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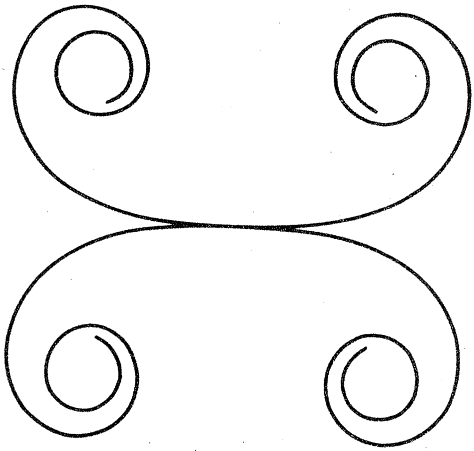

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the addresses shown above.

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Registered for posting as a periodical - "Category B"

Derek Holton is a Senior Lecturer in Mathematics at the University of Melbourne. For a number of years he has been interested in school mathematics at the primary and secondary level. This has led him to be Chairman of the HSC Pure Mathematics examination *(hiss! boo!)* and to be coorganiser with John Rickard of the Melbourne University School Mathematics Competition.

This latter competition has been running successfully now since 1971. Each year approximately 3500 students from just under 150 schools in Victoria enter the competition. Since its inception IBM have been very generous sponsors.

The solutions of all problems in the competition are due to be published in late 1980 by Prentice-Hall. These will appear under the title "*Problems! Problems!*" by D.A. Holton and J.A. Rickard.

His article in this issue describes how one of the 1980 competition problems arose, and how it may be solved.

Andrew Mattingly is now a second year Monash Science student, and he has made a number of contributions to Function over recent years. In this issue, he has contributed two articles, both involving use of the computer. Because of his outstanding results at H.S.C. and first year university, Andrew has been made a Faculty of Science Scholar. Such scholars are allowed to pursue a special course of study for their degree.

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THE FRONT COVER

Andrew Mattingly
Monash University

Consider a curve which starts at the origin, *O*, (in the *x*-*y* plane) and progresses into the first quadrant in such a way that at any point *P(x,y)* along the curve, the tangent to the curve makes an angle, θ, (described positively from the *x*-axis) which is proportional to the square of the distance, s, from *O* to *P* along the curve (see Fig. 1). In particular

$$\theta = \frac{\pi}{2} s^2 \quad . \tag{1}$$

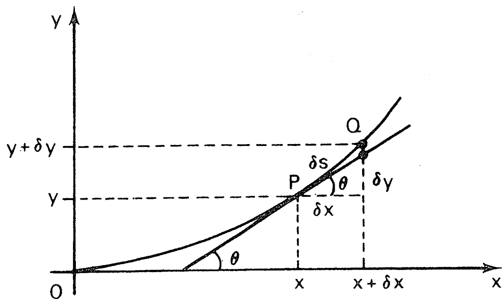


Figure 1.

From Fig. 1, if δs is small we approximate (using (1))

$$\delta x = \delta s \cos\theta = \delta s \cos(\frac{\pi}{2} s^2) \tag{2a}$$

$$\delta y = \delta s \sin\theta = \delta s \sin(\frac{\pi}{2} s^2) \, . \tag{2b}$$

Those readers familiar with calculus will recognise that letting $\delta s \to 0$ gives

$$\frac{dx}{ds} = \cos(\frac{\pi}{2} s^2) \, , \tag{3a}$$

$$\frac{dy}{ds} = \sin(\frac{\pi}{2} s^2) \, . \tag{3b}$$

Then

$$x(s) = \int_0^s \cos(\frac{\pi}{2} t^2) dt \qquad \text{(Fresnel's Cosine Integral)} \tag{4a}$$

$$y(s) = \int_0^s \sin(\frac{\pi}{2} t^2)\, dt \qquad \text{(Fresnel's Sine Integral).} \tag{4b}$$

We could let s be negative; then from (4a) and (4b) we obtain

$$x(-s) = -x(s) \text{ and } y(-s) = -y(s).$$

So our curve has symmetry about the origin. This curve is known as "Cornu's Spiral". Note that as s gets larger the curve nears $(\frac{1}{2}, \frac{1}{2})$ and likewise as s gets more negative it spirals in on $(-\frac{1}{2}, -\frac{1}{2})$ (see Fig. 2).

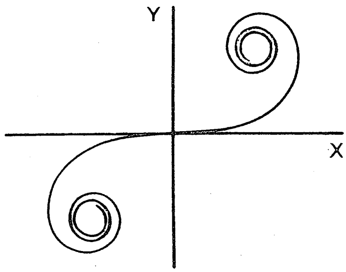


Figure 2.

The front cover depicts Cornu's Spiral with its reflection in the x- or y-axis. The integrals 4(a) and 4(b), and the curves, have been obtained using a computer.

Cornu's Spiral is used in optics, and in particular, in the prediction of diffraction patterns from various apertures. It is named after M.A. Cornu (1841-1902), who was Professor of Experimental Physics at the Ecole Polytechnique, Paris.

* * * * *

"Slowdown is Accelerating"

(Heading in *Journal of Commerce*).

* * * * *

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743\ .\ 999\ 999\ 999\ 999\ 2$$

is remarkably close to an integer !

BATHROOM FLOORS

D. A. Holton
University of Melbourne

I just happened to be having a shower at Echuca. There I was minding my own business with water running all over me when I dropped the soap on the floor. As I picked it up I noticed the floor tiles. Now tiles come in all shapes and sizes. There are your regular triangular, square and hexagonal tiles and your mixtures of octagonal and square tiles and all the other variations that you've seen here, there and everywhere.

But these wet tiles under my feet weren't any of the common ones I'd seen before. They were a variation on the square theme though and made up squares as shown in Figure 1. The different shading represents different colours.

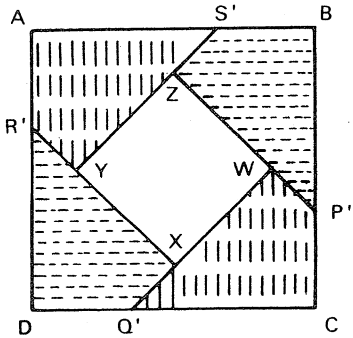


Figure 1.

It was clear that the tiles like the tile *WXYZ* were squares. Though it wasn't obvious to me immediately how you could prove that. Well, I could see a method of attack but I was hoping to see a neater method. But these tiles weren't initially of much interest to me.

It was the other tiles, all the same shape and size that caught my eye and jogged my memory. I'm getting old so it took me a little while to remember that I'd seen shapes like these at Christmas time. They had fallen out of a cracker and the challenge had been to put four of them together to make a square. Of course that's not at all difficult. But don't try doing it over Christmas dinner with an audience and the high probability that one of the younger children present will, without the benefits of excess Christmas spirit, do it before you can. But what is really going on here? How is the tiling constructed? Well clearly one way to produce the pattern is to start with a square *ABCD* of side length a and take an arbitrary point P on the side AB. Then draw the line

PP' with P' on BC, where the angle $P'PB$ is θ. Now if $AP = b$, we repeat the process starting at a point Q on BC which is a distance b from B and draw a line through Q to Q' on CD so that angle $Q'QC$ is θ. By performing a similar construction on the edges CD and DA and deleting certain parts of the newly constructed lines, we have the situation of Figure 1. (See Figure 2.)

