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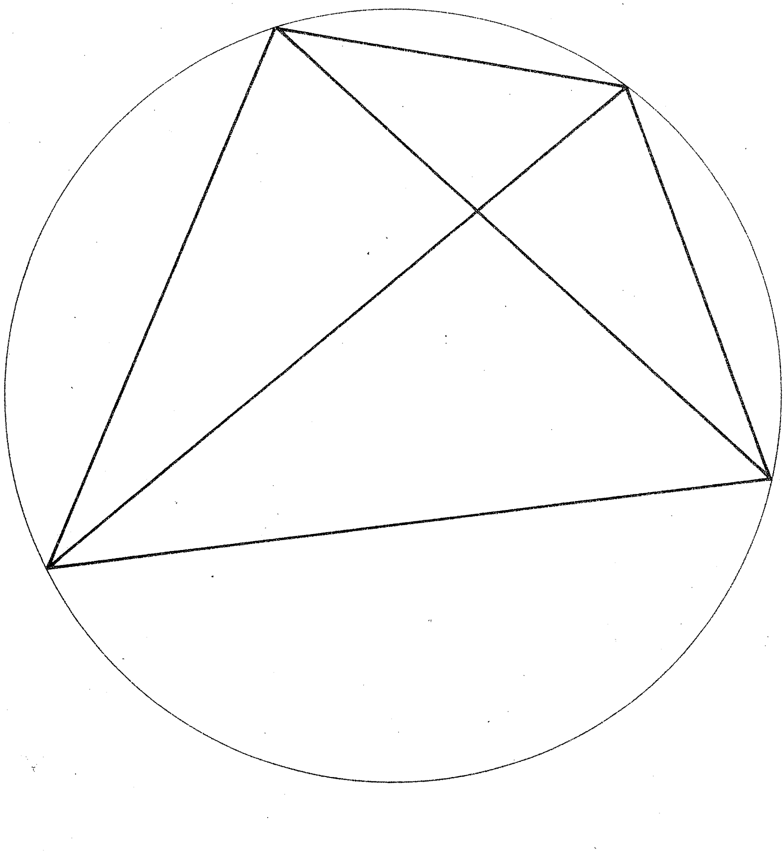

A SCHOOL MATHEMATICS MAGAZINE

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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The last issue concentrated largely on a single topic; this does the opposite. We have articles on a wide variety of subjects from complex numbers to negative mass. Then too there are the miscellanea, over 11 pages of them, from the numbers behind the space shuttle to what we think to be an unsolved problem. At last, too, we reveal the answer to Problem 3.3.5, with our special thanks to Mr Colin A. Wratten and Dr Rod Worley for their extremely thorough solutions.

CONTENTS

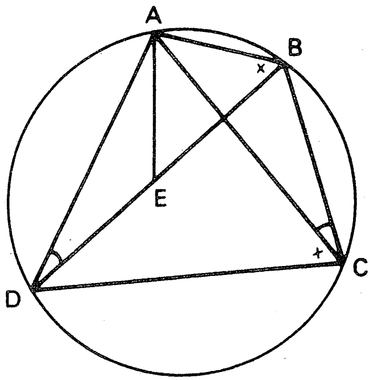
THE FRONT COVER

M.A.B. Deakin, Monash University

The cover diagram is the basic pattern behind a geometric result known as *Ptolemy's Theorem*, which states:

If a quadrilateral is inscribed in a circle, then the product of its diagonals equals the sum of the products of its opposite sides.

The theorem first appears in the *Almagest*, the main work of Claudius Ptolemy (100 A.D. - 178 A.D.). This is the astronomer after whom the Ptolemaic System (according to which the planets circled the earth) is named. Although this was later superseded by the Copernican, it was a great advance for its time.



In the notation of the diagram to the right, the theorem states:

$$AC.BD = AB.CD + BC.DA.$$

To prove it, we take the point E to be such that $\angle DAE = \angle BAC$. It follows that $\angle DAC = \angle EAB$. But, by a Euclidean theorem on angles in the same segment of a circle, $\angle ABE = \angle ACD$.

