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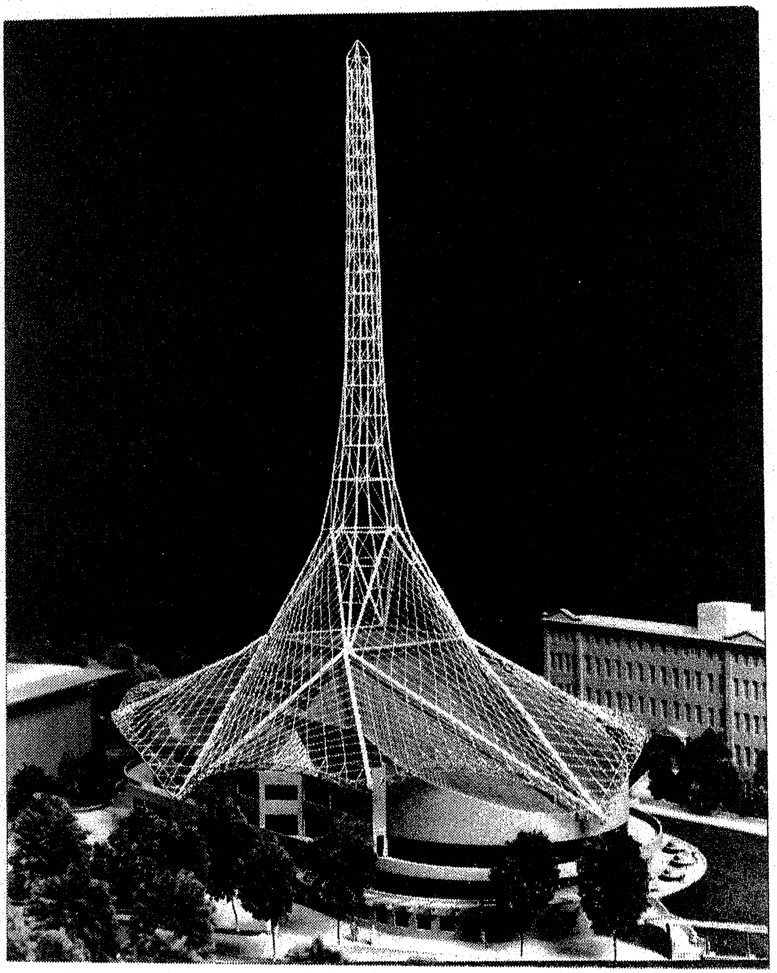

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Our first article this issue was written by Dr N. Barton of the University of New South Wales. He has just completed a major experiment for A.G. Thompson Pty. Ltd., the world's largest manufacturer of cricket balls. The results, going considerably further than those reported in this article, will eventually appear in the Journal of Fluid Mechanics. Meanwhile we hope his present article will be of considerable help to cricketers.

Sir Richard Eggleston is Chancellor of Monash University. He has had a distinguished career in the Law, being a judge of the Australian Industrial Court and of the Supreme Court of the A.C.T. He was also the first president of the Trade Practices Tribunal. He has written on the possibility of an innocent person being arrested when mistaken for a guilty person.

THE FRONT COVER

The illustration is of the spire to be erected at the Melbourne Arts Centre. It is reproduced here with permission, and was provided by Mr P. Greetham, Boronia Technical School. Mr Greetham has written an article concerning the way curves can be formed from straight lines, as in "curve stitching", and we plan to publish his article later in the year.

CONTENTS

THE SWERVE OF A CRICKET BALL*

N.G. Barton
University of New South Wales.

All cricketers and cricket followers know that a medium pace bowler can swing a new or well-preserved cricket ball in flight. In this article, I will explain the physical mechanism that allows cricket balls to swerve transversely in flight, and I will give some experimental details of the magnitude of the transverse forces on various cricket balls at varying speeds. Before proceeding, I must apologise for the minimal mathematical content in what follows, but a mathematical analysis of the airflow around a cricket ball is too complicated to be included. Therefore, I will leave you with a description of the physics of the situation, whilst assuring you that a mathematical analysis could be made (provided you had enough time and access to a large computer).

The swerve of cricket balls has been recognized for about a hundred years; indeed a certain Noah Mann was able to make the ball "curve the whole way" with his left-hand *under-arm* deliveries. Early cricketers were aware that new balls seemed "to favour the peculiar flight", although the mechanisms causing the effect most probably were not explained until the 1920's or even later.

Of course, swerve in flight occurs in many games apart from cricket - golf, tennis, table tennis, soccer and baseball are examples that spring immediately to mind. In each of these cases (and sometimes also in the case of cricket), the swerve is due to spin about a vertical axis imparted to the ball at release or projection. This totally separate phenomenon is called the Magnus effect and, I believe, was known to very early naval gunners who observed the swerve in flight of spherical cannon balls which picked up substantial spins when fired. (You may find an account of this in the book by Daish cited below.)

In cricket, however, swerve occurs even when the ball does not have a significant spin imparted to it. This is possible because a cricket ball is not spherically symmetrical, rather it has a prominent band of stitches (the seam of a 6-stitcher) which join the two hemispheres of the ball. A bowler is able to exploit the band of stitches to produce an asymmetry in the air flow past the ball; and it is this asymmetry that causes the ball to swerve in flight.

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Let us now examine what happens to the air near a rapidly moving cricket ball. The air is not just pushed aside by the ball only to rejoin it at the back in an otherwise undisturbed way. The reason for this is due to a subtle fluid mechanical property that was not analysed until the start of this century by Ludwig Prandtl in Germany. A real fluid such as air possesses viscosity (or inherent "stickiness") and, at any moving boundary such as the surface of a cricket ball, the air must have the same speed as the boundary. A little way from the ball however, the air would stream past the ball and would hardly be disturbed by the ball's progress. Clearly there must be a region, which turns out to be very thin, in which the air speed must adjust from nearly zero to the high speed of the ball. This thin region is known as a *boundary layer* and the viscosity or stickiness of the air is all-important in this region. Boundary layers are mathematically difficult to analyse, although now a whole branch of applied mathematics has been devoted to their understanding. Before we start to look at the boundary layers in the air flowing around a cricket ball, let us first record some of the properties of boundary layers around perfect spheres.

Consider the hypothetical situation illustrated in Figures 1(a) and (b) in which air is blowing around two identical smooth balls.

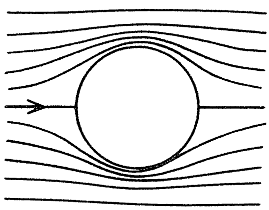

Figure 1(a)

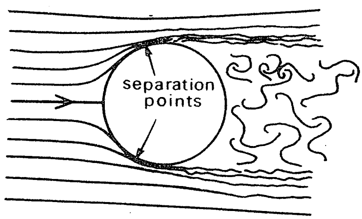

Figure 1(b)

In Figure 1(a), the air is assumed to be blowing very slowly, and the boundary layer which adjusts the air velocity from the free stream speed to zero at the surface of the sphere is thick and stays attached to the surface of the ball almost around to the rear. In Figure 1(b), the air is blowing quite fast and the boundary layer (which in fact is much thinner than illustrated on the figure) is observed to separate or blow off from the surface of the sphere at an angle of about 80° around from the front of the ball. (This angle can be predicted mathematically, but the mathematical techniques are far too complicated to be mentioned here.) Behind the ball in Figure 1(b) is observed a broad wake of irregularly moving (or *turbulent*) air. The point of separation of the boundary layer, moves quickly around from the rear of the ball to the

80° position as the air speed increases and, thereafter, the boundary layer continues to separate from the surface at the same position even at higher wind speeds.

A very strange thing now happens if the ball is slightly rough. As the air speed is increased, the boundary layer gets thinner and thinner until the roughness elements on the surface penetrate significantly into the boundary layer. When this occurs, the boundary layer is tripped into a turbulent state wherein the air moves irregularly in the layer in addition to the sweeping flow along the surface. The speed at which the boundary layer becomes turbulent is called the *critical* speed and, experimentally, the critical speed is high for smooth spheres and low for rougher spheres. The effect of transition to turbulence of boundary layers is very marked for it is found that turbulent boundary layers tend to stay attached to curved surfaces longer than the non-turbulent (that is *laminar*) boundary layers we had previously considered. Thus as the air speed past our hypothetical sphere is increased from zero, the boundary layer at first separates earlier and earlier from the surface until the 80° separation point is reached. Thereafter, the boundary layer separates from this point until the small surface roughness on the sphere is sufficient to trip the boundary layer into turbulence, and the separation point now moves around towards the back of the ball. The situation is sketched in Figure 2 in which the boundary layer is shown to be separating quite late from a rough sphere leaving a relatively thin turbulent wake behind the sphere.

