coxeter. The curvilinear triangles which fill the disc are not congruent, but they all have the same angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, and their size diminishes nicely as they approach the boundary. In fact, each triangle is a fundamental domain of a group of transformations which map the disc into itself. The transformations concerned are called *linear fractional transformations*, which we will not describe in detail here.



A diagram of the above type was first discovered by the Swiss mathematician H.A. Schwarz in 1872, while studying functions which are invariant under linear fractional transformations. (These generalize the functions cos z and sin z which are invariant when z goes to $z + 2\pi$.) About 10 years later, the great French mathematician Henri Poincaré† made the sensational discovery that if "distance" is defined in a new way for the disc then it can be considered as a "non-euclidean plane", in which the circular arcs perpendicular to the boundary are "straight lines", the triangles are "congruent", and the linear fractional transformations described above are "motions".

Poincaré imagines that the physical laws inside the disc are such that all objects shrink by a factor $1 - r^2$ when they travel a radial distance r from the centre, so that size nears 0 as they approach the boundary $r = 1$. Thus the inhabitants' steps will always fall short of the boundary, and if they are unaware of the shrinkage, they will consider themselves to live in an infinite space. And they certainly need not be aware of it, because the relative sizes of neighbouring objects always remain the same, in fact they can use rulers, tape measures, etc. to assign constant sizes to objects just as we do. In particular, the straight line between points A,B is the position of a tape measure between A,B which gives the smallest possible "length". It turns out that this "straight line" is the unique circle through A,B perpendicular to the boundary (or the diameter, if A,B are on the same diameter) - as desired.

We can also think of the Poincaré disc as a bird's eye view of a plane, with the disc boundary as horizon. However, this is not the euclidean plane, because for any "straight line" and a point P outside it there are many "parallels" to ℓ through P.

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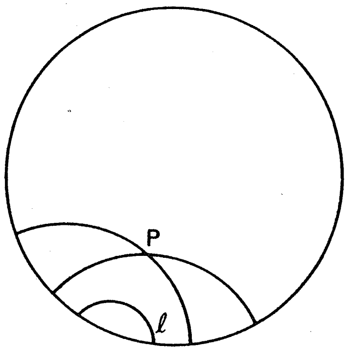
†Also attributed to the German Felix Klein.



It turns out that the geometry is simply that of Euclid with the "single parallel" axiom replaced by a "multiple parallel" axiom, a famous geometry first proposed by Bolyai and Lobachevsky (Hungary and Russia respectively) in the 1820's. Poincaré's disc not only provides a model of Bolyai-Lobachevsky geometry, but it shows that it is a *good* geometry to use when studying linear fractional transformations.

This observation has extraordinarily wide implications, with consequences in the theory of equations (both algebraic and differential), theory of numbers and relatively modern areas such as coding theory. There are also surprising connections with other parts of geometry, such as the study of curves on surfaces and in 3-dimensional space - just recently the problem of *classifying knots* has been solved using intricate constructions in the Poincaré disc.

But to return to tessellations: in contrast to the euclidean plane, there are *infinitely many* symmetry groups in the Bolyai-Lobachevsky plane. In particular, we can choose the fundamental domain to be any triangle with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ where p,q,r are integers such that $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. This is because Bolyai-Lobachevsky geometry allows the angle sum of a triangle to be any value less than π. We can see this in the Poincaré model: the angle sum approaches 0 as the vertices approach the disc boundary and approaches π (the euclidean value) as the size of the triangle approaches 0. Since it remains true that the sum of the angles at a point is 2π, a tessellation with congruent $(\frac{\pi}{p},\frac{\pi}{q},\frac{\pi}{r})$ triangles can be constructed with three types of vertices

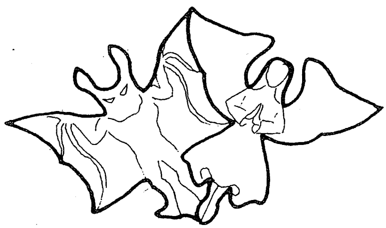
(a) where $2p$ angles of $\frac{\pi}{p}$ meet

(b) where $2q$ angles of $\frac{\pi}{q}$ meet

(c) where $2r$ angles of $\frac{\pi}{r}$ meet

at which the basic symmetries are "rotations" through angles of $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ respectively. There are also "translations" which carry any vertex onto any other of the same type.