It follows quickly from these observations that the *triangles* EAB and DAC are similar (i.e. scale models of one another). Hence

$$BA/CA = BE/CD$$

or

$$BA.CD = CA.BE.$$

In the same way, using the similar triangles ADE and CAB, we have

$$BC.DA = AC.DE.$$

Adding, we find

$$\begin{aligned} AB.CD + BC.DA &= AC.BE + AC.DE \\ &= AC(BE + ED) \\ &= AC.BD, \end{aligned}$$

which proves the theorem.

THE INTRODUCTION OF COMPLEX NUMBERS†

John N. Crossley, Monash University

Any keen mathematics student will tell you that complex numbers come in when you want to solve a quadratic equation $ax^2 + bx + c = 0$ when $b^2 < 4ac$. However, if one tries to find out how they first came into mathematics then the surprising answer is that they were first introduced in the process of solving not quadratic but cubic equations. When Bombelli (1526-1572) first discovered what we call complex numbers he wrote (in translation):

"I have found a new kind of tied cube root very different from the others."

By a "tied cube root" he means a cube root expression like $\sqrt[3]{(2 - \sqrt{-121})}$ where there is a square root under a cube root.

How did he come to consider such monstrous expressions? And why weren't complex numbers introduced in the context of solving quadratic equations as they are today? It is the purpose of this paper to try to answer these questions.

The Greeks

It is often said that the Babylonians (in the second millennium B.C.) and the Greeks (no later than about 300 B.C.) knew how to solve quadratic equations. In fact, if you look at what they wrote then there are no traces of x's, no ideas of polynomial equations of degree 2, 3, etc. What we do find are problems of the form: A rectangle has two adjacent sides whose total length is 10 and its area is 24. What are the lengths of the sides?

We think of this as $a + b = 10$, $ab = 24$ and then move to the equation $x^2 - 10x + 24 = 0$, regarding a, b as the roots and using the fact that the sum of the roots is 10 and the product 24. But this was not the Babylonian or Greek way. The Babylonians gave a recipe (formula is not quite the right word in

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†This paper is a version of a talk given to the Singapore Mathematical Society on 23.3.1979 and published by them in their journal *Mathematical Medley* in June 1980. We thank the Singapore Mathematical Society for permission to reprint Professor Crossley's talk. Related articles will be found in *Function*, Vol.5, Part 1 and Vol.3, Part 5.

its modern sense), the Greeks a geometrical construction, to give the answers. The Babylonians never considered (so far as we know) cases where problems had complex solutions. The Greeks could never have produced a complex solution because their constructions produced actual lines and you cannot draw a line of complex length (even though we do now use Argand diagrams for representing complex numbers).

al-Khwarizmi

From at least the seventh century A.D. Hindu mathematicians treated (the equivalent of) quadratic equations and they explicitly said that negative quantities do not have square roots. Much of their mathematics was transmitted to the Arabs but, curiously, the Arabs did not, so far as we know, use negative quantities.

al-Khwarizmi (9th century A.D.), from whose name we get the word "algorism" or "algorithm", wrote the first book on algebra - indeed he called his book *Hisāb al-jabr w'almuqābala* (830 A.D.) and that is why we use the Anglicized form of al-jabr even today.

This book classifies equations and shows how to solve them. Since al-Khwarizmi did not use negative numbers he classified quadratic equations in the following sorts of way:

Square equal to numbers,
Square plus roots equal to numbers,
Square equal to numbers plus roots, etc.