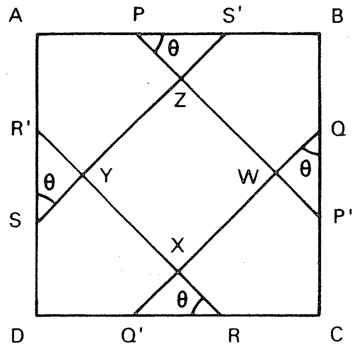


Figure 2.

Now since the angle at P' in triangle PBP' is $\frac{1}{2}\pi - \theta$, then the angle at W in triangle $QP'W$ must be a right angle. Hence PP' and QQ' are perpendicular to each other. Similar reasoning shows that the four angles of each at X, Y and Z are all right angles. So, clearly, the figure $WXYZ$ is a square. Or at least that is the conclusion reached by vast numbers of students in the Senior Division of the Melbourne University School Mathematics Competition. It didn't occur to them that $WXYZ$ might in fact be a *rectangle*!

To show that we do indeed have a square we can actually compute ZW in terms of a, b and θ. But since the answer is in terms of a, b and θ the same calculation will also apply to every other side of the quadrilateral $WXYZ$. Hence it is a square.

But surely there is a better way to do this than actually to compute the side lengths in terms of a, b and θ ?

Now motivated by our shower tiles we continued to have P' on BC. But, of course, depending on the size of θ, P' may end up on CD or AD. Do we still end up with a figure $WXYZ$ somewhere that can be shown to be a square?

And the tiles have also led us to believe that $WXYZ$ lies entirely inside $ABCD$ but this is far from being the case. If we choose θ correctly then Q and P' coincide. For any value of θ smaller than this (where $\tan\theta = \frac{b}{a-b}$), PP' and QQ' meet outside the initial square. So too do QQ' and RR', RR' and SS', SS' and PP'. Does this mean that $WXYZ$ can have an area larger than that of $ABCD$? What is the maximum size of the area of $WXYZ$? See Figure 3.

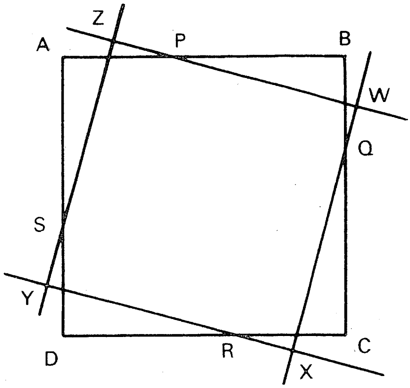


Figure 3.

Once you start on that theme, then you must surely ask what is the *minimum* size of the area of $WXYZ$? That's not hard to answer. It must be zero, since if $b = 0$ and $\theta = \frac{1}{4}\pi$, then PP', QQ', RR', SS' are all diagonals of $ABCD$. This forces W, X, Y, Z to all be the same point and so the square is degenerate.

But no matter what value b has, is there a value of θ which makes $WXYZ$ degenerate?

Now we were talking of putting four of the original tiles back together again to form a square. So how do we do that? Well, if you cut out four shapes identical to that of $BP'WZS'$ in Figure 1 you should be able to move them around until you end up with a square. But if you vary θ again what happens to the four sided figure $BP'WZS'$? Will it always be possible to put these four parts back together again to form a square?

****

Question 4, Melbourne University School Mathematics Competition, 1980:

Let $ABCD$ be a square of side length a. Let P be a point on AB a distance b ($0 < b < a$) from A. Draw the line PP' so that P' is on one of the sides of the square which is adjacent to AB and let the angle $P'PB$ be θ. Similarly draw the line QQ' so that Q is on the side BC, Q' is on one of the sides adjacent to BC and the angle $Q'QC$ is θ. Construct R, R' from CD and S, S' from DA in a similar fashion.

Let PP' intersect QQ' at W, QQ' intersect RR' at X, RR' intersect SS' at Y and SS' intersect PP' at Z, where W, X, Y, Z are all interior points of the square $ABCD$. Prove that $WXYZ$ is a square.

For what value of b and θ is this square degenerate? (That is, for what value of b and θ does this square have zero area?)

If $b = \frac{1}{2}a$ and P' lies on BC, show that the piece $PWQB$ together with the three similar pieces $QXRC$, $RYSD$, $SZPA$ can be put together to form a square.

Is is always possible to form a square with such pieces, regardless of the size of b and of the angle θ ?

****

So what is the article about?

Well first of all it shows you one way problems arise for the Senior Division of the Melbourne University Mathematics Competition. A lot of these questions come from books, but a reasonable number come from discussions with other mathematicians, from walking in the street or from having showers. And this is the way more serious mathematical problems arise. The type of research problem mathematicians indulge themselves in, may come to them via books, colleagues or "real life".

Secondly, I was attempting to show the "push-me-pull-me" aspect of mathematics. I could have easily proved that $WXYZ$ in Figure 1 was a square and left it at that. But mathematicians tend to look in under and around a problem, worrying it to death. By changing something here and moving something there they try to discover everything they possibly can about a situation. In this sort of way advances in mathematics are made.

****

I just happened to be in a bathroom in Kalamazoo (look that one up in your atlas) which had rectangular tiles all of the same size. Now I tried to imagine those rectangles stacked together to form a square. But although taking 5 horizontally and 3 vertically the 15 rectangles looked close to a square, any two adjacent sides didn't appear to be equal. Is it possible to find rectangles all of the same size, so that no matter how many you put together in a regular pattern with corresponding edges touching, they will never form a square? Can rectangles be found that can be put together in increasing numbers to make better and better approximations to a square, but which never actually do produce a square?

Can you find some more problems by looking at tiles?

THE MONTE CARLO METHOD

Andrew Mattingly
Monash University

With the advent of fast and accurate computers it is now possible to perform many thousands of simple computations in the space of less than one second.

In the field of probability, therefore, it is possible to estimate the probability of a given event (or probability of success) by actually performing the relevant experiment. This relies, of course, on our ability to construct a suitable numerical model of the experiment so that a computer *can* perform the experiment.

The computer is programmed to perform the experiment many times, recording the number of trials, N, and the number of successes, S. The probability of success, p, can be estimated according to the formula.

$$p = \frac{S}{N} .$$

To illustrate this technique, known as the "Monte Carlo Method" (sometimes "Las Vegas Method"), we consider a number of examples:

(a) Evaluating Areas Using a Dart Board.

In this example we wish to calculate the area (A) under the curve

$$y = -x \ln x \quad [\ln x \equiv \log_e x, \text{ natural logarithm of } x\ ]$$

between $x = 0$ and $x = 1$ (see Fig 1.)

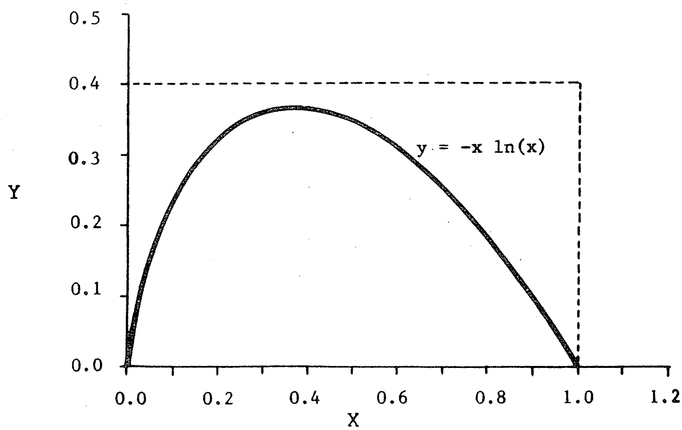


Figure 1.