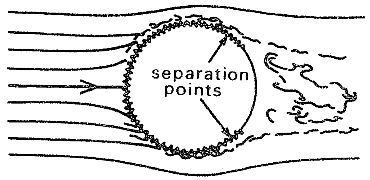

Figure 2

We are now in a position to consider the peculiarities of a cricket ball which enable it to swerve in flight. A cricket ball certainly is not a smooth sphere in view of the prominent band of stitches travelling around the ball. In addition, the surface of the cricket ball is slightly roughened by the internal stitches which hold the two pieces of leather comprising each hemisphere together, and by the trade marks and printing stamped on the surface of the ball. And, of course, the leather surface of the ball becomes scuffed up and roughened in play, although the surface can be smoothed out a little on one side if desired by vigorous polishing of the ball (generally on the trousers of the bowler).

Now the seam of a cricket ball sticks out about .5mm above the surface, whereas the laminar boundary layer on a smooth sphere bowled at medium pace is somewhat thinner, perhaps .2mm. Thus the seam is easily sufficient to trip the laminar boundary layer into turbulence, and the bowler merely has to ensure that the boundary layer on only one side of the ball becomes turbulent in order to produce a marked asymmetry in the flow. The bowler achieves this by slightly rotating the seam with respect to the air flowing past the ball as shown in Figure 3. It is then found that the boundary layer on one side of the ball is laminar and separates at the 80° position, whereas the boundary layer on the other side is turbulent and separates much later from the surface. The resulting air flow around the ball is clearly asymmetrical and it is found that there is a marked nett transverse pressure force acting on the ball.

A cricket ball will lose the transverse force acting upon it whenever the projection speed exceeds the critical speed for the smooth hemisphere of the ball. When this occurs, the boundary layers on both sides of the ball become turbulent, the separation points of both boundary layers become symmetrically placed, and the pressure forces balance on both sides of the ball. It is for this reason that an express bowler cannot swing a cricket ball in flight - he bowls above the critical speed for all but the newest balls. And, as the surface of the ball deteriorates during the course of play, the critical speed becomes lower and lower until eventually the effect is available only to those bowling at very gentle speeds. (This

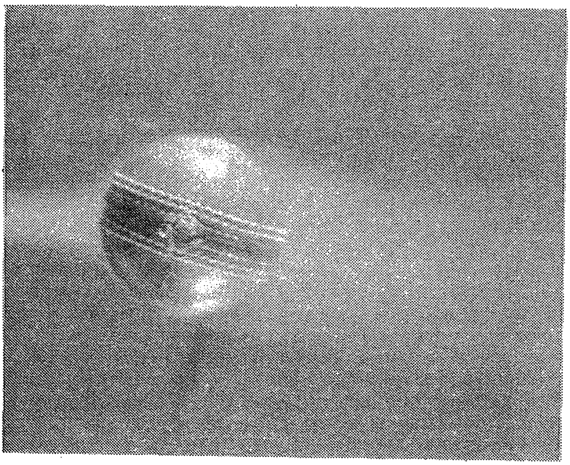

Figure 3

may help to explain why Doug Walters, who bowled at a very friendly pace, was able to pick up wickets with very old balls even when bowlers with bigger reputations could not.)

You may well ask how the bowler maintains the more or less constant orientation of the seam with respect to the airflow. This is achieved by imparting a small back spin along the line of the seam as the ball is released. Good swing bowlers regard this backspin as very important in stabilising the flight of the ball and the bowler's aim should be to bowl a ball whose seam does not wobble in flight.

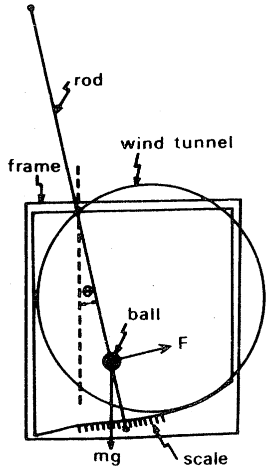

Figure 4

Now let us examine how large the transverse pressure force on a cricket ball can be, and what factors affect the magnitude of this force. Earlier this year, I designed a simple wind tunnel experiment to measure the transverse force on various cricket balls. Each cricket ball was skewered on a long thin metal rod which was pivoted on a frame clamped in front of the wind tunnel and free to swing transversely (see Figure 4). The experimental set up is designed so that the deflecting aerodynamic force is given by $F = mg \sin \theta$, where θ is the angle of deflection from the vertical.

Three balls were used in the experiments; they were a new ball and two balls about 10 and 40 (8 ball) overs old. In Figure 5, I have displayed the transverse force on the three balls as a function of wind speed when the seams were at approximately 30° to the air flow. The results were quite reproducible and they appeared to be independent of the atmospheric conditions. In accord with the experience of cricketers, the transverse force dropped to zero at air speeds greater than 30 m/sec. (respectively 28 m/sec., 26 m/sec.) for the new (respectively 10 over, 40 over) balls. Moreover, the transverse force at any given moderate air speed was the greatest and steadiest for the new ball, and the least and most variable for the oldest ball. The effects of varying the angle of the seams were then considered. Figure 6 shows the mean transverse force on the new ball for seam angles of 15° and 30° to the air flow, an also the mean transverse force on the 10 over ball at seam angles of 0°, 15° and 30°. The most surprising effect was found with the 10 over ball; in this case, the greatest transverse force was obtained with the seam at zero incidence. Clearly, the surface roughness of one side of the ball was sufficient by itself to trip the adjacent boundary layer into turbulence, although I am at a loss to explain why the transverse force should be greatest for this case.

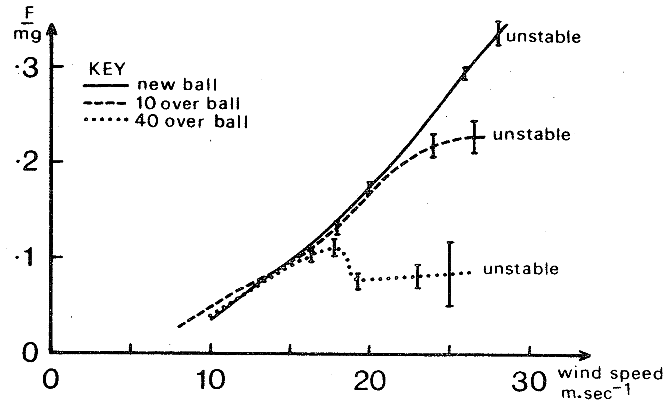

Figure 5

Transverse force as a function of speed. Bars indicate fluctuations. The transverse force became intermittent at the points marked "unstable" and dropped quickly to zero for higher speeds.

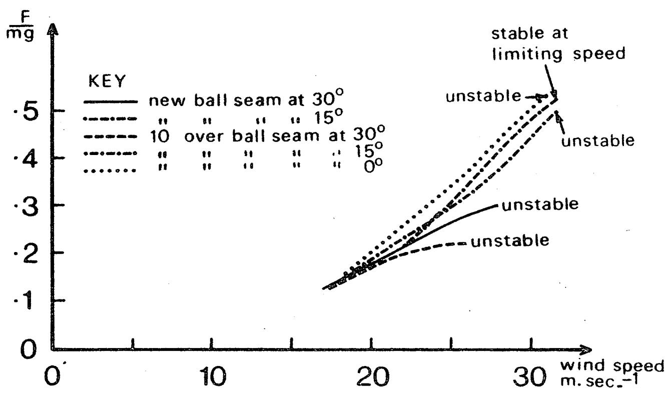

Figure 6

Transverse force as a function of speed for various seam angles.