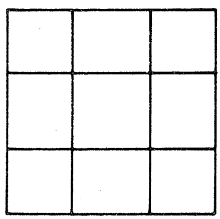
In our notation (with b, c positive):

$$x^2 = c,$$
$$x^2 + bx = c,$$
$$x^2 = c + bx.$$

He gave two kinds of solution. I shall call these geometrical and radical. Consider the equation

$$x^2 + 10x = 39$$

and consider the diagram



where the central square has side x and each oblong adjoining it has its other side $2\frac{1}{2}$. Then the figure without the corner squares has area $x^2 + 4(2\frac{1}{2}x) = x^2 + 10x$. If we add the corners, each of area $(2\frac{1}{2})^2$, we get the whole area to be

$$x^2 + 10x + 4(2\frac{1}{2})^2 = x^2 + 10x + 25.$$

But $x^2 + 10x = 39$ so $x^2 + 10x + 25 = 39 + 25 = 64$. Thus the area of the whole figure is 64 and, since it is a square, its side is 8. Hence $x = 8 - (2\frac{1}{2} + 2\frac{1}{2}) = 3$. This is the geometrical solution.

The radical solution is that given by (essentially) the well-known formula for solving a quadratic. Paraphrasing al-Khwarizmi we have "square half the coefficient of $x[(10/2)^2]$ and add the numbers [+39]. Total 64. Take its root, 8. Subtract half the coefficient of $x[10/2]$. Answer 3".

He neglected the other (in this case negative) root.

Omar Khayyam

Three hundred years later algebra had advanced considerably and cubics were being treated. It would appear that the work of Diophantos (who probably lived some time between 150 and 350 A.D.) had been rediscovered. Diophantos was sophisticated enough to consider not only cubes but also fourth, fifth, sixth powers. Just how much Diophantos influenced the Arabs we do not know, but Omar Khayyam (more famous for his Rubaiyat) wrote a book in which he gave a fine treatment of cubics.

The major breakthrough came from a theorem of Archimedes (The Sphere and the Cylinder,proposition 11.5). This gave a *geometric* method for solving a cubic equation. Omar Khayyam used this technique and said that if one wanted to solve a cubic equation then conics must come in (and not just ruler and compass). He did not justify this remark except practically. He solved cubics using properties of conics. What this amounts to is producing lines which solve cubic equations. Then, of course, if a numerical answer is required, the line must be measured.

Omar Khayyam's (not completely fulfilled) aim was: (a) to classify cubics and (b) to solve them all. He classified them in the same way as al-Khwarizmi, considering the various types such as

> cubes equal to squares, roots and numbers, ($x^3 = ax^2 + bx + c$ with a, b, c all positive).

He wished to give three types of solution; (i) geometrical solutions, (ii) radical solutions and (iii) integer solutions. He was successful in the first endeavour, for the geometric constructions he gave worked for all cubics (with real coefficients). He was not successful in the second, nor in the third. In the latter case he wished to find conditions on the coefficients which would ensure an integral solution. Diophantos employed similar considerations in his treatment of equations.

Not surprisingly, Omar Khayyam did not approach complex numbers for they had no place in the geometry.

The Italians

By about 1200 Arab culture was becoming better known in Europe. Fibonacci[†], otherwise known as Leonardo of Pisa, went across to North Africa where his father worked in the Customs. There he learned of Hindu-Arabic numberals (0,1,2,... ...,9) and he is generally regarded as one of the first to introduce these into Europe. He travelled a lot and learned a great deal of mathematics from the Arabs and in one of his works he did solve a cubic equation. He even showed that his equation $10x + 2x^2 + x^3 = 20$ did not have an integral nor a rational solution,[§] and he worked out an approximate answer to a high degree of accuracy (1.368 808). All this he did by Euclidean geometry, not using conics. Indeed, Omar Khayyam's work did not seem to have become known for a very long time.

From Fibonacci onward there were a lot of Italians working on algebra, but it was not until the end of the fifteenth century that great strides were made.

Luca Pacioli wrote a book in 1494 - the first printed book on algebra as opposed to arithmetic and in this book he spent a lot of time doing manipulations with square roots and cube roots. However, Pacioli was of the opinion that one could not solve cubic equations by radicals. (In a sense he was right: one cannot solve all cubic equations using only real roots even if one restricts the coefficients to be real.)