To do this we program the computer to generate pairs of random numbers. The first of each pair we constrain to be between *0* and *1* and the second to be between *0* and *0.4*. If we associate these pairs of numbers with coordinates in the $x - y$ plane it is clear that we are generating points within the dotted rectangle in Fig. 1. Physically this is like throwing darts at a dart board (i.e. it is impossible to predict the exact landing point of any dart thrown). If we now throw thousands of darts at our dart board, with $y = - x \ln x$ drawn on it, the number of darts landing in the area *A* divided by the total number thrown (assuming all darts land on the board) gives a good estimate of the probability of a dart landing in the region *A*. If the dart throwing is purely random this probability will equal the proportion, *p*, of the area of the dart board covered by *A*. Hence since the area of our dart board is $1 \times 0.4 = 0.4$, the area of *A* is given by $0.4p$.

Those readers familiar with Integration by Parts may care to verify that the actual area of *A* is *1/4*.

If we let the computer do the "dart throwing" and calculate the area of *A* as above, we find (see Fig. 2) that as we throw more and more darts our estimate of the area approaches *0.25*.

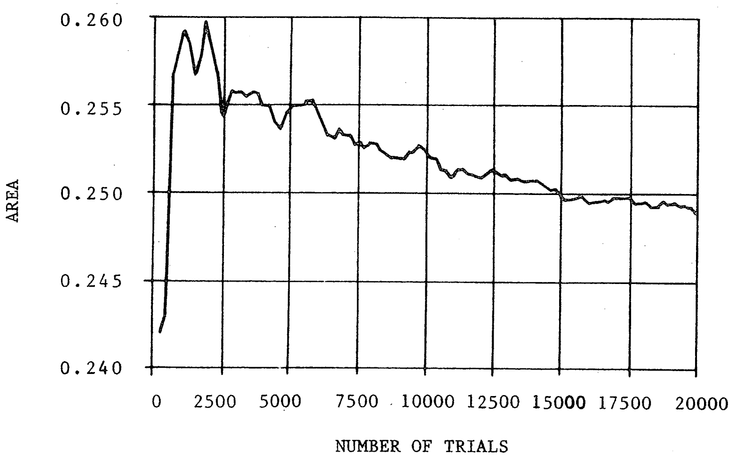
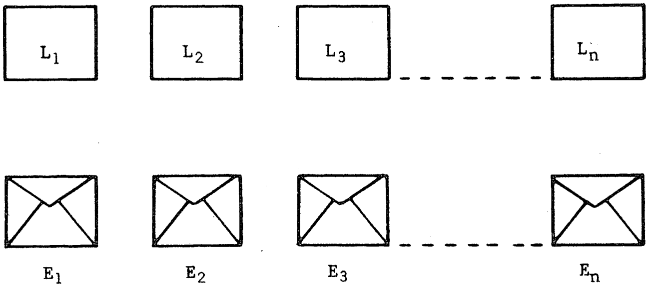


Figure 2.

(b) Estimation of "e" = 2.718281828459045 ...

A prominent businessman dictates *n* letters to be sent to *n* different people. His not-so-efficient secretary types up the letters and the envelopes but when putting the letters into the envelopes, she pays no attention to whom the envelopes are addressed. So, conceivably they are all in the wrong envelopes! Let us consider the number of ways this can happen. Take the *n* letters and n envelopes.



Let T_n denote the number of ways of putting each letter in a wrong envelope.

Now take the first letter L_1. It can go into any of $(n - 1)$ envelopes. Suppose it is put into E_k. Now L_k can go into E_1 in which case the remaining letters can be enveloped in T_{n-2} ways. Or L_k does not go into E_1, so, $L_2, L_3, \ldots, L_{k-1}, L_{k+1}, \ldots, L_n$ cannot go into their corresponding envelopes and L_k cannot go into E_1. This situation is equivalent to sorting $n - 1$ letters so that none are right, which can be done in T_{n-1} ways. Thus

$$T_n = (n - 1)\ [T_{n-1} + T_{n-2}]. \qquad (1)$$

Clearly $T_1 = 0$, since with one letter and one envelope the letter must go into that envelope (fool-proof).

And $T_2 = 1$ (i.e. there is one wrong way to put two letters into two envelopes). Using the "recurrence relation" (1) we find

$$T_3 = (3 - 1)[1 + 0] = 2$$

$$T_4 = 9$$

$$T_5 = 44$$

$$T_6 = 265, \ etc.$$

It can be shown (as the reader may verify) that T_n is given by

$$T_n = n![1 - (1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n-1}\frac{1}{n!})] \qquad (2)$$

where $n! = n(n-1)(n-2) \ldots \times 3 \times 2 \times 1$.

Now we can find the probability, P_n, of the secretary getting all n letters in wrong envelopes, by dividing T_n by the total number of ways of putting n letters in n envelopes.

As a moment's reflection should reveal, the total number of ways is $n!$, since the first letter can go into any of n envelopes; the second into any of $n - 1$; the third into any of $(n - 2)$, and so on to the n^{th} which goes in the remaining unfilled envelope. Thus

$$P_n = \frac{1}{n!} T_n$$

$$= [1 - (1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n-1} \frac{1}{n!})] \qquad \text{from (2)}$$

$$= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} \quad . \qquad (3)$$

The "exponential function", e^x, is given by the "power series"

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad (4)$$

and hence putting $x = -1$ we have

$$e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots \quad . \qquad (5)$$

Comparing (3) and (5) we see that as n tends to ∞, P_n tends towards $\frac{1}{e}$. Indeed for $n > 10$, $|P_n - \frac{1}{e}| < 10^{-7}$.

We now program the computer to sort 20 letters into twenty envelopes at random. This is done by generating a "jumbled" list of the numbers $1, 2, 3, \ldots, 20$. We say that all the letters are in their right envelopes if the random "jumbling" gives the list in the original order $1, 2, 3, \ldots, 20$. The computer repeats this jumbling procedure many times recording the number of times none of the numbers is in the right place, thus obtaining P_{20} to reasonable accuracy. Then we compute

$$e \approx \frac{1}{P_{20}} \quad . \qquad (6)$$

This procedure gives a value of e within 2% of the known value (see Fig. 3). For a better estimate more trials are needed.

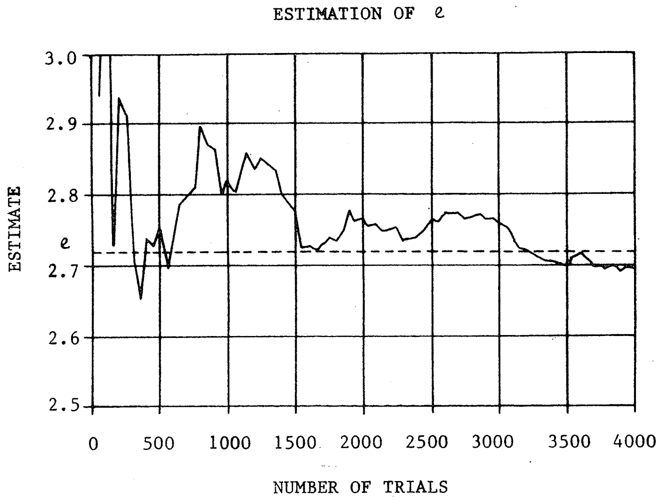


Figure 3.