How much can a cricket ball swerve in flight? The largest transverse force I measured was 53% of the weight of the ball for the new ball with the seam at 15° and at 31.5 m/sec. At this speed, the ball would be in flight for about .5 sec. and, if this force were to be uniform throughout the flight, the ball would be deflected about 65 cm from its original line. Greater deflections may be possible, but I could not detect them without using a better wind tunnel. Deflections of this size could also be obtained for the 10 over ball with the seam at zero incidence. For the 40 over ball, the greatest transverse force was 15% of the weight of the ball at 18 m/sec., leading to a deflection of about 47 cm from the original line of flight in .8 sec.

To conclude this essay, I must stress that the results appeared to be independent of atmospheric conditions. To the accuracy of the apparatus, the deflecting force did not vary significantly over a number of days in which the relative humidity varied from about 50% to greater than 76%. Now this result appears to contradict the commonly held belief that cricket balls swing more on humid days, a belief for which a number of possible explanations have been advanced. The explanations range from the false supposition that humid air is heavier than dry air, to the more plausible ones such as humidity causing the seam to swell thereby ensuring greater turbulence in one boundary layer, and humidity causing condensation which makes the "smooth" side of the ball even smoother. My belief, based on my experiments, is that neither of the last two explanations is satisfactory, although to be certain I need further experimental results over a wider range of humidities. The one possible hypothesis I can offer to explain the enhanced swing on humid days is that the surface of cricket balls tends to get scuffed up less on humid days. A glance at the graphs for the 10-over ball which had one side "rough" and the other "smooth" shows that surface roughness alone can cause very large deflecting forces even with the seam at zero incidence. Thus the surface condition of the ball is very important, and if cricket balls retained their shine longer on humid days, this could go a long way towards explaining the popular belief.

Further Reading

C.B. Daish, "The physics of ball games", English Universities Press, (1972).

R.A. Lyttleton, "The swing of a cricket ball", Discovery 18 (1957), 186-191.

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AS A MATTER OF INTEREST

Neil Cameron,
Monash University

Many borrow money for such purposes as the purchase of a house, car or furniture from banks, credit unions, finance companies, etc.. Different interest rates are quoted and different *kinds* of interest rates are used. Here we investigate two kinds, *nominal* and *flat* rates.

Consider a loan of $\$A$ to be paid in equal instalments of $\$P$ at p equally spaced periods per year. If the *nominal* rate is given as $100R\%$ p.a., what is meant is that the interest is calculated on outstanding balance at the end of a period at the compound interest rate of $r = R/p$ per period. For example if the nominal rate is 18% p.a. and instalments are monthly ($p = 12$) then $r = 0.015$; if instalments are quarterly ($p = 4$) then $r = 0.045$.

After one period, interest accumulated on the loan is $\$Ar$ so that after the first instalment is paid the amount $\$A_1$ of loan outstanding is given by

$$A_1 = A + Ar - P = A(1 + r) - P.$$

Similarly the amount $\$A_2$ outstanding after the second instalment is given by

$$\begin{aligned} A_2 &= A_1(1 + r) - P \\ &= A(1 + r)^2 - P(1 + r) - P \\ &= A(1 + r)^2 - P[(1 + r) + 1], \end{aligned}$$

and, in general, if $\$A_m$ is the amount of loan outstanding after the mth instalment then

$$A_m = A(1 + r)^m - P[(1 + r)^{m-1} + \ldots + (1 + r) + 1].$$

But $r\,[(1 + r)^{m-1} + \ldots + (1 + r) + 1]$

$$\begin{aligned} &= (1 + r)\,[(1 + r)^{m-1} + \ldots + (1 + r) + 1] - [(1 + r)^{m-1} + \ldots + (1 + r) + 1] \\ &= [(1 + r)^m + \ldots + (1 + r)^2 + (1 + r)] - [(1 + r)^{m-1} + \ldots + (1 + r) + 1] \\ &= (1 + r)^m - 1, \end{aligned}$$

so $A_m = A(1 + r)^m - P\dfrac{[(1+r)^m - 1]}{r}$.

We note then that the loan is paid off after n periods where $A_n = 0$, that is,

$$\frac{P}{A} = \frac{r(1 + r)^n}{(1 + r)^n - 1}$$

The total interest paid is $\$ I$ where

$$I = nP - A = \left[\frac{nP}{A} - 1\right] A$$

$$= \left[\frac{nr(1+r)^n}{(1+r)^n - 1} - 1\right] A.$$

Expressed in terms of R, p and $N = n/p$, the number of *years* taken to repay the loan, this is

$$I = \left(NR \frac{(1 + \frac{R}{p})^{Np}}{(1 + \frac{R}{p})^{Np} - 1} - 1 \right) A.$$

The *flat* interest rate is then $100F\%$ p.a. where

$$F = \frac{I}{NA} = \frac{R(1 + \frac{R}{p})^{Np}}{(1 + \frac{R}{p})^{Np} - 1} - \frac{1}{N},$$

which depends on both N and p as well as R.

To demonstrate the effective difference between nominal and flat interest rate consider the following example. Suppose you wish to borrow $1000 towards the purchase of a used car and to repay the loan in monthly instalments over a year. Thus $A = 1000$, $p = 12$ and $N = 1$. You are told that the interest rate is 16.2% p.a.. If this is nominal then instalments will be $90.83 and the flat interest rate is only about 9.0%; however, if 16.2% p.a. is the *flat* rate, instalments will be $96.83 and you will pay about 80% more interest than in the former situation!

A small difference in nominal interest rate can also have a greater effect than might be imagined. For instance if, in the above example, the nominal interest rate is 18% rather than 16.2%, instalments will be $91.68, the flat interest rate will be about 10.0% and you will pay about 11% more interest.

Lending authorities such as credit unions fix the nominal interest rate yet we can see that flat interest rate depends on the whole pattern of payment (N and p) as well as R. Let us investigate this further.

For fixed R(as well as A) it is intuitively clear that $\$ I$, total interest paid, increases the longer you take to pay off the loan (i.e. with increasing N) and the less payments made annually (i.e. with decreasing p). Flat interest *rate* is another matter. This does increase with decreasing p but is least for a middle range of N, the number of years taken to repay the loan. For $R = 0.18$, table I demonstrates this phenomenon.

		N			
	F	1	2	3	10
	4	0.115	0.106	0.105	0.117
p	12	0.100	0.099	0.101	0.116
	52	0.094	0.097	0.099	0.116

Table I (R = 0.18)

In the commonly used range for R of 0.15 to 0.21, F/R remains virtually constant for given N; in particular the minimum is achieved for payments completed after about 10, 20 and 40 instalments when paying quarterly, monthly and weekly respectively. The graph of F/R versus N is very flat near the minimum as Figure 1 shows (for the case of monthly payments).

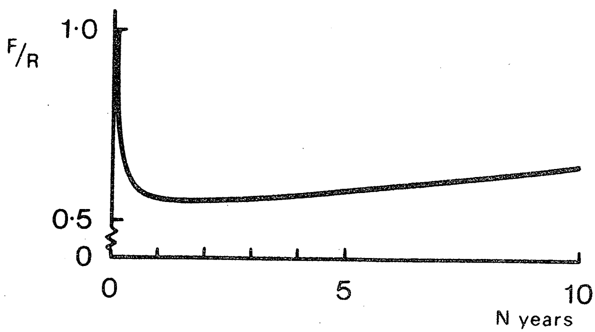

Figure 1 (R = 0.18, p = 12)

Table II gives the minimum value of F/R for various values of p.

p	1	4	12	52	365
min. F/R	0.66	0.59	0.55	0.52	0.51

Table II (R = 0.18)

We see that F approaches a minimum value of 0.5 for increasing p. On the other hand F is as large as R for $n = Np = 1$ and increases (slowly) towards R as N increases.

How should we interpret this behaviour of F as a proportion of R? Given the freedom to do so, should one try to arrange payments of the loan to achieve minimum flat interest rate?

Using calculus, show that if R and p are fixed, the minimum F is achieved for n satisfying the condition

$$\left[\frac{(1+\frac{R}{p})^n - 1}{n}\right]^2 = R\log_e(1+\frac{R}{p})(1+\frac{R}{p})^n .$$

If $R = 0.18$ and $p = 12$, use a calculator to check that the value $n = 20$ is a good approximation to the 'best' n, $n = 24$ is not bad but $n = 60$ is terrible.