The amount of symmetry is so overwhelming that to our euclidean eyes all non-euclidean tessellations look much the same. Escher's main concern was to find interesting shapes for the fundamental domains, which he did in a series of pictures entitled *Circle Limit* I, II, III, IV. His favourite number, III, is a very elegant design using fish, but mathematically disappointing because it depends on some lines which are not "straight", namely, circles not perpendicular to the disc boundary. Number IV (also known as "Heaven and Hell" or "Angels and Devils", and available as a poster)† is mathematically correct as well as being an interesting artistic statement. The fundamental domain is a black devil and white angel side by side



the underlying triangle of each being $(\frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{4})$.

Circle Limit IV is the best picture we have of the world imagined by Poincaré (see overleaf).

If one stands close to the large poster it is possible to accept the disc boundary as "horizon" and the remarkable spaciousness of the Bolyai-Lobachevsky plane becomes apparent. Not only do parallel lines diverge in many directions, but population growth explodes in all directions too. Concentric rings of angles and devils around any point grow rapidly in size, each ring containing more than twice as many as its predecessor. This shows that population can grow indefinitely at its natural exponential rate in the Bolyai-Lobachevsky plane - something which is not possible in euclidean space - so in some ways this imaginary world works better than the real one.

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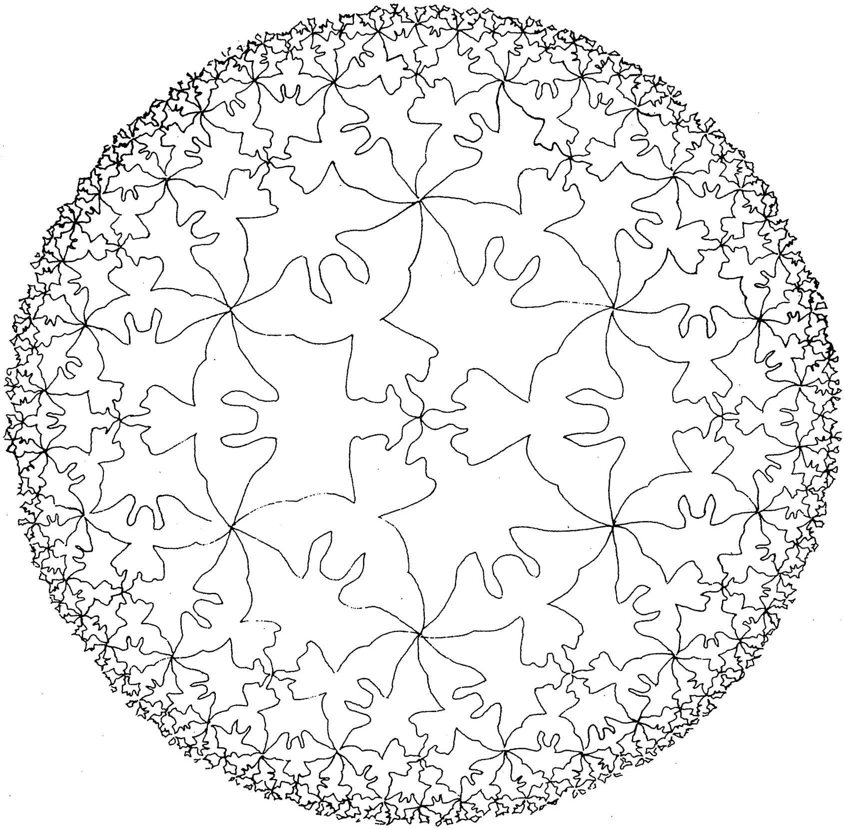
†See advertisement on back cover.



A diagrammatic version of *Circle Limit* IV.

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MORE MATHEMATICAL SWIFTIES

"The lines intersect", stated Tom and Jane simultaneously.

"Why isn't π equal to 22/7?", inquired Tom irrationally.

"What is the area of the unit circle?", Tom asked piously.

"$\frac{1}{x}\sin\frac{1}{x}$ oscillates near $x = 0$", Tom yelled wildly.

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EINSTEIN'S PRINCIPLE OF EQUIVALENCE

Gordon Troup, Monash University

Consider two observers moving relative to one another. Each uses a frame of reference, or coordinate system, to follow the motion of a particle. If we make due allowance for their relative motion, we must arrive at consistent descriptions. The first observer will express his law of motion in terms of three spatial coordinates x,y,z and a time coordinate t; the second will use x',y',z' and t'.