(c) Estimation of π - Buffon's Needle Problem.

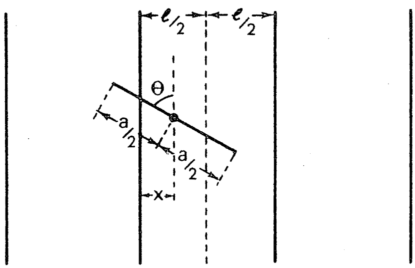


Figure 4.

Many parallel lines are drawn (each spaced a distance l apart) on a table top. A pin of length $a(< l)$ is thrown onto the table. We say that the result of this experiment is a success if the pin cuts one of the lines (see Fig. 4).

Let us calculate the probability, P, of success. From Fig. 4 the needle cuts a line if x, the distance of the centre of the needle to the nearest line, is less than $\frac{a}{2}\sin\theta$, where θ is the acute angle the needle (or its extension) makes with the parallel lines. So we write the touching condition

$$x \leqslant \frac{a}{2} \sin \theta \ . \qquad (7)$$

From our definitions of x and θ we have that

$$0 \leqslant x \leqslant \frac{\ell}{2} \qquad (8a)$$

$$0 \leqslant \theta \leqslant \frac{\pi}{2} \ . \qquad (8b)$$

We now plot x against θ

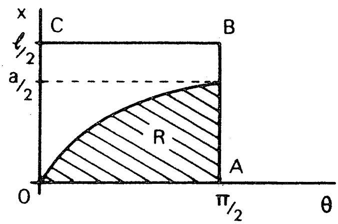


Figure 5.

Possible values of x and θ lie in the rectangle $OABC$. Those ordered pairs (θ, x) such that

$$x \leqslant \frac{a}{2} \sin \theta$$

lie in the shaded region R. Hence the probability, P, of the needle landing on a line is

$$P = \frac{\text{area of } R}{\text{area of } OABC} \ .$$

The area of R can be shown (by integration) to be equal to $\frac{a}{2}$ thus

$$P = \frac{a}{2} / \left(\frac{\pi}{2} \times \frac{\ell}{2}\right) = \frac{2a}{\pi \ell} \ . \qquad (9)$$

So

$$\pi = \frac{2a}{\ell P} \ . \qquad (10)$$

The computer is now programmed to generate random values of x and θ satisfying (8a) and (8b) respectively. It then tests to see whether (7) is satisfied. This is repeated many times, giving a good estimate of P. Then, using (10), an estimate of π is obtained (see Fig. 6).

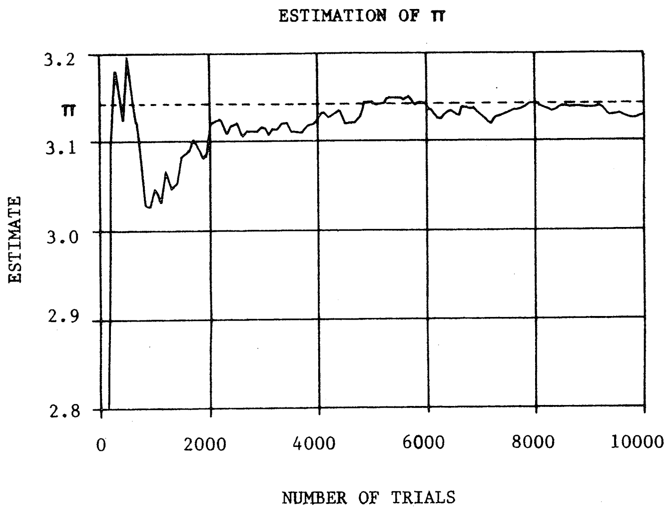


Figure 6.

These examples illustrate some of the simplest applications of the Monte Carlo Method.

Acknowledgements. The author wishes to thank Miss C. Hussár and Dr J.J. Monaghan for their help and encouragement.

[*The name "Monte Carlo" was the code name used by mathematicians Ulam, von Neumann and others during World War II. They were using the method to study the penetration of materials by neutrons, in connection with the development of the Atomic bomb at Los Alamos. Eds.*]

*   *   *   *   *

Some Puzzles. For answers, see p 29.

1. What is the next number in the sequence *20, 40, 60, 80, 100?*

2. What have the numbers *173, 8, 15, 92, 33, 45* in common?

3. *N E 1 4 10 S ?*

*   *   *   *   *

2+2 = 3 ?

M.A.B. Deakin,
Monash University

I once saw the following graffito:

$2 + 2 = 3$ for small values of 2.

To justify: let two numbers x, y be each rounded to 2, the nearest integer. Then

$$1.5 < x < 2.5$$

$$1.5 < y < 2.5 .$$

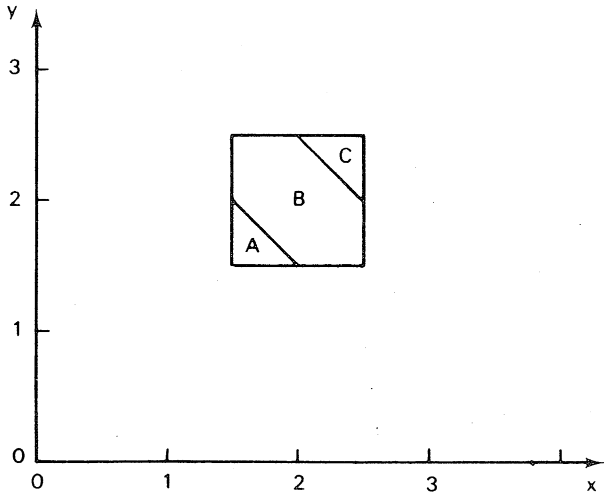
Then $x + y$ is best approximated by 3 if

$$3 < x + y < 3.5. \qquad \text{(Region } A \text{ on graph)}$$

Similarly $x + y$ is best approximated by 5 if

$$4.5 < x + y < 5. \qquad \text{(Region } C \text{ on graph)}$$

The graph shows that the probability that $2 + 2 = 4$ (Region B) is $3/4$.



THE SEVEN-POINT GEOMETRY

G. B. Preston
Monash University

Geometry is about points, lines, planes, and other curves and surfaces, and their relation to one another. Let us stick to geometry in a single plane: so we are just looking at points, lines and curves in a plane.

If we think of ordinary Euclidean geometry, then two properties of points and lines in a plane are:

I. There is a unique line passing through any two distinct points.

II. Any two distinct lines (unless they are parallel) intersect in a unique point.

The relationship between lines and points expressed in properties I and II is symmetrical except for the caveat about parallel lines. Let us consider a variant of I and II that forgets about this special case. Let us also change our terminology slightly to emphasize the resultant symmetry and regard as equivalent the two statements "a point is on a line" and "a line is on a point". We also say that a point and line are *incident with* one another if each (i.e. either) is on the other. Investigating which lines meet each other and which points are on which lines is investigating their *properties of incidence*.

The variants of I and II we wish to consider are:

PI. There is a unique line on any two distinct points.

PII. There is a unique point on any two distinct lines.

There is a perfect symmetry between the properties of lines and points satisfying PI and PII, and this is usually expressed by saying that, in relation to PI and PII, lines and points play a *dual* role.