REFLECTIONS ON GRAVITY

Kim Dean
Urban - Campagna, U.S.A.

Last year's prestigious *Prix le Bon* went to the little known Welsh scientist, Dai Fwls ap Rhyl, for work on the relation between gravity and symmetry violation. In its basis, the theory is apparently trivial, as Dr Fwls has had the courage to query basic assumptions. He begins with the well-known fact that mirrors induce lateral inversion(i.e. interchange left and right), and also dorso-ventral inversion(front and back interchange, as the image faces the viewer).

The third inversion does not occur - the viewer is not shown upside down. This, in Dr Fwls' theory, is due to the action of gravity in the vertical direction. Whenever symmetry breakdown occurs (as in the fact that time runs forward), a gravitational or gravity-like force must act, he maintains. Furthermore, as the laws of Physics reflect, in part, our own thought processes, these are subject also to similar forces, whose discovery promises to revolutionise Psychology.

All in all, this new theory promises to stand existing scientific notions on their heads.

1/4/80

BAYES AND THE ISLAND PROBLEM

Sir Richard Eggleston
Monash University

In my book, Evidence, Proof and Probability, I constructed a problem about an island on which a murder had been committed. There were a hundred people on the island, and no one had entered or left since the murder. It was taken as certain that the murderer had a motive, and that he had access to a weapon of a particular kind. He must also have had the opportunity to commit the murder. Given that *A* has all three characteristics, and that the probability of an innocent person having all three (motive, opportunity, and access to the right sort of weapon) is one in 250 (.004), what is the probability that *A* is the murderer?

As posed in the book, the problem was stated on the assumption that *A* was discovered as a result of a random search by the police, and that the search stopped when *A* was found to have the characteristics.

There are several ways of approaching this problem. One is to calculate the probability of there being no innocent person on the island who has all three characteristics (in which case *A* is certainly the murderer); the probability of there being one such innocent person (in which case the probability that the first person found to have the characteristics would be the murderer would be .5); then the probability of there being two such persons (probability of first person found being the murderer is .33), and so on.* But there are two methods of calculating the number of innocent persons. If we treat the innocent persons as a population distinct from the murderer, we simply calculate the probability of finding 0, 1, 2, ... persons in that population having a combination of traits which occurs with frequency .004.

---

* The probability that x persons out of 100 have all three characteristics is the binomial probability

$$\binom{100}{x}(0.004)^x(0.996)^{100-x}$$

It is assumed that the characteristics are such that they occur independently in different people; so that the argument that follows would not apply, for example, to such characteristics as blood groups if the members of the population were closely related to each other.

Joel Yellin, who reviewed the book for an American journal, derived a relatively simple formula for the sum of all the probabilities that the person found is the murderer.

$$S = \frac{1-(1-P)^N}{NP}$$

For the figures given above (N = 100; P = .004), S = .83.

On the other hand, some writers consider that in calculating the probability of there being any given number of innocent persons having the required characteristics, we should take into account the fact that we know that there is one person in the population who has those characteristics, namely, the murderer. According to this view, we discard the case where no person has the required characteristics (hereinafter denoted by "C"), and confine ourselves to the remaining probabilities, viz.: that there is one person having C (who will certainly be the murderer); that there are two persons having C (when the probability will be 0.5 that the first person found will be the murderer); that there are three persons ... and so on. Since this series will cover all the possible cases (the case of no person having C being excluded) it is necessary to rescale the remaining probabilities so that they add to unity. Since, for the figures given in the example, the probability that no person will have C is approximately 0.670, all the other probabilities must be multiplied by $\frac{1}{1 - .670}$ (= 3.03) to achieve this result.

This approach gives quite a different result, as the following table shows. In the table, Column 1 indicates the number of innocent suspects; column 2 the probability of there being that number of innocent persons having C, calculated according to the first method;* column 3 is the factor by which we must multiply column 2 to find the probability that A is guilty in that case; column 4 gives the result of the multiplication; column 5 gives the factor (3.03) by which we must multiply column 2 to give the probability of there being 1, 2, 3, or more persons having the characteristics, according to the alternative approach described in the preceding paragtaph. Columns 7 and 8 then provide similar calculations for the second method. It will be seen that with figures of the order involved, we can ignore for practical purposes the calculation for more than four innocent suspects. The difference between the two results is quite marked, Method II

* Strictly speaking, the calculation for column 2 should be based on 99 persons rather than 100, but the difference in the result is negligible for N of this magnitude.

giving a total probability of guilt of .904 as against a total probability of .825 if we follow Method *I*.

TABLE 1

1 Number of Suspects	2 Pr.	3 Factor	4 Col.2 × Col.3	5 Factor	6 Col.2 × Col.5	7 Factor	8 Col.6 × Col.7
0	.670	1	.670				
1	.269	0.5	.135	3.03	.815	1	.815
2	.053	0.33	.018	3.03	.161	0.5	.081
3	.007	0.25	.002	3.03	.021	0.33	.007
4	.0007	0.20	.00014	3.03	.0021	0.25	.0005
Totals:	.9997		.825		.999		.904

In discussing the island problem with Professor Lindley (formerly Head of the Department of Statistics and Computer Science at University College London), he suggested yet another approach, using Bayes' Theorem. This can be stated as follows

Let H = The general data (before discovering E).

E = A has the required characteristics

G = A is guilty.

By Bayes' Theorem

$$\frac{Pr(G|H \;\&\; E)}{Pr(Not\text{-}G|H \;\&\; E)} = \frac{Pr(E|G \;\&\; H)}{Pr(E|Not - G \;\&\; H)} \times \frac{Pr(G|H)}{Pr(Not - G|H)} \dagger$$

† Bayes theorem is often expressed in a form which omits reference to the general data. The above formula should then become

$$\frac{Pr(G|E)}{Pr(Not - G|E)} = \frac{Pr(E|G)}{Pr(E|Not - G)} \times \frac{Pr(G)}{Pr(Not - G)}$$

I do not like doing this, partly because all evaluations of probability must be based on *some* data, so that "$Pr(G)$" is really meaningless; but more importantly, because when we are trying to analyse an actual case, the necessity to specify "H", the general data, in each expression is a safeguard against the possibility of an unnoticed shifting of ground. However, when "H" appears after the conditionality sign "|" in *every* expression it can be omitted, as long as care is taken not to change the general data on which the reasoning is based.

For the island problem we take $Pr(G|H)$ as .01 (on the basis that, without further facts, any one of the hundred inhabitants is equally likely to be guilty); $Pr(E|G \& H)$ is 1 (since the murderer must have the required characteristics) and $Pr(E|Not\text{-}G \& H)$ is .004. The right hand side then becomes $\frac{1}{.004} \times \frac{.01}{.99} = \frac{2.525}{1}$, and $Pr(G|H \& E) \simeq .72$. Lindley derives the formula $Pr(G|E \& H) = [NP + (1-P)]^{-1}$ for this solution.

It will be seen that this approach makes no assumption as to the method of search, nor does it involve any problem of the calculation of the distribution of innocent suspects. In fact, when I first propounded the problem the mathematicians whom I consulted expressed some doubt as to whether it was proper to assume that the distribution of suspects would be binomial. Dr Watterson, when he posed a similar problem in terms of Cinderella and the traditional "glass" slipper (actually a mistranslation of the French "glacé") in the October 1977 issue of "Function", was one who shared those doubts. I therefore sought a solution which would give a minimum figure for the probability of guilt, assuming the distribution most favourable to the accused. To achieve this, I set up a model assuming that the situation was repeated over a set of ten islands, and ascertained by inspection that the most favourable possible distribution for each accused would arise when each innocent suspect was found on a separate island. Thus for ten islands of 100 people each we would expect to find four innocent suspects. If each were on a separate island, there would be four islands in which the probability of finding the murderer first would be .5, and six in which it would be 1. No distribution which assumed that any island had more than one innocent suspect could give a lower overall probability of guilt than that. Hence I concluded that the minimum probability of guilt for the figures chosen was $\frac{(6 \times 1) + (4 \times .5)}{10} = .8$, and for any greater number of islands (i.e. as the number of islands tended to infinity) this figure would remain constant.