Galileo was the first to produce a principle of relativity - the statement that under the transformation

$$x' = x - Vt, \quad y' = y, \quad z' = z, \quad t' = t \tag{1}$$

(where V is the relative velocity of the observers - along the x-axis), the laws of classical dynamics are unaltered.

Newtonian mechanics obeys Galilean relativity, but electricity and magnetism do not. This apparent paradox was resolved by Einstein, in his special theory of relativity. The appropriate transformation is

$$\begin{aligned} x' &= (x - Vt)(1 - V^2/c^2)^{-\frac{1}{2}} \\ y' &= y, \quad z' = z \\ t' &= (t - Vx/c^2)(1 - V^2/c^2)^{-\frac{1}{2}}, \end{aligned} \tag{2}$$

where c is the speed of light (3×10^8 metres per sec.). This set of equations is known as the Lorentz† transformation.

When V is much smaller than c, the Lorentz transformation (2) is approximated by the Galilean transformation (1). Already we can see that the Lorentz transformation (2) involves an inter-play between space co-ordinates and time co-ordinates.

But it is not the intention of this article to examine the consequences of special relativity. The aim is to show how, by remarkably simple (to us now!) reasoning, Einstein arrived at his *principle of equivalence*, the foundation of his general theory of relativity.

Special and Galilean relativity apply to observers using *inertial frames*, namely those in which Newtonian dynamics applies in constant relative motion. General relativity

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†Dutch Physicist (1853 - 1928), Nobel Laureate 1902, whose work anticipated and prepared the way for much of Einstein's special theory of relativity.

applies to accelerated frames, as well as to inertial frames.

We shall start by considering two observers, one of whom is being accelerated.

Let the first observer be the one using an inertial frame (I) with coordinates (x,y,z,t), and the second be the one whose frame (N), with coordinates (x',y',z',t'), is accelerated. Suppose that N is subject to a uniform acceleration A along the x-axis of I, and is coincident with I at a moment $t = t' = 0$, when the initial velocity of N is also zero.

By using the well-known expressions for uniformly accelerated motion, we find

$$\begin{aligned} x' &= x - \tfrac{1}{2}At^2 \\ y' &= y, \quad z' = z, \end{aligned} \qquad (3)$$

and we will *assume* the Galilean condition $t' = t$. (In view of our experience with good watches on trains and in cars, the assumption is a good one - provided any speeds involved are very much less than the speed of light.)

Fixing our attention now on the x and x' components, we have

$$\frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{d}{dt}(x - \tfrac{1}{2}At^2) = \frac{dx}{dt} - At \qquad (4)$$

for the velocity transformation, and

$$\frac{d^2x'}{dt'^2} = \frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} - A \qquad (5)$$

for the acceleration transformation. Thus even a constantly accelerated frame does *not* keep accelerations constant. Because Newton's laws give us the value of a particle's *acceleration*, it follows that accelerated frames are *not* inertial frames.

Nevertheless, it is sometimes convenient to use accelerated frames, so it becomes necessary to know the consequences of being in such a frame. Let us consider the results of some experiments performed in a constantly accelerated laboratory. (See Figure 1.)

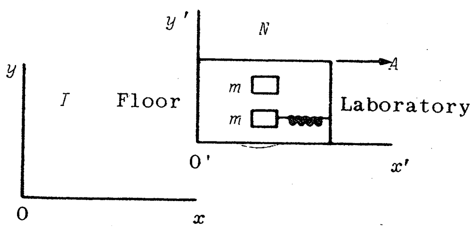

Figure 1

With respect to the fixed inertial frame I, the non-inertial frame N is moving along $0x$ with a constant acceleration A. Consider a laboratory fixed in N as shown; an observer in it cannot see out of the laboratory, but can communicate with I, whose observer can see into the laboratory.

The observer in N performs the following experiments.

(1) He releases a mass m from "rest" in N.

(2) He suspends a mass m from a spring balance as shown.

(3) He observes a pulse of light which is emitted from I parallel to $0y$.

The observer in I describes the results as follows. On release, the mass m continues moving along parallel to $0x$ with the instantaneous velocity it had just before release, since no forces now act and the law of inertia holds. However, the laboratory N accelerates past the mass m, which is therefore struck by the floor.

The mass m on the spring requires a force F to keep it accellerating. This is given by $F = mA$, and so the spring will be stretched by the amount necessary to make the tension force in it equal in magnitude to mA.