At first sight it might be thought that PI and PII are simply not true: we know that there is no point on two distinct parallel lines. But jumping to this conclusion depends on giving a special interpretation to the words 'point' and 'line' in PI and PII, for example, in assuming that what is meant are the lines and points of Euclidean geometry in a plane.

To show that both PI and PII can hold for some interpretation of line and point it suffices to give an example. If PI and PII are each to impose conditions on our lines and points, there must be at least two lines and at least two points. There is an easy example with just three points and just three lines, the *3-point geometry*.

Let the three points be denoted by P_1, P_2, and P_3, and then define the lines, L_1, L_2 and L_3 by

$$L_1 = \{P_2, P_3\},$$
$$L_2 = \{P_3, P_1\},$$
$$L_3 = \{P_1, P_2\},$$

i.e. each line is a (specific) set of points. Let us drop the curly brackets for sets containing a single element, so that P will denote, if the context requires it, the set $\{P\}$. Then corresponding to the above equations we have

$$P_1 = L_2 \cap L_3,$$
$$P_2 = L_3 \cap L_1,$$
$$P_3 = L_1 \cap L_2.$$

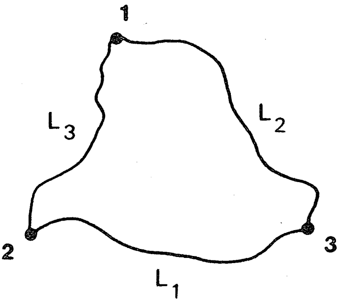
We have in fact chosen our notation for points and lines so that $P_i \in L_j$ if and only if $P_j \in L_i$.

The latter equations show that any two distinct lines determine, or are on, a unique point; so property PII holds for this interpretation of line and point. The defining equations for L_1, L_2 and L_3 show that any two distinct points determine, or are on, a unique line; so PI also holds.

Properties PI and PII are usually called *axioms of incidence*, and an interpretation of lines and points is an example of a *model*, in which the axioms of incidence are satisfied, and in which there are at least two points and at least two lines, I shall call a *geometry*. We can state what we have shown in the following theorem.

THEOREM 1. *There is a geometry, the 3-point geometry, with exactly three points and three lines. In this geometry each point has precisely two lines on it and each line has precisely two points on it.*

The diagram overleaf illustrates the 3-point geometry. On this diagram the points are indicated simply by numbers *1*, *2*, and *3*, and the lines are suggested by the curves connecting each pair of points. Note that there are no points (of our geometry) on these lines (curves) other than the points denoted by *1*, *2*, and *3*.



3-point geometry

The geometry of Theorem 1 is in fact characterized by the statement in the second sentence of the Theorem, in the sense of the next Theorem.

THEOREM 2. *Let G be a geometry in which each line has precisely two points on it and each point has precisely two lines on it. Then G has exactly three points and three lines and it is (effectively) the 3-point geometry.*

Proof. Since G is a geometry it has at least two points. Let P_1 and P_2 be two of its points. By assumption there is precisely one line L_3, say, on P_1 and P_2. Again since G is a geometry, it has at least two lines. Let L be a line distinct from L_3. It meets L_3 in a unique point which must be either P_1 or P_2, since these are the only points on L_3. Suppose, without loss of generality that L meets L_3 in P_1, and let us rename L as L_2. Then, by assumption, L_2 has precisely one point other than P_1 on it. It cannot be P_2, for then L_2 would equal L_3, contrary to assumption. Hence G has a further point P_3, say, and it lies on L_2. Then P_2 and P_3 have a unique line on them. It cannot be L_2, for P_3 and P_1 are the only points on L_2, and similarly it cannot be L_3. So this is a new line; let us denote it by L_1.

P_1, P_2, P_3 and L_1, L_2, L_3 then satisfy the incidence relations indicated by the diagram of the 3-point geometry. So effectively they form the 3-point geometry.

We show now that G can have no further points and lines. For suppose that P is a further point of G. Then there must be a line on P_1 and P, which has to be distinct from L_2 because P_1 and P_3 are the only points on L_2, and similarly must be distinct from L_3. Thus P_1 has three lines on it, contrary to assumption. This contradiction shows that there can be no further point in G other than P_1, P_2, and P_3.

A similar argument shows that G has no lines other than L_1, L_2, and L_3. The proof is complete.

The 3-point geometry just provides a way of talking about some of the properties of the sides and vertices of the usual triangle of Euclidean geometry; and we could equally well do this without bothering about axioms of incidence. We now give a second example of a geometry, in our sense, that cannot be embedded in Euclidean space. This model will have exactly seven points and seven lines.

Let $P_1, P_2, \ldots, P_7$ be the points and define seven lines, $L_1, L_2, \ldots, L_7$, by the equations

$$\begin{aligned} L_1 &= \{P_1, P_2, P_3\} , \\ L_2 &= \{P_1, P_4, P_5\} , \\ L_3 &= \{P_1, P_6, P_7\} , \\ L_4 &= \{P_2, P_4, P_6\} , \\ L_5 &= \{P_2, P_5, P_7\} , \\ L_6 &= \{P_3, P_4, P_7\} , \\ L_7 &= \{P_3, P_5, P_6\} . \end{aligned}$$

Then, as may be verified by working through the 21 possible pairs, any two of these lines intersect in a single point: for example $L_5 \cap L_6 = P_7$ and $L_4 \cap L_7 = P_6$. Hence axiom PII is satisfied by those lines and points. Similarly, by going through all 21 cases, we show that each pair of points lies on a line, unique because PII holds. Thus axiom PI holds. Thus we have a model of a geometry.

Note that, as for the 3-point geometry, we have enumerated the points and lines so that $P_i \in L_j$ if and only if $P_j \in L_i$. Hence, for any distinct j, k, l, we have

$$L_i = \{P_j, P_k, P_l\}$$

if and only if

$$P_i = L_j \cap L_k \cap L_l .$$

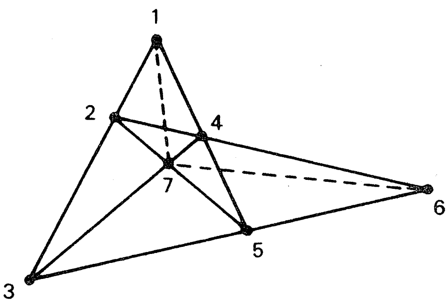
A moment's thought shows that this implies that there are precisely three lines on every point. By definition of the lines, there were precisely three points on each line. Hence we have the following analogue, for this new geometry, called the *seven-point geometry*, of Theorem 1.

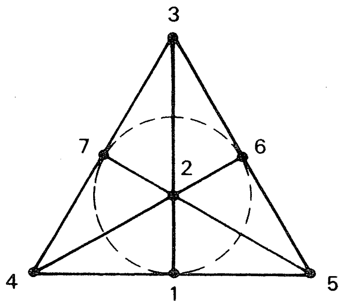
THEOREM 3. *There exists a geometry, the 7-point geometry, with exactly seven points and seven lines, satisfying the axioms of incidence PI and PII. In this geometry each line has precisely three points on it and each point has precisely three lines on it.*

We have again the analogue of Theorem 2. We omit the proof because it is very similar to that of Theorem 2.

THEOREM 4. *Let G be a geometry in which each line has precisely three points on it and each point has precisely three lines on it. Then G has exactly seven lines and seven points and is effectively the 7-point geometry.*

Either diagram below illustrates the 7-point geometry.