Shortly afterwards, probably around 1510 or slightly later, Scipio dal Ferro did solve cubics. Whether he knew how to solve all types is unclear. Basically his treatment was the modern one. After removing any x^2 term by an appropriate substitution (in $x^3 + ax^2 + bx + c = 0$ put $y = x + a/3$) one is left with either $x^3 = px + q$ or $x^3 + px = q$ where p, q are both non-negative. Neglecting the easy case of p or $q = 0$, in essence, Scipio's treatment was to compare the equation with $x^3 = (u^3 + v^3) + 3uvx$ which he obtained from $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$, where $x = u + v$.

The problem then is to find numbers u, v such that $u^3 + v^3 = q$ and $3uv = p$. If we write $a = u^3$, $b = v^3$ then the problem is to find numbers a, b such that $a + b = q$ and $ab = p^3/27$. But this problem is one whose solution was known to the Greeks, as we noted near the beginning of this paper. Having found a, b we have

$$x = \sqrt[3]{a} + \sqrt[3]{b}$$

where $a, b = +q/2 \pm \sqrt{(q/2)^2 - p^3/27}$. And now we see where Bombelli's tied cube roots come in.

[†] For more on Fibonacci, see *Function*, Vol.1, Part 1.

[§] For more on this point, see p.31 of this issue.

In fact Cardan published, in his Ars Magna of 1545, the solution of a quadratic equation with complex roots but Cardan regarded these as sophistic and useless. It appears from other writings that Cardan did not have any clear grasp of complex numbers and how they worked. It was left to Bombelli, who wrote his Algebra in the 1550's but did not publish it until 1572, the year of his death, to give a full, formal and clear treatment of these new numbers.

Bombelli was adept at manipulating expressions involving radicals. Presumably he employed the same rules for his tied cube roots and also performed calculations such as in (in our notation) $(2 + i)^3 = 2 + 11i$. In treating one cubic equation he came up with the expression $\sqrt[3]{(2 + \sqrt{0 - 121})} + \sqrt[3]{(2 - \sqrt{0 - 121})}$ from which he obtained $(2 + \sqrt{0 - 1}) + (2 - \sqrt{0 - 1})$ which equalled 4. What he then did was to substitute this solution back into his cubic and it worked!

Bombelli had also been suspicious of these new numbers but having employed them for a while, he came to accept them and overcame his misgivings. For him the proof of the pudding seems to be in the eating!

Even though Bombelli gave rules such as (in modern, but not very different, notation)

$$(+i).(+i) = -,$$
$$(+i).(-i) = +$$

it was still quite a long time before complex numbers were totally accepted. The final justification of the formula for the solution of the cubic did not come until 1686 when Leibniz showed by substituting the formal solution back in the cubic that the formula always gave a solution. By that time the use of letters for variables became common practice and this allowed a general treatment which was not possible a century or so before.

Conclusion

Thus we see that the geometric context of problems which we regard as polynomial equations militated against the introduction of complex numbers for a very long time, and it was not until a more 'algebraic' approach was adopted and the solution of cubic equations was given a recipe or 'formula', that it transpired that numbers formally defined did lead to solutions - even when, in the intermediate stages, those numbers were imaginary. Even then it was a long time before these new numbers were formally justified - first by Leibniz in the sense that they did give proper solutions of the cubic and later, in the nineteenth century, by Hamilton when he reduced complex numbers to pairs of real numbers with specially defined operations for addition and multiplication - but that is beyond the scope of our present essay.

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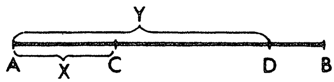
CAN A LINE SEGMENT BE CONVERTED INTO A TRIANGLE?

Ravi Phatarfod, Monash University

If you cut a stick into three pieces, you can form them into a triangle provided that each pair of pieces has combined length greater than the third. What is the probability of this happening when the two cutting points of the stick are chosen independently and at random? The 'standard' solution of this problem is fairly simple (and is given below); however, if we were now to extend the problem to taking n points at random and ask what is the probability that the $n + 1$ segments can form a $(n + 1)$-sided polygon, the solution to the problem is far from simple. The purpose of this article is to provide an alternative solution to the first problem, which can be very easily extended to the case of n points chosen at random.