The question then arose, why should Professor Lindley's solution give a result lower than my minimum? In an analysis of all the possibilities in the case where $N = 2$, Professor Lindley himself supplied the clue to the answer. In the following table we assume a total population of only two persons, whom we call Suspect A and Suspect B. M denotes that the person so labelled is the murderer, C denotes that he possesses the characteristics which we know the murderer has, and which occur randomly in the population with probability P.

TABLE 2

	Suspect A	Suspect B	Probability	Pr M found	Pr $\bar{M}$ found
1	CM	C	P	$0.5P$	$0.5P$
2	C	CM	P	$0.5P$	$0.5P$
3	CM	$\bar{C}$	$(1 - P)$	$(1 - P)$	0
4	$\bar{C}$	CM	$(1 - P)$	$(1 - P)$	0
5	$\bar{C}M$	C			
6	C	$\bar{C}M$			
7	$\bar{C}M$	$\bar{C}$			
8	$\bar{C}$	$\bar{C}M$			

We can ignore cases 5, 6, 7 and 8, as these do not fulfil the essential condition, that the murderer has C. Which of the first four cases we consider depends on how we frame the problem. For the application of Bayes theorem, we made no assumptions about the method of search, but simply took the fact that A has C, and assessed the probability of A's guilt. We are therefore confined to cases 1, 2 and 3, since in case 4, A does not have C. For cases 1, 2 and 3, the probability of the cases in which A is guilty (cases 1 and 3) is represented by $P + (1 - P)$, and the probability of case 2, in which he is not guilty, by P. The probability of A's guilt is therefore:

$$\frac{P + (1 - P)}{P + (1 - P) + P} = \frac{1}{1 + P}$$

This agrees with the result given by Bayes' Theorem, since, if $N = 2$, Lindley's formula gives $[2P + (1 - P)]^{-1} = (1 + P)^{-1}$

On the other hand, if we postulate a random search in which the police are looking for anyone they can find who has C, we must take account of case 4 also, since in that case the search will discover suspect B, and he will certainly be the murderer, since suspect A does not have C. The last two columns of the table give respectively the probability that the murderer is found first, and the probability that an innocent suspect is found first in a random search. The former column totals $(2 - P)$, and the final column totals P. The probability of finding the murderer first is therefore:

$$\frac{(2 - P)}{(2 - P) + P} = 1 - \frac{P}{2}$$

which accords with Yellin's formula, which for $N = 2$ gives:

$$\frac{1 - (1 - P)^2}{2P} = 1 - \frac{P}{2} .$$

It will be seen that in the above calculation we treated C as occurring by chance amongst innocent suspects, but not in the murderer. The kind of characteristics that we were considering made this a reasonable supposition. But if we had been dealing with characteristics such that C occurred by chance in the murderer also, the result would have been the same, since the probability column for cases 1 and 2 would read "P^2" and for cases 3 and 4 "$P(1 - P)$" and P would be a common factor in both numerator and denominator.

We are still left, however, with the question whether Method *I* or Method *II* is appropriate for the calculation of the probabilities in the case of a random search. Where C occurs by chance amongst innocent suspects, but not in the case of the murderer, Method *I* seems to me to be clearly preferable. But what of the case where C occurs by chance in the case of the murderer also? Suppose, for example, that the murderer has left traces at the scene of the crime from which it can be determined that he belongs to a particular blood group, and the population of the island is such that we can ascertain the frequency with which that blood grouping can be expected to occur in that population. Method *II*, which has powerful supporters (Watterson, Cullison, Kingston), has an obvious appeal to a mathematician. If we know that there is one person who has C we would expect to find the probability that there is only one, or two, or more, by excluding $x = 0$, as we have done in column 6 of Table 2. But the answer is not so simple. Let us pose two problems:

(1) *A* tosses two coins. He asks *B* to look at them. *B* does so and reports that he has looked at both coins, and that there is one head, but he is not saying which one it is, or whether the other coin is a head or a tail. How should *A* evaluate the probability that there are two heads?

(2) *A* tosses two coins. He inspects one and finds that it is a head. Before he has looked at the other, how should he evaluate the probability that there are two heads?

If we apply Method *II* to the first question, we should analyse it as follows:

The possible outcomes are "Two heads", "One head and one tail", and "Two tails". These have probabilities of 1/4, 1/2, and 1/4 respectively. Omitting the case of two tails (no heads) and rescaling the other cases to make them add to unity, we get probabilities of 1/3 and 2/3 for "Two heads" and "One head and one tail", respectively. Since *B*'s answer has eliminated "Two tails", the probability of "Two heads" has become one in three.

The validity of this reasoning, however, depends on the fact that on the given data, *A* cannot eliminate the possibility of either coin being a tail. If it is established that one of the coins cannot be a tail, then the only possible outcomes are *HH* and *HT*, rather than *HH*, *HT* and *TH*. In our example of the random search, the question of probability does not arise until we have identified a specific person as having *C*, and the question is, what is the probability of there being no other person having *C*, or one or more, other persons having *C*, on the island?

The true analogy is therefore example (2) rather than example (1). Most people (but not all - see the note at the end of this article) would agree that the answer to the second question is that the probability of two heads is 1/2, which is the probability that the coin not seen is a head. The difference between the two cases can be seen if we imagine ourselves being asked to bet on the probability of two heads before two coins are tossed. If the bet is expressed in the form "I will back two heads, provided that if there are two tails the bet is off" the person making such a bet would need to get odds of two to one in his favour, since the analysis of question (1) would be applicable. But if he made the bet in the form "I will back two heads, provided that if the first coin looked at is a tail, the bet is off" then he could bet at even money, since the effect would be to eliminate all *TH* and *TT* cases, leaving only *HH* and *HT*.

The result seems to be that we should prefer the approach which I have termed Method *I* to the alternative Method *II*.

It seems to follow by parity of reasoning that if, in the island case, we had examined (say) 49 of the inhabitants and found that they did not have *C*, and then found one having *C*, we could calculate the probability of guilt taking account only of the remaining unknown 50 and the one *C* already found. Professor Lindley has expressed this in the following terms "Generally, with n islanders and k interrogated and found [not to have *C*] before finding Smith with [*C*], the probability of guilt is $(1 + (n - k - 1)p)^{-1}$," (this, of course, is the Lindley formula given above, modified to take account of the elimination of some members of the population; it is to be further noted that it assumes that it is possible to determine that a given member of the population does *not* have *C*, which may not always be the case).

What of my calculation of the minimum value of the probability of guilt? In the case under consideration the result (.8) lies between the Method *I* result and the Bayes result. It is obviously inapplicable in cases where $NP > 1$, but how does it fare within the range where $NP < 1$?

Actually, since we are excluding the murderer from the calculation of the probability that other (innocent) persons having *C* exist, we should use the formula $1 - \frac{(N - 1)P}{2}$ rather than $1 - \frac{NP}{2}$. Yellin's formula, expanded would read:

$$\frac{1 - (1 - NP + \frac{N(N-1)P^2}{2} - \dots + \dots \pm P^N)}{NP}$$

If we ignore all the terms represented by "$- \dots + \dots \pm P^N$", the expression simplifies to $1 - \frac{(N-1)P}{2}$, and since, if $NP < 1$, these subsequent expressions are negative in total, this expression is a minimum for the Yellin formula.

Finally, in considering the possible uses of these methods, one must bear in mind that the Yellin formula, and my approximation, are dependent on there being a random search. In practice, a police search will hardly ever be random. Police place great reliance on "modus operandi" files, in which the characteristic methods of known criminals are recorded, and any search based on these files would naturally be biased against known criminals. In a murder case which I tried in Canberra, the police were about to arrest the accused because of information received from the "modus operandi" files in Sydney, when the accused virtually gave himself away by his conduct, was arrested for another offence and ultimately charged with the murder. There are great problems involved in the attempt to assess guilt by the use of probability theory (not least in explaining to the jury - and for that matter to the judge - the calculations involved). So that if we felt disposed to abandon the random search approach, we would find it necessary to explain Bayes theorem to the tribunal, and the possibilities of error and confusion are manifold. The main uses of probability theory in such cases, in my view, will be to enable the exposure of erroneous reasoning, not to attempt to provide the tribunal with a mathematical calculation of the probability of guilt.