Let the pulse of light enter the laboratory at time $t = 0$, and travel for a time Δt *parallel to* $0y$. The distance travelled parallel to $0y$ will be $c\Delta t$; meanwhile the laboratory will have moved a distance $\frac{1}{2}A(\Delta t)^2$ parallel to $0x$.

The observer in N describes the results as follows. On release, the mass m accelerates towards the floor with acceleration $- A$.

The spring is stretched by just the amount that it would be stretched if there were a 'weight force' $-mA$ acting towards the floor.

The light beam travels a distance $c\Delta t$ parallel to $0'y'$, and a distance $-\frac{1}{2}A(\Delta t)^2$ parallel to $0'x'$. This relationship characterises a *parabola*. (Refer to a particle projected horizontally in a uniform vertical gravitational field of acceleration A downwards; see figure 2.)

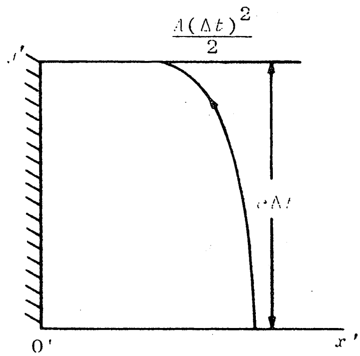

Figure 2

Since the observer in *N* cannot see out of the laboratory, he cannot tell

(i) whether the laboratory is being uniformly accelerated parallel to Ox (as the observer in *I* *knows* it is); or

(ii) whether he (*N*) is *stationary* in a uniform gravitational field of acceleration - *A* parallel to O'x'.

Thus in every dynamical experiment *N* performs he must postulate the existence of the force $-mA$ on a mass m if, for him, the law of inertia is to hold. From the viewpoint of the *inertial frame I*, this force is *non-existent*. It is therefore called a 'pseudo-force', 'fictitious force', or (better) an 'inertial force', since it is very real to anyone experiencing it, as in a rapidly decelerating car. Hence:

(i) in a uniformly accelerated frame the effects of the acceleration are equivalent to those of a uniform gravitational field of the same acceleration, but oppositely directed;

(ii) the acceleration a_N *relative* to the non-inertial frame *N* is equal to the acceleration a_I *relative* to the inertial frame *I* *minus* the acceleration *A* of the non-inertial frame:

$$a_N = a_I - A \quad \text{or} \quad a_I = a_N + A. \tag{6}$$

Einstein's 'Principle of Equivalence', on which is founded his General Theory of Relativity, states that it is impossible to distinguish between uniform acceleration of a laboratory and an oppositely directed, uniform gravitational acceleration (field) of the same magnitude by experiments performed solely in the laboratory. A consequence of this is that the path of a light ray should be *curved* in passing through a gravitational field. The results of measurements on the bending of starlight passing near the sun are not inconsistent with Einstein's General Relativistic prediction. Very accurate measurements of the delay of radar echoes from Venus which pass by the sun, and of the radio signals from deep space probes, have agreed very well with Einstein's predictions. A further prediction is that frequencies of electromagnetic radiation emitted by similar sources will differ if the sources are in different gravitational fields, or if they are at different heights in a constant gravitational field. This has also been verified. Finally, the General Theory of Relativity predicts gravitational waves, whereas the Newtonian theory does not. It *may* be that these waves have been detected; the work still goes on to make better and better gravitational wave detectors, since most scientists working in the field of gravitational waves now doubt that Weber in his pioneering work did in fact detect such waves. In Australia, a group in Perth is working on gravitational wave detectors.

In February of this year, the astronomers Taylor, Fowler and McCulloch (a Tasmanian) reported observations on a binary pulsar which indicate that the system is losing energy by gravitational radiation. (See *New Scientist*, 8 February 1979, or *Nature*, also 8 February, 1979.)

The considerations leading to the equivalence of the results of dynamics experiments in a uniformly accelerated frame to the results of the same experiments in a frame subject of a uniform gravitational field (or equal, but oppositely directed acceleration) are comparatively simple. Yet Einstein was the first to publish these considerations. He displayed his genius by extending this equivalence to the results of *all* physical experiments, and then insisting that the equivalence was a two-way one so that, for example, light should be bent on passing through a gravitational field.

Of course, Einstein's theory of general relativity is not written in the simple mathematical language used above. It is written in terms of tensor calculus - with which Einstein was not particularly familiar when he first published his early works on general relativity, but to which he contributed later on. Nevertheless, after he pioneered the way, we may grasp the basis of his theory - the equivalence principle - by means of comparatively simple considerations.