First, let us consider the standard solution of the first problem.

Let AB be a straight line of unit length (clearly, the length of AB does not matter - so we take it to be of unit length), and let the points chosen at random be denoted by C and D. The variables X and Y, measuring the lengths of AC and AD respectively, are independent and have uniform distribution of probability (i.e. all values are equiprobable). In the figure shown, C precedes D and so $X < Y$, but we could just as well have D preceding C, in which case $Y < X$.

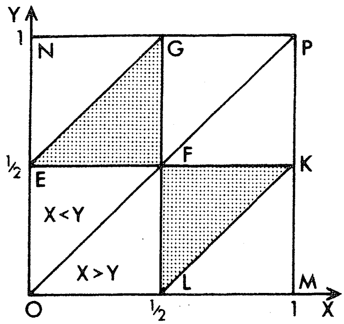


The domain of all possible positions of the point (X,Y) is a square $OMPN$ with the sides $OM = ON = 1$. Let us find the positions of the point (X,Y) when the segments can form a triangle. First, let us suppose C precedes D; this means we have $X < Y$ and we are restricting attention to the region OPN. The lengths of the three segments are $AC = X$, $CD = Y - X$, and $DB = 1 - Y$. Then, in order that AC, CD and DB can form a triangle, the following inequalities must be fulfilled:

$$X + (Y - X) \geqslant 1 - Y \quad \text{i.e.} \quad Y \geqslant \tfrac{1}{2},$$

$$X + (1 - Y) \geqslant Y - X \quad \text{i.e.} \quad Y \leqslant X + \tfrac{1}{2},$$

$$(Y - X) + (1 - Y) \geqslant X \quad \text{i.e.} \quad X \leqslant \tfrac{1}{2}.$$

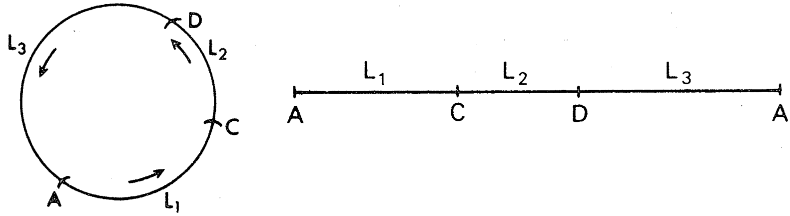


This means (X,Y) belongs to the triangle *EFG*. If on the other hand, D precedes C, we get by symmetry the triangle *FKL*. The required probability is

$$\frac{\text{Area } EFG + \text{Area } FKL}{\text{Area } OMPN} = \tfrac{1}{4}.$$

The extension even to the case of 3 cuts is rather tricky. Four segments can form the sides of a quadrilateral provided each triple of segments have combined length greater than the fourth. If X,Y,Z are the distances of the cut-points from A, we have to consider the six congruent regions of the unit cube defined by $X < Y < Z$ and similar inequalities obtained by permutations of X, Y and Z, and within each such region determine the subset of the points (X,Y,Z) such that the condition for the formation of a quadrilateral is satisfied.

To obtain the alternative solution, we first consider the concept of points chosen at random on a circle. Consider three points, A, C, and D chosen independently and at random on a circle of unit circumference.



Imagine now the circle being cut at the point A, and the circle laid out as a straight line. We have three intervals formed, AC, CD, and DA, and the situation is identical to that of choosing two points C, D at random on a straight line of unit length. We can therefore state the following (somewhat loosely worded) proposition.

PROPOSITION 1. *Taking two points at random on a straight line of unit length is equivalent to taking three points at random on a circle of unit circumference.*

Now let the lengths of the three intervals formed by choosing two points at random on a straight line of unit length be denoted by L_1, L_2 and L_3. The next proposition is about the probability distribution of the three lengths L_1, L_2, L_3.