7-point geometry

On the diagrams, the point P_1 is indicated by a dot labelled *1*, etc. The lines of the geometry are indicated by the lines drawn. Note in particular the single line $\{P_1, P_6, P_7\}$ shown by the sequence of dashes.

These diagrams suggest a fact that may be proved, namely that it is impossible to embed the 7-point geometry in a Euclidean plane in such a fashion that the points of the 7-point geometry are represented by points in the plane, that the lines in the 7-point geometry are represented by straight lines in the plane and such that all the incidence relations of the 7-point geometry correspond to incidences of the representing points and lines.

Let us give another theorem about the 7-point geometry. We have already used the term triangle to refer to the 3-point geometry. Let us say, in general, that three points in a geometry form a *triangle* if they do not lie on a line.

THEOREM 5. *Let L, M, N be three distinct lines in a 7-point geometry intersecting at a point P. Let A, B, and C be points on L, M, and N, respectively, such that A, B, and C form a triangle. Let A', B', and C' be points on L, M, and N, respectively, that are different from A, B, C, and P. Then A', B', C' are collinear, i.e. they lie on a line.*

Proof. Let us denote the lines determined by $\{B, C\}$, $\{C, A\}$, and $\{A, B\}$, by L', M', and N', respectively. Then we easily check that L, M, N, L', M', and N' are six distinct lines. For example, if L were equal to N', then B would be on L, so that $L \cap M$ would contain P and B, two distinct points; which is impossible.

Now consider the line K, say, determined by A' and B'. This is different from each of L, M, N, L', M', and N', since otherwise K would have four distinct points on it, namely, A' and B', and two of A, B, C, P, each of which is distinct from A' and B'. Hence K is the unique seventh remaining line of the geometry. Similarly, the line determined by B' and C' is also this line. Hence A', B' and C' are collinear.

The 3-point geometry and the 7-point geometry are each examples of what are called *finite projective geometries*. There is in fact a finite projective geometry with $n^2 - n + 1$ points and $n^2 - n + 1$ lines, for each $n \geqslant 2$, $n = 2$ giving the 3-point geometry, and $n = 3$ giving the 7-point geometry. In the $(n^2 - n + 1)$-point geometry each line has precisely n points and each point has precisely n lines on it.

Because of the dual nature of points and lines satisfying axioms PI and PII, any result that can be proved for projective geometries, where proof depends solely on PI and PII, and on any other dual pairs of properties, or self-dual properties, has a corresponding *dual* result. This is the so-called "Principle of Duality" for projective geometry. For example, in each of the 3-point and 7-point geometries the constructions given had, corresponding to each true relation $P_i \in L_j$, i.e. P_i lies on L_j, the corresponding true dual relation L_i lies on P_j, i.e. $P_j \in L_i$. Hence everything on which the proof of Theorem 5 rests, has a dual that holds. Hence, without further proof, we have the dual theorem (in which a *trilateral* is three lines whose intersections form a triangle):

THEOREM 6. *Let P, Q, and R be three distinct points in a 7-point geometry lying on a line N. Let K, L, and M be lines on P, Q, and R, respectively, such that K, L, and M form a trilateral. Let K′, L′, and M′ be lines on P, Q, and R, respectively, that are different from K, L, M, and N. Then K′, L′, M′ are concurrent, i.e. they lie on a point.*

The reader is advised to draw pictures to illustrate Theorem 5 and its dual Theorem 6, or alternatively to interpret them on the diagram we have given of the 7-point geometry. The theorems apparently state essentially different properties of the geometry. Yet separate proofs are not required: one follows from the other solely by appeal to the principal of duality.

Here is the 13-point $(4^2 - 4 + 1 = 13)$ geometry. Its points are $P_1, P_2, \ldots, P_{13}$ and its lines $L_1, L_2, \ldots, L_{13}$ are defined by

$$
\begin{aligned}
L_1 &= \{P_1, P_2, P_3, P_4\}\\
L_2 &= \{P_1, P_5, P_9, P_{11}\}\\
L_3 &= \{P_1, P_7, P_8, P_{12}\}\\
L_4 &= \{P_1, P_6, P_{10}, P_{13}\}\\
L_5 &= \{P_2, P_5, P_6, P_7\}\\
L_6 &= \{P_4, P_5, P_{10}, P_{12}\}\\
L_7 &= \{P_3, P_5, P_8, P_{13}\}\\
L_8 &= \{P_3, P_7, P_{10}, P_{11}\}\\
L_9 &= \{P_2, P_{11}, P_{12}, P_{13}\}\\
L_{10} &= \{P_4, P_6, P_8, P_{11}\}\\
L_{11} &= \{P_2, P_8, P_9, P_{10}\}\\
L_{12} &= \{P_3, P_6, P_9, P_{12}\}\\
L_{13} &= \{P_4, P_7, P_9, P_{13}\} .
\end{aligned}
$$

Notice that we have arranged once again that

$$P_i \in L_j \text{ if and only if } P_j \in L_i .$$

We leave the reader to verify that PI and PII are satisfied.

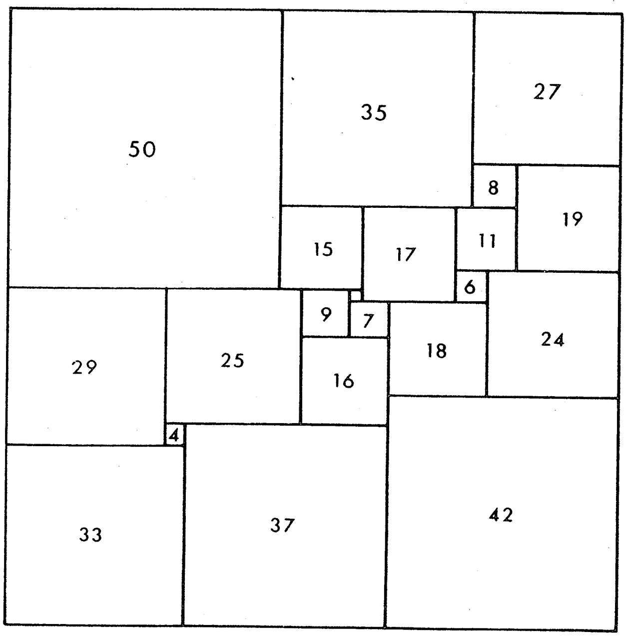
Does the analogue of Theorem 4 hold for the 13-point geometry?

Does the analogue of Theorem 5 hold for the 13-point geometry?

Finally, for the reader who wants to find out more about projective geometry I recommend strongly the very well written book - it requires no previous knowledge for its reading - by C.W. O'Hara and D.R. Ward: *An Introduction to Projective Geometry*, Oxford University Press. This book was written for a sixth-form (12th year) audience.

* * * * * * *

A PERFECT SQUARE



A "perfect square" is a square which has been partitioned into smaller squares all of different size. Up to about 1940, it was thought that this was impossible to achieve. Later, it was shown that at least 21 smaller squares would be necessary. The diagram above was discovered by A.J.W. Duijvestijn at Eindhoven in 1978. It contains exactly 21 unequal squares, whose side-lengths are indicated. The outer square has side 112 units. What is the size of the small square in the middle?