Example: (a) By working out the problem *in the accelerating frame*, show that a man in a lift, which is in free fall, will be essentially weightless.

(b) Show the same thing for an astronaut in an earth satellite with a circular orbit. *Hint*: see Equation (6).

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PROJECT STEP

By now, your school should have received copies of a recent report on the progress of Project STEP (Secondary-Tertiary Education Planning). This Monash-based study is the most comprehensive survey to date of Victorian school students and their subject choices. Its aim is to provide realistic planning data for the development of tertiary courses.

The preliminary report is of especial interest to readers of *Function* because its main concern is with mathematics. Among the major patterns noted in the data are:

(1) There has been a swing away from the study of mathematics at Year 12 - the trend to General Mathematics (in place of Pure and Applied) has been followed by a tendency to drop the subject altogether;

(2) Girls, more so than boys, tend to drop mathematics at Year 12, although overall more girls than boys sit HSC and they also have the better pass-rate.

The project officer, Mr B.J. Walsh, asks if students may not have too much flexibility in their choice of subjects. He suggests that many courses actually chosen by students at HSC level lead to very little freedom of choice at the tertiary level. In the Foreword, Mr Alan Wilkinson, the Personnel and Public Affairs Director of Shell (Australia), whose company funded publication of the well-produced report, expresses concern at the trend away from mathematics.

LETTERS TO THE EDITOR

RECORDS TUMBLE

$2^{44497} - 1$ is prime

In quick succession the largest known prime has grown larger. Last year came the surprising announcement (L.A. Times November 16, 1978) that two freshman students of USC Hayward, Laura Nickel and Curt Noll, had completed a high school computing project by showing that $2^{21701} - 1$ (a number of 6553 digits) is prime. Curt Noll continued this work, and next in February, 1979, found that $2^{23209} - 1$ (6987 digits) is prime. His good fortune is illustrated by the more recent announcement (L.A. Times May 31, 1979) that the next prime in the sequence $2^m - 1$ is $2^{44497} - 1$ (a number of 13 395 digits); this finding is due to Harry Nelson and David Slowinski who apparently had access to considerably faster computing facilities than did Noll; the latter is reported to have remarked that it would have taken him 16 years to duplicate this finding. So now we know of 27 Mersenne primes (primes of the shape $2^m - 1$) and by the same token, of 27 perfect numbers: a number is said to be perfect if it is equal to the sum of its divisors other than itself, and it has long been known (Euclid) that $2^{p-1}(2^p - 1)$ is perfect if $2^p - 1$ is prime. The complete list of known Mersenne primes is $2^m - 1$, with

m = 2 3 5 7 13 17 19 31 61 89 107 127 521 607
1279 2203 2281 3217 4253 4423 9689 9941 11213
19937 21701 23209 44497.

A.J. van der Poorten
Macquarie University

[*Things are moving in this area of mathematics. This is the third time we have reported the highest known prime. See* Function, *Vol.2, No.4, and Vol.3, No.1. We wonder how long this new record will last. Eds.*]

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A MESSAGE FROM THE CRYPT

On a recent trip to England, I bought a copy of Cocker's Decimal Arithmetick (John Hawkins' edition) of 1685. This book shows how to use decimal fractions (introduced not long before by Simon Stevin), how to extract square and cube roots and how to manipulate them and also how to solve quadratic equations. Near the beginning of this book, however, there appears the strange message reproduced opposite.

Anixo guo Anamſiggino Jorammi Lehkeg *Lòfoxrofè-holrii Torpiemgig Im Xonifafu Diſohèmiemgi Pnwi-naſigfêho.*

G I H,

IT you Lèpeag fo zegfod gone ot youh glahe rouhèg im lehugims fre toppodims fèheafige you dipp frem ze fre zeffeh azpe fo juws rod I rave glemf nime, amw it ny laimeg freheim nay ze Lhotifazpe fo fre luzpixk I rave ny digr, zuf it mof, if ig mof a foow frims mod imweew I wo gay go.

G I H, I au

Oxfoz. 30. 1684.
Thon Pomwom

Youh runzpe gehvamf
Jorm Radking.

One should not underestimate the information which is provided by the physical appearance of the message.

John N. Crossley
Monash University

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PROBLEM SECTION

As in earlier cases, we begin with solutions - some submitted to us, others (regrettably) supplied by the editors. We leave some earlier ones open, and print some new ones. One reader queried us on PROBLEM 3.2.3, asking if we were serious. Yes, we are. Some reasonable assumptions have to be made. The problem read "Prove that, of all the teenagers in the world, at least two have the same number of teenage friends". We might begin by disallowing the usage "A is a friend of A", and we need another slightly less obvious understanding. After this, the problem is entirely mathematical in character, and not difficult if you approach it from the right direction.