PROPOSITION 2. *The lengths of the three intervals, L_1, L_2, L_3 have the same probability distribution, namely for $0 < t < 1$,*

$$Pr\{L_i > t\} = (1 - t)^2, \quad i = 1,2,3. \tag{1}$$

This result is surprising, since intuitively we might have expected that the length of the middle interval, L_2 would have a distribution different from the one for the end intervals L_1, L_3. However, if we consider the equivalent situation of choosing three points at random on a circle and cutting the circle at the point A, then from reasons of symmetry, the lengths of the three intervals would have identical probability distributions. To obtain the distribution, we note that $L_1 > t$ if and only if C and D fall in the interval $(t,1)$, and the probability of this event is $(1 - t)^2$.

We now give the alternative solution of the problem posed at the beginning of the article. First, note that the condition that the three segments can form a triangle is equivalent to the condition that none of the L_1, L_2 or L_3 exceeds $\frac{1}{2}$. Now using (1) we have

$$Pr\{L_1 > \tfrac{1}{2}\} = Pr\{L_2 > \tfrac{1}{2}\} = Pr\{L_3 > \tfrac{1}{2}\} = \tfrac{1}{4}.$$

Now comes the important point: the events $\{L_1 > \frac{1}{2}\}$, $\{L_2 > \frac{1}{2}\}$, and $\{L_3 > \frac{1}{2}\}$ are *mutually exclusive* (no two of them can occur together). Hence,

$$Pr\{\text{some } L_i > \tfrac{1}{2}\} = Pr\{L_1 > \tfrac{1}{2}\} + Pr\{L_2 > \tfrac{1}{2}\} + Pr\{L_3 > \tfrac{1}{2}\} = \frac{3}{4}.$$

Now since the event $\{\text{no } L_i > \frac{1}{2}\}$ is the complement of the event $\{\text{some } L_i > \frac{1}{2}\}$, we deduce that the required probability that the three segments form a triangle is $1 - \frac{3}{4} = \frac{1}{4}$.

It is now easy to see how we can generalise the result. We have, first of all, the proposition that choosing n points independently and at random on a straight line of unit length is equivalent to taking $n + 1$ points at random on a circle of unit circumference. Also, the lengths of the $n + 1$ intervals so formed, $L_1, L_2, \ldots, L_{n+1}$, have identical distributions, and

$Pr\{L_i > t\} = (1 - t)^n$ for $0 < t < 1$, and $i = 1,2,\ldots,n+1$, so that putting $t = \frac{1}{2}$, we have $Pr\{L_i > \frac{1}{2}\} = (\frac{1}{2})^n$, $i = 1,2,\ldots,n+1$. We also note that, as before, the events $\{L_i > \frac{1}{2}\}$ $(i = 1,2,\ldots n+1)$ are mutually exclusive, so that $Pr\{\text{some } L_i > \frac{1}{2}\} = (n + 1)(\frac{1}{2})^n$. This gives us the probability of the event $\{\text{no } L_i > \frac{1}{2}\}$, (the condition for the $n + 1$ segments to form a polygon) to be equal to $1 - (n + 1)(\frac{1}{2})^n$.

Consider this formula when $n = 1,2,3$. We get the answers $0,\frac{1}{4},\frac{1}{2}$ respectively. Of course, for the case $n = 1$, there is no question of forming a figure of 2 sides. The answer (zero probability) is the probability of getting both the segments of length less than $\frac{1}{2}$. As n gets larger and larger, the answer, given by the formula, approaches one. This implies that as we cut the line segment at random into more and more parts, it is more and more certain that all the segments would be of length less than $\frac{1}{2}$, i.e. that the $n + 1$ segments would form a polygon.

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A PROBLEM IN GRAVITATION†

P. Baines, CSIRO

Question. What would be the behaviour of two particles in proximity, one of which has *negative mass*?

Answer. Neither Newtonian Mechanics nor General Relativity Theory precludes the existence of negative mass, and although it has never been observed, the concept leads to some interesting results.

Suppose we have two particles, masses m_1, m_2 where $m_1 < 0$, $m_2 > 0$, and apply Newtonian mechanics.