NOTES ON THE TWO COIN PROBLEMS

There are several versions of the two coin problems that I have posed above. The fullest analysis, in the legal literature, of the probability of finding a guilty person having *C* in a population in which *C* occurs with frequency *P*, is that of Cullison (Houston Law Review, Vol. 6:471 at pp. 486-88). He espouses Method *II* rather than Method *I*. He imagines a case in which a card is drawn from a pack made up by drawing 156 cards from an infinitely large heap of cards in which the distribution of any particular card (e.g. the two of spades) is the same as in a normal pack. Thus the pack of 156 cards may contain one or more two's of spades, or none at all. *A* draws a card at random from the pack of 156, looks at it, tells *B* it was a two of spades, and returns it to the pack. *B* then turns the cards over until he comes to a two of spades. What is the probability that it is the one that *A* saw? He then explains that we are seeking to calculate a conditional probability - the probability that *B*'s two of spades is identical to the one *A* saw ($b \equiv a$) on the condition that the pack contains at least one two of spades (denoted by "$\geqslant 1$"). Cullison then shows that, applying the "multiplication rule":

$$P(b \equiv a| \geqslant 1) = \frac{P(b \equiv a)}{P(\geqslant 1)} .$$

Then, in order to illustrate the difference between the unconditional probability $P(b \equiv a)$ and the conditional probability $P(b \equiv a| \geqslant 1)$, he supposes that the pack contains only two cards (instead of 156) and that all that *A* recalls about the card that he saw is that it was black. This pack, he says, can be composed in four possible ways. each having probability 1/4: *BB*, *BR*, *RB*, and *RR*. Since *RR* is excluded by the fact that *A* saw a black card, Cullison says that the conditional probability $P(b \equiv a| \geqslant 1)$ would be 5/6, since in the case *BB* there is a 50-50 chance of *B* turning over *A*'s card, and in the cases *BR* and *RB* , if *B* turns over the cards until he comes to a black card, it will be certain to be the card *A* saw. The total probability is therefore

$(\frac{1}{3} \times \frac{1}{2}) + (\frac{2}{3} \times 1) = \frac{5}{6}$. But the analysis in the text shows that this is incorrect. Given that *A* saw only one card (which is an essential condition of the problem) the only possible cases are "*A*'s card black, the other red" and "Both cards black". Both have a probability of 1/2, and the probability of *B* turning over the card that *A* saw is

$(\frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times 1) = \frac{3}{4}$.

A somewhat similar problem involving cards is posed by Keynes (Treatise p.54) who derived it from Von Kries. "Two cards, chosen from different packs, are placed face downwards on the table; one is taken up and found to be of a black suit: what is the chance that the other is black also?" According to Keynes, while one would naturally reply that the chance is even, this solution is relatively unpopular with writers on the subject. He says the alternative, or text-book theory assumes that there are three equal possibilities, one of each colour, both black, both red. If both are black, we are twice as likely to turn up a black card than if only one is black. Accordingly, when we have turned up one black card, the probability that the other is black is twice as great as the probability that it is red. Hence the probability of both being black is $\frac{2}{3}$. Keynes says this is Poisson's solution.

If we take the statement "chosen from different packs" as implying that the packs are normal packs of 52 cards, 26 red and 26 black, the two-thirds solution is obviously absurd. Since the two packs are independent, the probability of a black card from the second pack is unaffected by the outcome of the draw from the first pack, and for a normal pack will be 1/2. What Poisson actually said (Recherches, p.96, §(35)) was quite different. In his example, we *find* two cards face down on the table, and on turning one over, observe that it is red. He says that we can only make two hypotheses about the colours of the cards: that they are both red, or that one is red and one is black. He then says, if we ignore absolutely the origin of the cards, these two hypotheses are equal *a priori*, and after the observation the probability of both being red is $\frac{2}{3}$, as shown in an earlier paragraph in his book (No. 32). Reference back to paragraph 32 reveals that Poisson was there speaking of a case in which there is an urn, containing a known number of balls, either black or white, but in unknown proportions, so that they could be all black, or all white, or in any intermediate proportions. Poisson proceeds on the assumption that all possible combinations are equally likely, and shows that if there are only two balls, and a white ball is drawn from the urn, the probability that the remaining ball is also white is in fact $\frac{2}{3}$. But this result depends on the assumption that all possible combinations are equally likely, so that in the case of a draw of two cards, both red is as likely as one red and one black. This no doubt accounts for the careful framing of the example - one finds two cards on a table, and one ignores absolutely from whence the cards originate. That Poisson would not have agreed with the solution of the problem posed by Keynes is shown by his further discussion of the question, where he goes on to say that the case would be different if the two cards had been chosen from the same pack (in his case a *piquet* pack, containing 16 red and 16 black). In that case the probability of both black is $\frac{16.15}{32.31}$ and of one black and one red $\frac{16.16}{32.31} \times 2$, since there are two ways of getting the latter combination. Hence in this case, if the first card turned over is red, the probability of the second being red also is $\frac{15}{31}$. It is obvious that Poisson would have answered the Keynes problem by saying that the chance was even, instead of $\frac{2}{3}$ in favour of both black. In fact he says that his result of $\frac{15}{31}$ verifies itself, for it is evident that this is the same as if, after having drawn a red card from the entire pack, we asked what is the probability of drawing another red card from the remainder of the pack containing 15 red and 16 black. The answer in each case is $\frac{15}{31}$.

There is a similar problem posed by Bertrand (p.2) (see Károly Jordan, p.208) in which a random choice is made between three chests, each containing two drawers.

One chest has a gold coin in each drawer, one a silver coin in each, and the third, a gold coin in one drawer and a silver coin in the other. If I open a drawer at random in a chest chosen at random, and find a gold coin, what is the probability that the other coin is also gold? Here the probability is $\frac{2}{3}$. If I chose the *GG* chest, I had two chances of getting *G* in the first drawer opened, whereas if I chose the *GS* chest, I had only one. This is confirmed by Bayes' Theorem, as follows (reference to general data omitted - see footnote † on p.15 above):

Let G = a gold coin is in the first drawer opened.

$2G$ = the chest containing 2 gold coins is selected,

$\overline{2G}$ = the complement of $2G$.

Then $$Pr(2G|G) = \frac{Pr(G|2G)\,Pr(2G)}{Pr(G|2G)\,Pr(2G) + Pr(G|\overline{2G})\,Pr(\overline{2G})}$$

$Pr(2G) = \frac{1}{3}$

$Pr(G|2G) = 1$

$Pr(G|\overline{2G}) = \frac{1}{4}$

$Pr(\overline{2G}) = \frac{2}{3}$

Therefore $$Pr(2G|G) = \frac{1 \times \frac{1}{3}}{(1 \times \frac{1}{3}) + (\frac{1}{4} \times \frac{2}{3})}$$

$$= \frac{2}{3}.$$

Similar reasoning can be applied to solve the two coin problems (reference to general data omitted as before):-

Let $1H$ = the event that B looks at one coin only and finds that it is a head,

$2H$ = the event that two heads are tossed,

$\overline{2H}$ = the complement of $2H$.

By Bayes' Theorem -

$$Pr(2H|1H) \quad \frac{Pr(1H|2H)\,Pr(2H)}{Pr(1H|2H)\,Pr(2H) + Pr(1H|\overline{2H})\,Pr(\overline{2H})}$$

$Pr(1H|2H) = 1$

$Pr(2H) = \frac{1}{4}$

$Pr(1H|\overline{2H}) = \frac{1}{3}$

(i.e. the probability that B will see a head if he looks at only one coin. The possible outcomes, excluding $2H$, are HT, TH and TT, so B has 2 chances out of 6 of seeing a head).

$Pr(\overline{2H}) = \frac{3}{4}$.

Therefore $$Pr(2H|1H) = \frac{1 \times \frac{1}{4}}{(1 \times \frac{1}{4}) + (\frac{1}{3} \times \frac{3}{4})} = \frac{1}{2}.$$

If, on the other hand, we let $1H$ = the event that E looks at both coins and sees at least one head, then

$$Pr(1H|\overline{2H}) = \frac{2}{3},$$

since the only one of the three cases in which he will *not* see at least one head is *TT*.