SOLUTION TO PROBLEM 3.1.1

This problem read:

Two of a 3-man jury each independently arrive at a correct decision with probability p. The third flips a coin. The decision of the majority is final. What is the probability of the jury's reaching a correct decision?

This problem is referred to as "the problem of the flippant juror". A simple writing out of all cases produces the value p

for the required probability - i.e. the value is the same as that expected if the decision rested with a single "non-flippant" juror.

PARTIAL SOLUTION TO PROBLEM 3.1.2

If the product of four consecutive integers is a square, what are the numbers?

There are three cases to consider:

(1) One of the integers is zero - this clearly leads to four solutions;

(2) All the integers are positive - this case is discussed below;

(3) All the integers are negative - this case reduces trivially to case (2).

In case 2, we may choose a rational number x, such that

$$(x - 3/2)(x - 1/2)(x + 1/2)(x + 3/2) = m^2$$

for integral m. This equation may be solved to give $x^2 = \frac{5}{4} \pm \sqrt{1+m^2}$ from which it follows that $\sqrt{1 + m^2}$ is integral, and $1 + m^2$ is a perfect square. This is impossible for $m > 0$.

Hence the only solutions are those for which one of the integers is zero.

We leave open, for the present, the second part of the problem which asked for four consecutive odd numbers whose product is a square.

SOLUTION TO PROBLEM 3.1.5

This problem asked how many different pairing arrangements could be produced from $2n$ players in a tennis tournament.

Line the players up and pick off pairs from one end. This can be done in $(2n)!$ ways. The order in which the pairs are taken is irrelevant. There are $n!$ such orderings. Nor does it matter in which order the members of any individual pair are selected. There are 2^n arrangements of pairs. Thus the required answer is $(2n)!/(n!\ 2^n)$, which is the answer given in the problem.

SOLUTION TO PROBLEM 3.1.6

This version of a well-known brainteaser read as follows:

Hanging over a pulley is a rope with a weight at one end. At the other, there is a monkey of equal weight. The rope weighs 250 gm per metre. The combined ages of the monkey and its father total 4 years and the weight of the monkey is as many kilograms as his father is years old. The father is twice as old as the monkey was when the father was half as old as the monkey will be when the monkey is three times as old as the monkey was. The weight of the weight plus the weight of the rope is half as much again as the difference between the weight of the weight and the weight of the weight plus the weight of the monkey.

How long is the rope?

Caecilia Potter (Year 11, Siena Convent) solved the problem correctly. She put a for the monkey's age, b for that of its father, the weight of the monkey and the weight of the weight. (All of these are easily seen to be equal.) If now L is the length of the rope in metres and w its weight in kilograms, we easily find

$$L = 4w, \quad b = 2w, \quad a + b = 4,$$

i.e. three equations in four unknowns. To find a fourth equation, we need to unravel the complex sentence beginning "The father is twice as old as the monkey was when ...". This first "when" refers to a time x years previously, and the final "when" to a time y years before the description. The sentence may now be disentangled to produce three further equations:

$$b = 2(a - x), \quad b - x = \frac{9}{2}(a - y), \quad b - y = 3(a - y).$$

There are now six equations in six unknowns, and solving for L, we find $L = 5$. The rope is 5 metres long.

[This problem has stayed in my mind because of a painful memory. Twenty years ago, I was a cadet industrial engineer with Mt Isa Mines and learned of this problem through a book of business mathematics in their library. Two of my senior colleagues and I had a competition to solve it. I finished in 30 minutes, the next man in 45 and the third gave up after an hour. The second man won the competition because I, having reached the conclusion $b = 2\frac{1}{2}$, *had written* $L = 2b = 2 \times 2\frac{1}{2} = 3$. *Very sad! M.D.]*

SOLUTION TO PROBLEM 3.2.1

This problem asked for a proof that every year contained at least one Friday 13th. To show this, note that only 14 different types of year can occur. A year may be leap or non-leap and may start on any day of the week. We simply check each possibility. The calculation may be greatly reduced, however, by using the techniques described in Dr Sonenberg's article in *Function*, Vol.1, No.1. In a non-leap year, the 13th of the months January, February, March, etc. occur on the 13th, 44th, 72nd, etc. days of the year. Taking a set of remainders after division by 7, we find that each of the numbers 0, 1, 2, 3, 4, 5, 6 occurs at least once. A similar result is found for leap years. As this remainder determines the day of the week on which the 13th falls, the proof is complete.