Then the force F on the first particle is given by

$$F = G\frac{m_1 m_2}{r^2},$$

where r is the distance between them and G is the universal gravitational constant. The m_1 to m_2 direction is regarded as positive. Let a_1, a_2 be the accelerations of the first and second particles respectively. The equations of motion then are

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†This article was first published 20 years' ago, in Dr Baines' student days. It appeared in the journal *Matrix*, published by the Melbourne University Mathematical Society. It is reprinted here with the permission of the author and the publishers. We have slightly altered the notation to accord with that now used.

$$m_1a_1 = G\frac{m_1m_2}{r^2}$$

$$m_2a_2 = -G\frac{m_1m_2}{r^2}$$

(the minus sign in the second equation coming from Newton's third law), and so we get two *positive* accelerations:

$$a_1 = Gm_2/r^2$$

$$a_2 = -Gm_1/r^2.$$

Hence if we consider the particles to be initially at rest, both will subsequently move in the *same* direction, and if the masses are equal in magnitude, the negative mass particle will take off after the other with a (common) constant acceleration.

Subsequent analysis of the problem shows that if $m_2 < -m_1$ the distance r steadily increases, while if $m_2 = -m_1$ the distance r is constant, as noted above, and if $m_2 > -m_1$ the distance r will decrease and the particles will collide. Subsequent behaviour is left as an exercise for the imagination of the reader.

The above assumes the equality of inertial and gravitational mass, both positive and negative, and General Relativity (which takes this as a principle) leads to the same results.

In conclusion we may notice that it is not possible for all observed mass to be negative instead of positive, since negative mass repels all other mass (both positive and negative) while positive mass attracts. In a universe containing only negative mass therefore, all gravitational forces would be repulsive.

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BARNES-MATH

According to Earle Hackett (*The Body Programme*, *3AR* 30.5.1981), the Dorset poet William Barnes sought to re-Saxonize the English language. Hackett quoted two mathematical examples (Pythagoras' theorem and the pons asinorum - *Function*, Vol.3, Part 3) of Barnes-speak, and we were tempted to try another. Here is our cover theorem in what may resemble what Barnes had in mind.

An inrounded fourwinkel's pairwise twixtfolded acrosslying sides' sum is its crossbraces' tweenfold.

Or words to that effect.

Barnes-speak had a brief vogue among super-patriots during World War II. (E.g. "pushwainling" for "perambulator"). They apparently did not recognise the fact that it is structurally and verbally more Germanic than present-day English.

THE RELIABILITY OF A WITNESS

Doug Campbell, Monash University

When a witness appears in a court of law, the judge and jury will have to consider both

* the evidence that he gives, and
* how reliable the witness seems to be.

While it is rarely possible to present statistical evidence to assist in assessing how reliable a witness is, it is an open question as to whether statistical evidence would be properly interpreted if it were available.

Consider the following situation.

1. In a certain city, 85% of taxis are blue and the other 15% are green.

2. One night, a taxi was involved in an accident, in conditions of poor visibility.

3. The only witness to the accident was Fred.

4. Fred had been tested, and was found to be 80% reliable in telling the colour of a taxi in those weather conditions.

5. Fred said the taxi in the accident was green.

What reliability can be attached to Fred's evidence?

(a) Fred is said to be an "80% reliable" witness. This may mean that, as a reporter, Fred is right 80% of the time, whichever colour was the taxi, i.e.

> If he sees a blue taxi, he will report that it was blue with a probability of 0.80, and he will get it wrong and report that the taxi was green with a probability of 0.20.
>
> If he sees a green taxi, he will report that it was green with a probability of 0.80, and that it was blue with a probability of 0.20.

(b) A taxi drives past Fred. What will he report?

> If a blue taxi drives past, there is a probability of 0.80 that Fred will report "blue" and of 0.20 that Fred will report "green".