CONTINUED FRACTIONS

Rod Worley
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The first way one learns to find the greatest common divisor of two integers is to factorize both numbers into their prime factors and take the common prime factors. For example, to find the greatest common divisor of 45024 and 5712 we write

$$45024 = 2^5.3.7.67$$

$$5712 = 2^4.3.7.17$$

and so the greatest common divisor is $2^4.3.7 = 336$.

However finding the prime factors is not always easy, and there is a better way, known as Euclid's algorithm, of finding the greatest common divisor. To apply it we divide the smaller into the larger. 45024 divided by 5712 gives 7 with remainder 5040. Thus

$$45024 = 7 \times 5712 + 5040.$$

Now divide the remainder (5040) into the smaller (5712)

$$5712 = 1 \times 5040 + 672,$$

and the new remainder into the previous remainder, and so on:

$$5040 = 7 \times 672 + 336,$$

$$672 = 2 \times 336 + 0.$$

Then the last non-zero remainder is the greatest common divisor, because it divides both 45024 and 5712, and every divisor of both 45024 and 5712 must also divide it. (Can you show this?)

We rewrite the above equations as

$$45024/5712 = 7 + 1/(5712/5040)$$

$$5712/5040 = 1 + 1/(5040/672)$$

$$5040/672 = 7 + 1/(672/336)$$

$$672/336 = 2 \qquad (1)$$

which we combine to give

$$\frac{45024}{5712} = 7 + \cfrac{1}{1+\cfrac{1}{7+\frac{1}{2}}} \quad .$$

The expression on the right is called a continued fraction, and we write it either as $<7,1,7,2>$ or $7 + \frac{1}{1} + \frac{1}{7} + \frac{1}{2}$.

In what follows we shall use the symbols $[x]$ and $\{x\}$ to denote the integer and fractional parts of the number x . For example:

If $x = 3.27$ then $[x] = 3$ and $\{x\} = .27$.

If $x = 6.45$ then $[x] = 6$ and $\{x\} = .45$.

If $x = 7$ then $[x] = 7$ and $\{x\} = 0$.

Thus $[x]$ is the integer between $x - 1$ (excluded) and x (included), and $\{x\} = x - [x]$.

Looking closely at equations (1) we see that they can be written in the form

$$x_0 = [x_0] + \{x_0\}$$
$$x_1 = [x_1] + \{x_1\}$$
$$x_2 = [x_2] + \{x_2\}$$
$$x_3 = [x_3] + \{x_3\},$$

where $x_1 = 1/\{x_0\}$, $x_2 = 1/\{x_1\}$ *etc.*
This gives us the basic method of writing a number as a continued fraction.

Given a number x , form in turn

$$a_0 = [x], \quad x_1 = 1/\{x\}$$
$$a_1 = [x_1], \quad x_2 = 1/\{x_1\}$$
$$a_2 = [x_2], \quad x_3 = 1/\{x_2\}$$

and so on. In the case of a rational number x this process must stop when some $\{x_i\}$ becomes zero. However for an irrational number x, every $\{x_i\}$ is also irrational, so cannot be zero, and the process never stops. Thus a rational number x can be written as the continued fraction

$$x = \langle a_0, a_1, a_2, \ldots, a_n \rangle = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$$

and an irrational as

$$x = \langle a_0, a_1, a_2, \ldots, a_n, \ldots \rangle \ .$$

As an example, take $x = \sqrt{7}$

$$\sqrt{7} = 2 + (\sqrt{7}-2) \quad : \quad a_0 = 2,\ x_1 = \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{3}$$

$$\frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3} \quad : \quad a_1 = 1,\ x_2 = \frac{3}{\sqrt{7}-1} = \frac{3(\sqrt{7}+1)}{6}$$

$$\frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2} \quad : \quad a_2 = 1,\ x_2 = \frac{2}{\sqrt{7}-1} = \frac{2(\sqrt{7}+1)}{6}$$

$$\frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3} \quad : \quad a_3 = 1,\ x_3 = \frac{3}{\sqrt{7}-2} = \frac{3(\sqrt{7}+2)}{3}$$

$$\sqrt{7}+2 = 4 + (\sqrt{7}-2) \quad : \quad a_4 = 4,\ x_5 = \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{3}$$

and so on. Since $x_5 = x_1$ the process repeats indefinitely.

$$\sqrt{7} = \langle 2, 1, 1, 1, 4, 1, 1, 1, 4, \ldots \rangle$$
$$= \langle 2, \overline{1, 1, 1, 4} \rangle$$

where the bar indicates repeat.

It can be shown that a quadratic surd always has a continued fraction that eventually repeats indefinitely. Conversely it is also true that a continued fraction that repeats indefinitely represents the root of a quadratic equation. As a simple example, consider

$$x = \langle 1, 2, \overline{1, 2} \rangle \ .$$

We note that

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

can be written as

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{x}}$$

which gives $x = 1 + \frac{x}{2x+1} = \frac{1+3x}{2x+1}$, so

$$2x^2 + 1x = 1 + 3x$$

$$2x^2 - 2x - 1 = 0;$$

so x is the positive root of a quadratic equation. In fact

$$x = \langle \overline{1, 2} \rangle = \frac{1 + \sqrt{3}}{2} \ .$$

One reason for continued fractions being of interest is that if $x = <a_0, a_1, a_2, \ldots, a_n, \ldots>$ then taking a finite part $<a_0, a_1, \ldots, a_n>$ of this continued fraction gives a rational number $\frac{p_n}{q_n}$ which is a very good approximation to x. We can show

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2} \qquad (2)$$

[indeed, $\left| x - \frac{p_n}{q_n} \right| = 1/(a_{n+1} q_n + q_{n-1}) q_n$]. Take for example

$$\pi = <3, 7, 15, 1, 292, \ldots> .$$

Then the approximation.

$$\pi \doteq <3, 7> = \frac{22}{7}$$

gives an error less than $1/(15 \times 7^2) = 1/725 = .0013 \ldots$.
Even better is the approximation

$$\pi \doteq <3, 7, 15, 1> = \frac{355}{113}$$

which has an error less than $1/(292 \times 113^2) = \cdot 000\ 000\ 26 \ldots$.
For comparison we have, to 9 decimal places,

$$\pi = 3.141592653 \ldots$$

$$355/113 = 3.141592920 \ldots$$

$$22/7 = 3.142857142 \ldots$$

A particularly interesting continued fraction is

$$e = <2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, \ldots>$$

which, except for the initial 2, has a nice pattern.

While a continued fraction may be evaluated in the obvious manner, such as

$$<1, 3, 5, 4> = 1 + 1/(3+1/(5+1/4))$$

(evaluating the innermost bracket first) there is a more convenient way. First set up the table

n	a_n	p_n	q_n
		1	0
0	a_0	a_0	1

then fill in every subsequent line as follows:

(i) Put the next value of n in the column headed n.

(ii) Put the corresponding value of a_n in the column headed a_n.

(iii) Under p_n, put the value obtained by multiplying the bottom entry of the p_n column (which is p_{n-1}) by the value of a_n, and adding the second to bottom entry of the p_n column.
This means that $p_n = a_n p_{n-1} + p_{n-2}$.

(iv) Repeat the procedure for the q_n column.

An example is given, in which extra "working" columns have been inserted to help explain the procedure. We evaluate $<3,7,15,1>$.

n	a_n	p_n working	p_n	q_n working	q_n
			1		0
0	3		3		1
1	7	$(7\times3)+1$	22	$(7\times1)+0$	7
2	15	$(15\times22)+3$	333	$(15\times7)+1$	106
3	1	$(1\times333)+22$	355	$(1\times106)+7$	113

Then $<3,7,15,1> = p_n/q_n = 355/113$.