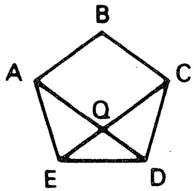
SOLUTION TO PROBLEM 3.2.5

This considered a regular pentagon $ABCDE$, whose diagonals AD and EC meet at Q. Readers were asked to show that the segment lengths AD, AQ, QD satisfied the equation

$$\frac{AD}{AQ} = \frac{AQ}{QD}$$

and hence to show that the ratio AD/AQ equals $\frac{1}{2}(1 + \sqrt{5})$.

Stephen Tolhurst (Year 12, Springwood H.S., N.S.W.) solved this problem. He writes:



Let a be the side of the pentagon, and let $\angle EAD = \alpha$. Then $\angle EDA = \alpha$ as the triangle AED is isosceles. The triangles AED and EDC are congruent and therefore $\angle CED = \angle ECD = \alpha$. It follows that $\angle DQC = \angle AQE = 2\alpha$, and $\angle EQD = \pi - 2\alpha$, as $\angle DQC = \angle QDE + \angle QED$. Then $\angle AEQ = \angle QDC = \pi - 3\alpha$. But, since $ABCDE$ is a regular pentagon, $\alpha = \pi/5$.

Now apply the sine rule to the triangle ADE to find $AD = (a \sin \frac{2\pi}{5})/(\sin \frac{\pi}{5})$. Similarly from triangles AEQ and QDE respectively, we find $AQ = a$, $QD = (a \sin \frac{\pi}{5})/(\sin \frac{2\pi}{5})$. It follows that

$$\frac{AD}{AQ} = \frac{AQ}{QD} . \qquad (*)$$

Stephen then gives a trigonometric argument to show that $AD/AQ = \frac{1}{2}(1 + \sqrt{5})$. He finds $\cos \frac{\pi}{5} = \frac{1}{4}(1 + \sqrt{5})$ in the course of his investigation. A somewhat more direct argument is possible. Put x for each of the ratios in equation (*) and note that the left-hand side may be written as $1 + \frac{1}{x}$. Then $1 + \frac{1}{x} = x$, from which we find $x = \frac{1}{2}(1 \pm \sqrt{5})$ and clearly the positive root is the one required.

[*We have slightly amended Stephen's letter - most notably by changing degrees to radians. Degrees are arbitrary units, whereas radians are natural mathematically. The French, and some other continentals, measure angles in* grades, *of which a hundred (cent) make up a right angle. The existence of this* centigrade *scale is the reason that we now use the term* Celsius *for the metric temperature scale. Eds.*]

SOLUTION TO PROBLEM 3.2.6

What is the probability of three coins all falling alike when tossed together? One argument has it that they all fall heads with probability 1/8, and tails with the same probability. The answer on this argument is $\frac{1}{4}$. Another view is that as some two coins must fall alike, the third could fall the same or different. On this view, the probability is $\frac{1}{2}$. Who is correct, and why?

Stephen Tolhurst also answered this, pointing out that the first argument is correct. "The second argument arises if the coins are indistinguishable, i.e. *HHT* is the same as *THH* or *HTH*." In the case of coins, one expects to be able to mark them (say with a felt-tip pen) in such a way as to distinguish the coin without affecting the results of the experiment. Nevertheless, this expectation is not a mathematical result and it does require checking. Stephen did this, subjecting, as he writes, "three images of Her Majesty to a most undignified procedure". In 100 such trials, he found 30 cases in which the coins all agreed. He asked how good this was as evidence for the figure of $\frac{1}{4}$ as opposed to $\frac{1}{2}$, and how many trials should be carried out before the results may be considered reasonable.

Stephen's letter was passed on to the editor who posed the question. Here is his reply.

Stephen is wise to be not firmly convinced by his own reasoning and hence to resort to experimental verification. In physics, elementary particles behave as if they are indistinguishable. If we were dealing with photons rather than coins, there would be *four* equally likely cases *HHH*, *HHT*, *HTT*, *TTT*, not eight as for coins.

For the experimental verification, let us define a "trial" as being the tossing of three coins, and call a trial a "success" if all three coins fall with the same face showing. Let p denote the probability of success in one trial. If we perform n independent trials, we can be fairly sure that the observed proportion of successful trials will be close to the true probability, p, provided we make n large enough.