The calculation then becomes

$$Pr(2H|1H) = \frac{1 \times \frac{1}{4}}{(1 \times \frac{1}{4}) + (\frac{2}{3} \times \frac{3}{4})} = \frac{1}{3}.$$

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SOME POWER - FULL SUMS

It would take quite a bit of accurate calculation to check some of the curious equalities below. Can anyone write us an article describing how to program a computer to check those involving very many digits? Can anyone discover a similar equality using the power 6?

$5^2 = 4^2 + 3^2$ (Pythagoras)

$6^3 = 5^3 + 4^3 + 3^3$ (Euler)

$353^4 = 315^4 + 272^4 + 120^4 + 30^4$ (Dickson)

$144^5 = 133^5 + 110^5 + 84^5 + 27^5$ (Wu)

$102^7 = 12^7 + 83^7 + 85^7 + 58^7 + 64^7 + 90^7 + 35^7 + 53^7$ (Selfridge)

$1827^8 = 1067^8 + 1066^8 + 1065^8 + \ldots + 961^8 + 960^8 + 958^8$
$+ 379^8 + 227^8 + 137^8 + 93^8 + 65^8 + 47^8 + 36^8$
$+ 26^8 + 21^8 + 15^8 + 14^8 + 10^8 + 9^8 + 8^8$
$+ 6^8 + 5^8 + 3^8 + 2^8$ (Wu)

There are 108 consecutive terms between 1067^8 and 960^8 inclusive, and 127 altogether, on the right. By comparison, the next example by Wu Tze Chen is quite short. Only 90 terms on the right!

$$
\begin{aligned}
9339636^9 = {} & 8445344^9 + 8441982^9 + 7779668^9 + 2582016^9 + 1398592^9 + 759812^9 + \\
& 500938^9 + 339562^9 + 221892^9 + 168100^9 + 50430^9 + 43706^9 + 40344^9 + \\
& 36982^9 + 30258^9 + 26896^9 + 20172^9 + 13448^9 + 7092^9 + 7089^9 + \\
& 7086^9 + 7083^9 + 7080^9 + 7077^9 + 7074^9 + 7071^9 + 7068^9 + \\
& 7065^9 + 7062^9 + 7059^9 + 7056^9 + 7053^9 + 7050^9 + 7047^9 + \\
& 7044^9 + 7041^9 + 7038^9 + 7035^9 + 7032^9 + 7029^9 + 7026^9 + \\
& 7023^9 + 7020^9 + 7017^9 + 7014^9 + 7011^9 + 7008^9 + 7005^9 + \\
& 7002^9 + 6999^9 + 6996^9 + 6993^9 + 6990^9 + 6987^9 + 6984^9 + \\
& 6981^9 + 6978^9 + 6975^9 + 6972^9 + 6969^9 + 6966^9 + 6963^9 + \\
& 6960^9 + 6957^9 + 6954^9 + 6951^9 + 6948^9 + 6945^9 + 6942^9 + \\
& 6939^9 + 6336^9 + 3362^9 + 3069^9 + 1599^9 + 918^9 + 615^9 + \\
& 405^9 + 237^9 + 174^9 + 135^9 + 108^9 + 72^9 + 63^9 + \\
& 54^9 + 42^9 + 36^9 + 33^9 + 15^9 + 9^9 + 6^9 .
\end{aligned}
$$

"Absolute values have absolutely no value" C.W.
(Frustrated year 11 student, P.L.C.)

COMPUTER EXPERT MAKES FORM PAY

A British computer scientist has worked out a mathematical method of gambling on horse races and football pools which he claims reduces the odds by about a third.

Professor Frank George says he and his colleagues have collected winnings of up to £10,000 (about $A20,000).

"By studying the form scientifically I claim you can greatly increase your chances of winning money," Professor George said in an article in this week's New Scientist magazine.

He says it all started when his daughter said: "Dad, you think you're so bloody clever, why can't you win the pools?"

"The Age" 21/3/1980.

PROBLEM SECTION

SOLUTION TO PROBLEM 3.3.2

Do the hour, minute, and second hand of a clock coincide at any time between 12 noon and 12 midnight?

(a) If so, when? If not, when do they most nearly do so?

(b) When do the hands come closest to trisecting the clock face?

The answers to (a) and (b) depend on what is meant by closest. We answer (a) by requiring that the angle between the two outer hands be minimum, and (b) when the sum of the three absolute deviations from 120° for the three sectors is minimum. For (a), the hands don't exactly coincide, but come closest just after 3.16 pm and a little before 8.44 pm. We explain the reasoning below. Similar reasoning (which we do not give below) shows that at 2.54 pm (and 34.55 seconds) and at 9.05 pm (and 25.45 seconds) the hour and second hands are exactly 120° apart, and the minute hand differs from them by $120 \pm 0.17°$.

The hour hand travels at 30°/hour, the minute hand at 6°/minute and the second hand at 6°/second. Thus h hours, m minutes and s seconds after noon, (where h and m are integers, but s is not necessarily an integer), the hour hand is pointing at an angle $H°$ clockwise from the 12 o'clock position, where

$$H = 30 \times (h + \frac{m}{60} + \frac{s}{3600}). \qquad (1)$$

Similarly the minute hand will be at $M°$ where

$$M = 6 \times (m + \frac{s}{60}), \qquad (2)$$

and the second hand at $S°$ where

$$S = 6s. \qquad (3)$$

In order that the hour hand and the minute hand should coincide, we need

$$H = M,$$

that is

$$30(h + \frac{m}{60} + \frac{s}{3600}) = 6(m + \frac{s}{60}),$$

which reduces to

$$m + \frac{1}{60} s = \frac{60}{11} h. \qquad (4)$$

Remembering that h and m have to be integers, and that $s < 60$ must hold, if we try $h = 1, 2, \ldots, 10$ we get the following values for m and s.

TABLE 1

h	1	2	3	4	5
m	5	10	16	21	27
s	$27\frac{3}{11}$	$54\frac{6}{11}$	$21\frac{9}{11}$	$49\frac{1}{11}$	$16\frac{4}{11}$

h	6	7	8	9	10
m	32	38	43	49	54
s	$43\frac{7}{11}$	$10\frac{10}{11}$	$38\frac{2}{11}$	$5\frac{5}{11}$	$32\frac{8}{11}$

For instance, when $h = 1$, equation (4) says

$$m + \frac{1}{60} s = \frac{60}{11} = 5\frac{5}{11} \text{ minutes,}$$

so that $m = 5$ minutes and $\frac{1}{60} s = \frac{5}{11}$, i.e. $s = \frac{300}{11} = 27\frac{3}{11}$ sec. [The $h = 11$ case actually yields 11 hours 60 minutes, i.e. 12 o'clock.]

The equation $H = S$, to find when the hour and second hand coincide, reduces to

$$s = \frac{1}{719}(3600h + 60m). \qquad (5)$$

Using the values in Table 1 for h and m we find the values for s as in (5). See Table 2. (The quantity d will be discussed shortly).

TABLE 2

h	1	2	3	4	5
m	5	10	16	21	27
s	5.42	10.85	16.36	21.78	27.29
d	-.33	-.67	-.08	-.42	+.17

h	6	7	8	9	10
m	32	38	43	49	54
s	32.71	38.22	43.64	49.15	54.58
d	-.17	+.42	+.08	+.67	+.33

The first thing to notice is that the s values in Table 2 are different from those of Table 1. This shows that we cannot solve $H = M = S$ simultaneously; the three hands never coincide exactly (except at noon and midnight).

In Table 1, the second hand is never really close to the common position of the hour and minute hand, so the time at which the <u>three</u> hands are closest together is <u>not</u> one of the times in Table 1. We should look for cases of close coincidence when $H \neq M$. These are two important cases to consider, as in Figure 1 (a), (b).

Figure 1

Irrespective of whether the minute hand has not, Figure 1a, or has, Figure 1b, passed the hour hand, it is easy to see that the angular-spread of the hands is smallest just as the <u>second</u> hand is passing the <u>hour</u> hand. This, in view of the relative speeds of the three hands. Hence it is when $H = S$, that is, the cases studied in Table 2 when we should expect to find the hands closest in the sense that the two outer hands include the smallest angle.