An interesting example is obtained when we take $<1,1,\bar{1}> = \frac{1+\sqrt{5}}{2}$ and evaluate the approximations $<1,1,1,\ldots,1>$ (n one's).

n	a_n	p_n working	p_n	q_n working	q_n
			1		0
0	1		1		1
1	1	$(1\times1)+1$	2	$(1\times1)+0$	1
2	1	$(1\times2)+1$	3	$(1\times1)+1$	2
3	1	$(1\times3)+2$	5	$(1\times2)+1$	3
4	1	$(1\times5)+3$	8	$(1\times3)+2$	5
5	1	$(1\times8)+5$	11	$(1\times5)+3$	8

and so on. Clearly both p_n and q_n come from the Fibonacci sequence (F_n) defined by

$$F_1 = 1 \qquad F_2 = 1$$

$$F_n = F_{n-1} + F_{n-2} \qquad (n \geqslant 2),$$

in which the first two terms are *1* and every term thereafter is the sum of the previous two terms. Because of (2), p_n/q_n gets closer to $\frac{1+\sqrt{5}}{2}$ as n gets larger. Thus the quotient F_n/F_{n-1} of consecutive terms of the Fibonacci sequence tends to $\frac{1+\sqrt{5}}{2}$ as n tends to infinity. Fibonacci sequences were discussed in *Function* Vol. 1, Part 1, 1977, by Christopher Stuart.

* * * * *

A Soothsayer's Lament ?

In Goscinny and Uderzo's *Asterix and the Soothsayer* (1972), the Romans apprehend a Gaulish "soothsayer", who predicts correctly (and quite by chance) that the next throw of a pair of dice will give the total *VII*. His prediction puts him in a spot, as the Romans are under orders to imprison all Gaulish soothsayers. He argues as follows:

> If I were a real soothsayer, I should have known that the dice would make *VII*, so I would have said *VIII*, and then you wouldn't have believed I was a real soothsayer because the dice said *VII* and not *VIII*!

But note a further subtlety (or is it an oversight by Goscinny?). He was given the instruction to predict the fall of the dice by picking a number between *I* and *XII*. Picking the number *I* would, of course, have assured him of getting the answer wrong. *VII* is the most likely outcome - a fact probably known to Goscinny.

* * * * *

Answers to p 14 Puzzles.

1. *100:* my kettle boils water at *100°C*.
2. Successive room numbers at the Dublin Hilton.
3. Any one for tennis?

PROBLEM SECTION

SOLUTION TO PROBLEM 4.2.1

What is the meaning, and value, of the continued fraction,

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}} \qquad ?$$

This continued fraction is the limit of the sequence

$$1,\ 1 + \frac{1}{1},\ 1 + \cfrac{1}{1 + \cfrac{1}{1}},\ 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}},\ \dots .$$

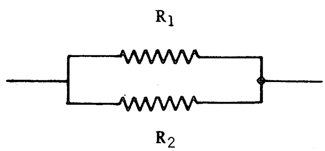
Ravi Sidhu (Townsville) noted that if we let x denote the value of the continued fraction, we can see that the fraction has the form

$$x = 1 + \frac{1}{x},$$

which leads to the quadratic $x^2 - x - 1 = 0$. The positive solution is $x = (1 + \sqrt{5})/2 = 1.618\dots$.

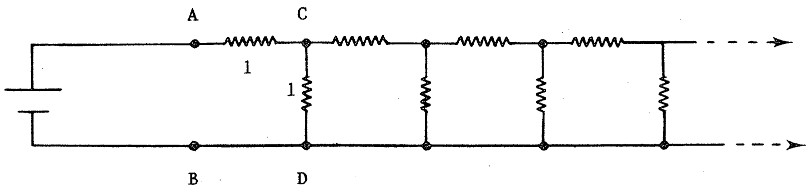
For a discussion of continued fractions generally, see the article by Rod Worley in this issue. It is interesting that Ravi's continued fraction is useful in studying electrical circuits. Two resistors in series, R_1 and R_2, have combined resistance $R_1 + R_2$, while two in parallel have combined resistance $1/(\frac{1}{R_1} + \frac{1}{R_2})$.

SERIES



PARALLEL

Now consider the infinitely long electrical circuit below, in which all resistors are *1* ohm.



If we let x denote the resistance between A and B, then x is the combined effect of AC and CD in series. But between CD, there are two circuits in parallel, one of *1* ohm resistance and one of resistance x again. Thus

$$x = 1 + 1/(\frac{1}{1} + \frac{1}{x}) .$$

This is exactly of the form of the continued fraction above, and $x = 1.618...$ (ohms). This idea came from Tom Morley (University of Illinois).

Solution to Problem 4.2.2

Eleven men toss their hats in the air; the hats are picked up randomly. Each man who received his own hat leaves, and the remaining men toss the hats again. The process continues until every man has received his own hat again. How many rounds of tosses are expected ?

If there are n men and n hats in a particular round, each man has probability $\frac{1}{n}$ of getting his own hat in that round. The expected number of hats to be correctly matched with owners in that round is $n \times \frac{1}{n} = 1$. Thus the expected number of rounds required to correctly match the initial *11* hats with *11* owners is *11*. [Although this solution may not be fully convincing, the answer is, in fact, correct!]

Solution to Problem 4.2.3.

Show that, whatever number base is used, *21* is not twice *12*. Find, for each possible number base not exceeding *10*, every instance of a number consisting of two different non-zero digits which is a multiple of the number obtained by interchanging the digits.

Let b be the base. Then $21 = 2 \times b + 1$ and $12 = 1 \times b + 2$. If $21 = 2 \times 12$, then $2 \times b + 1 = 2 \times (1 \times b + 2)$, that is, $1 = 4$, which is impossible no matter what the base b. The more general problem has the following solutions: $31 = 2 \times 13$ (base 5), $51 = 3 \times 15$ (base 7), $52 = 2 \times 25$ (base 8), and $71 = 4 \times 17$ (base 9).

PROBLEM 4.4.1.

There are seven persons and seven committees. Each committee is to have three persons. Can you share the committees out to the people so that each person is on the same number of committees ? Does this problem have any connection with the Seven-Point Geometry article in this issue ?

PROBLEM 4.4.2.

Determine the continued fractions for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{11}$, $\sqrt{23}$, and find which numbers equal the continued fractions

$$\langle 1, 3, \overline{1, 3} \rangle \quad , \quad \langle 1, 1, 2, 2, \overline{1, 1, 2, 2} \rangle .$$

(See the article Continued Fractions in this issue.)

PROBLEM 4.4.3.

In a Chinese game, six dice are tossed. Among various possible outcomes, "two pairs" (e.g. the dice might fall $2, 1, 5, 2, 5, 3$) are rated more highly than "one pair" (e.g. $2, 1, 5, 6, 5, 3$). What are the probabilities for getting "two pairs", "one pair"? Do you think the relative ratings are sensible ?

PROBLEM 4.4.4.

Can you arrange squares of sides $1, 4, 7, 8, 9, 10, 14, 15, 18$ to form a rectangle of sides 32 and 33 ? (Compare the Perfect Square article in this issue.)

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