Let X denote the number of successful trials out of n. Then X has a binomial probability distribution, mean np and variance $np(1 - p)$. The proportion X/n has a mean p and variance $\frac{p(1-p)}{n}$. With a high probability, perhaps around 95%, X/n will be within two standard deviations of its mean:

$$\left|\frac{X}{n} - p\right| \leqslant 2\sqrt{\frac{p(1-p)}{n}} .$$

The standard deviation is largest when $p = \frac{1}{2}$, so we can be pretty confident that

$$\left|\frac{X}{n} - p\right| \leqslant 2\sqrt{\frac{\frac{1}{2}(1-\frac{1}{2})}{n}} = \frac{1}{\sqrt{n}} .$$

Taking $n = 100$ trials (as did Stephen), X/n should be within $\frac{1}{\sqrt{100}} = \frac{1}{10}$ of the true p value. This should distinguish between the two possibilities $p = \frac{1}{4}$ and $p = \frac{1}{2}$; in fact Stephen's observation of $X/n = 30/100$ *is* within 1/10 of $\frac{1}{4}$ but *not* within $\frac{1}{10}$ of $\frac{1}{2}$, supporting the argument that $p = \frac{1}{4}$. In order confidently to estimate p to within $\frac{1}{100}$, say, we would need $\frac{1}{\sqrt{n}} = \frac{1}{100}$, i.e. $n = 10\,000$.

We leave you with some further problems.

PROBLEM 3.4.1

Five sets of traffic lights are spaced along a road at 200 metre intervals. For each set, the red signal lasts 30 sec., the green 28 sec. and the amber 2 sec. The lights are synchronised in such a way that a car travelling at 36 k.p.h. and just catching the first light, just catches the other four. The width of the cross-street at each light is 20 metres. Find all the speeds at which it is possible to travel without being held up at any of the lights.

PROBLEM 3.4.2

Many hallways have light switches at either end, allowing the light to be operated from each. How can the wiring be arranged to achieve this?

PROBLEM 3.4.3 (Submitted by Andrew Mattingly, Science I, Monash University.)

Let Δ_n denote the number of ways of putting n letters into n addressed envelopes so that every letter goes into a wrong envelope. Derive a formula from which Δ_n may be calculated.

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PERILS OF THE DOUBLE NEGATIVE

President Carter's unpopularity has reached an all-time low.

ABC News, 3AR, 16.7.79.

1979 PRIZE FOR TAUTOLOGY

You get this feeling of *dêjà vu*, after the things have happened.

John Gribbin, author of *Timewarp*,
ABC Science Show, 9.6.79.

HOW THE MANAGER KEPT HIS WORD

(See *Function*, Vol.3, No.3, p.19.)

The manager brings Professor Fist a cake just after eleven o'clock and says:

"This is how I surprise you. I leave it so late you no longer have any reason to believe I'll give you the party, and so I can do what I promised."

Do you think his explanation is satisfactory?

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FAMOUS MEN DISCUSS MATHEMATICS

The advancement and perfection of mathematics are intimately connected with the prosperity of the State.

Napoleon (1769 - 1821)

There is no smallest among the small and no largest among the large; but always something still smaller and something still larger.

Anaxagoras (ca. 500 B.C. - 428 B.C.)

No more fiction for us: we calculate; but that we may calculate, we had to make fiction first.

Friedrich Wilhelm Nietzsche (1844 - 1900)

A quantity which is increased or decreased by an infinitesimally small quantity is neither increased nor decreased.

Johannes Bernoulli (1667 - 1748)

Those impostors, then, whom they call mathematicians, I consulted without scruple, because they seemed to use no sacrifice, nor pray to any spirit for their divinations.

St Augustine (354 - 430)

It is true that the mathematician who is not somewhat of a poet, will never be a perfect mathematician.

Karl Theodor Weierstrass (1815 - 1897)

From a nightmare of Friedrich Engels (1820 - 1895)

... last week in a dream I gave a fellow my shirt buttons to differentiate and the fellow ran away with them.

The closely held consolation of some rationalizing mathematicians that dy and dx are in fact only infinitely small ... is a chimera.

Karl Marx (1818 - 1883)

I take space to be absolute.

Isaac Newton (1642 - 1727)

I hold space to be something purely relative as time is.

Gottfried Wilhelm Leibniz (1646 - 1716)

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