When $H = S$, the angular distance, in degrees, between the minute hand and the (coincident) hour and second hands is, by (2) and (3),

$$M - S = 6m - \frac{59}{10} s$$

$$= (3960m - 21240h)/719 \quad \text{using (5).}$$

Measuring the angular distance in minutes of time on the clock face, it is

$$d = \frac{1}{6}(M - S) = (660m - 3540h)/719.$$

For the combinations of h and m considered in Table 2, the distances d are as tabulated in Table 2. Notice the distance is closest to 0 when the time is 3 hours 16 minutes 16.36

seconds, or when the time is 8 hours 43 minutes 43.64 seconds. In either case, the spread of the three hands is only $|d| \approx 0.08$ (minutes of time) or roughly 0.5° (angle).

The conclusion isn't changed if we consider other h,m combinations. Only those which differ by up to ± one minute from those in Table 2 are worth considering; we do find we can reduce the (h=2, m=10, d=-.67) difference by considering h=2, m=11 when d=0.25, and similarly replacing h=9, m=49, d=.67 by h=9, m=48, d=-.25.

SOLUTION TO PROBLEM 3.4.1.

Five sets of traffic lights are spaced along a road at 200 metre intervals. For each set, the red signal lasts 30 sec., the green 28 sec. and the amber 2 sec. The lights are synchronised in such a way that a car travelling at 36 k.p.h. and just catching the first light, just catches the other four. The width of the cross-street at each light is 20 metres. Find all the speeds at which it is possible to travel without being held up at any of the lights.

The following solution makes use of ideas contributed by Stephen Tolhurst, then in year 12, Springwood High School, N.S.W. The problem was not completely un-ambiguous, so let us interpret it to mean that the traffic lights go Red, Green, Amber, Red,... . Also, suppose that cars can start crossing an intersection only if the lights are green. Finally, suppose that the road width (20m) is included in the 200m between lights.

Measure time in seconds, with $t = 0$ being the time at which the first set of traffic lights changes from green to amber. From the given information, we can deduce that the first light is green for times t such that

$$-28 + 60n_1 \leqslant t \leqslant 60n_1, \quad \text{for } n_1 \text{ some integer.} \qquad (1)$$

The second, third, fourth and fifth lights are green when

$$-8 + 60n_2 \leqslant t \leqslant 20 + 60n_2, \qquad (2)$$

$$12 + 60n_3 \leqslant t \leqslant 40 + 60n_3, \qquad (3)$$

$$32 + 60n_4 \leqslant t \leqslant 60 + 60n_4, \qquad (4)$$

$$52 + 60n_5 \leqslant t \leqslant 80 + 60n_5, \quad \text{respectively,} \qquad (5)$$

where n_2, n_3, n_4, and n_5 are arbitrary integers. Let s m/sec denote the speed of a car, and t_1 the time it reaches the first traffic light. Suppose that $-28 \leqslant t_1 \leqslant 0$ so that the car catches the first green light with $n_1 = 0$. In order for it to catch all the other lights at green, the four times

$$t_1 + \frac{200}{s},\ t_1 + \frac{400}{s},\ t_1 + \frac{600}{s},\ t_1 + \frac{800}{s}$$

must satisfy (2), (3), (4), (5) respectively. Rearranging these we find we must have

$$(-8 + 60n_2 - t_1)/200 \leqslant \frac{1}{s} \leqslant (20 + 60n_2 - t_1)/200$$
$$(12 + 60n_3 - t_1)/400 \leqslant \frac{1}{s} \leqslant (40 + 60n_3 - t_1)/400$$
$$(32 + 60n_4 - t_1)/600 \leqslant \frac{1}{s} \leqslant (60 + 60n_4 - t_1)/600$$
$$(52 + 60n_5 - t_1)/800 \leqslant \frac{1}{s} \leqslant (80 + 60n_5 - t_1)/800$$

It can be checked that, assuming $-28 \leqslant t_1 \leqslant 0$, these inequalities are impossible to satisfy unless

$$n_3 = 2n_2,\ n_4 = \frac{3}{2}\, n_3 = 3n_2,\ n_5 = \frac{4}{3}\, n_4 \doteq 4n_2,$$

when they will all be satisfied so long as

$$(52 + 240n_2 - t_1)/800 \leqslant \frac{1}{s} \leqslant (80 + 240n_2 - t_1)/800 .$$

For instance, if $t_1 = 0$ so that the car <u>just</u> catches the first light, its speed, s (m/s), to catch the other lights must satisfy

$$\frac{800}{80 + 240n_2} \leqslant s \leqslant \frac{800}{52 + 240n_2}$$

for some integer n_2. This yields speeds (k.p.h.) in the intervals

[36, 55.4], or [9, 9.9], or [5.1, 5.4], or [3.6, 3.7], or [2.8, 2.8], ... (taking $n_2 = 0, 1, 2, \ldots$ in turn).

Similarly, if the car arrives at the first light just as it turns green, $t_1 = -28$, and we need

$$\frac{800}{108 + 240n_2} \leqslant s \leqslant \frac{800}{80 + 240n_2} ,$$

corresponding to speeds (k.p.h.) in the intervals

[26.7, 36], or[8.3, 9], or[4.9, 5.1], or[3.5, 3.6], or [2.7, 2.8],... .

PROBLEM 4.2.1

(Submitted by Ravi Sidhu, Townsville)

What is the meaning, and value of the "continued fraction"

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots\cdots}}}} \quad ?$$

PROBLEM 4.2.2

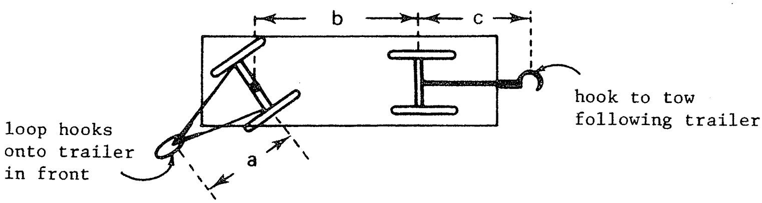
Eleven men toss their hats in the air; the hats are picked up randomly. Each man who receives his own hat leaves and the remaining men toss the hats again. The process continues until every man has received his own hat again. How many rounds of tosses are expected?

PROBLEM 4.2.3

Show that, whatever number base is used, 21 is not twice 12. Find, for each possible number base not exceeding 10, every instance of a number consisting of two different non-zero digits which is a multiple of the number obtained by interchanging the digits.

PROBLEM 4.2.4

Baggage trains used at airports, railway stations etc. have a small tractor which pulls a train of 4-wheeled trailers, each connected to the one in front. The back axle of each trailer is fixed, and the front axle pivots, being steered by the towing bar connecting the trailer to the one in front. An underneath view of a trailer is shown below.



Problem: how should the dimensions a, b and c be proportioned so as to make the train follow as nearly as possible the path taken by the tractor?

MONASH SCHOOLS' MATHEMATICS LECTURES, 1980

Monash University Mathematics Department invites secondary school students studying mathematics, particularly those in years 11 and 12 (H.S.C.) to a series of lectures on mathematical topics. The lectures are in the Rotunda Lecture Theatre R1 at 7 p.m.

2nd May	"Black holes"	Dr C. McIntosh
9th May	"The sundial"	Dr C. Moppert

We hope to publish some of the talks in *Function* later this year, for students who are unable to attend in person.

LEAP YEAR (FROM THE AGE, 29/2/1980)

"These are strange times. Today we gain a day; next Sunday we pick up an extra hour; and not long ago we lost a year.

February 29 is known as Leap Day, which is simply a way to disguise the mathematical contortions which determine the length of our days.

A 24-hour day is a figment of our imagination, even if it is socially convenient. A real day - the mean solar day, scientists call it - lasts 23 hours, 56 minutes and 4 seconds.

Every four years we lump together all those extra 3 minute 56 second bits of time and call them a leap day.

To cover our embarrassment about it, we have even developed a strange ritual to make it all right. Leap Day gives us Leap Year and for some reason that entitles women to propose marriage to men."

Well, that may be <u>one</u> explanation! Does anyone know the <u>correct</u> explanation of leap year?