The probability that a taxi is blue is 0.85. Hence

Pr(Fred will call a taxi "blue" and the taxi is blue)

$$= 0.85 \times 0.80 = 0.68,$$

and Pr(Fred will call a taxi "green" and the taxi is blue)

$$= 0.85 \times 0.20 = 0.17.$$

Similarly,

Pr(Fred will call a taxi "green" and the taxi is green)

$$= 0.15 \times 0.80 = 0.12,$$

and Pr(Fred will call a taxi "blue" and the taxi is green)

$$= 0.15 \times 0.20 = 0.03.$$

So

Fred reports as	Taxi is Blue	Taxi is Green	Total	Proportion of observations correct
Blue	.68	.03	.71	$\frac{.68}{.71} = .96$
Green	.17	.12	.29	$\frac{.12}{.29} = .41$

From this, we conclude that, if Fred says the taxi was blue, he is 96% reliable, whereas if he says it was green he is only 41% reliable.

So much for our 80% reliable observer!

In case you find the law too abstract, consider an analogous situation in medicine.

There are two fatal diseases, *A* and *B*, which give rise to identical symptoms.

Of people with those symptoms, 85% suffer from disease *A*, and 15% from disease *B*.

The only known method of distinguishing between disease *A* and disease *B* is a blood test, but it is known that this test is only 80% reliable.

The treatment for disease *A* will bring about a complete cure, for a patient suffering from disease *A* but will shorten the life-expectancy of a patient suffering from disease *B*, and vice versa.

If the blood test tells you that you have disease *B*, what treatment (if any) would you request?

This article is based on a seminar presented by Jonathon Cohen at the University of Melbourne and discussions with Professor Frank Jackson and others.

LETTER TO THE EDITOR

A CARD TRICK

Nine cards are dealt face down. Somebody looks at one of the cards and places it on top of the pack of nine cards. These are then placed under the deck, face down. The dealer now makes four piles of cards as follows. He turns cards up counting 10, 9, 8, 7, ... and stops if one of the numbers he calls coincides with the card. If there is no coincidence by the time he reaches 1, he puts a card face down on this pile. In this manner he builds four piles and counts out that many cards. The last card he turns up is the card that was originally selected and put on top of the pile of nine cards.

Here is why this trick works.

If there is coincidence at a in a pile, the number of cards in the pile is $11 - a$. If the pile goes to completion, it has 11 cards. This corresponds to $11 - 0$. The sum of the numbers appearing is $a + b + c + d$. The number of cards in the four packs is $11 - a + 11 - b + 11 - c + 11 - d$ which equals $44 - (a + b + c + d)$. On counting $a + b + c + d$, the card arrived at will be the 44th card in the deck which is the card on top of the pack of 9, there being eight cards remaining below it.

Garnet J. Greenbury,
123 Waverley Road, Taringa, Q'ld.

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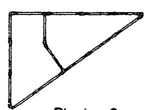
MATCH TRICK NO.17

A number of readers have shown interest in the match puzzles circulating widely in the community. Through the courtesy of the Wilkinson Match Company, we have been issued with a complete set of these and will discuss some of the mathematically more interesting ones in *Function*. Here we consider No.17 (reproduced at right). The main point of mathematical interest here is that the two matches internal to the triangle do indeed have the properties stated or implied in the directions given. That is to say, they are perpendicular to the top side and to the hypotenuse as shown and do indeed just meet. These statements can be verified using relatively straightforward coordinate geometry. That the area is bisected can best be verified by taking the right hand region, bisecting it along its axis of symmetry and so calculating its area.

MATCH TRICK No. 17



Placing 2 matches as shown divide the triangle into halves (2 areas of 3 units each). Using 3 matches instead of 2 divide the area into 2 halves, each with 3 units.

For the solution to the initial problem, turn to p.32.

PROBLEM SECTION

SOLUTION TO PROBLEM 3.3.5

This problem has been outstanding for two years, and was restated in Vol.5, Part 1. It read:

Consider the set $\{2^n$, where $0 \leqslant n \leqslant N\}$ - i.e. the first $N + 1$ powers of 2. Let $p_N(a)$ be the proportion of numbers in this set whose first digit is a. Find $\lim_{N\to\infty} p_N(a)$. Is the first digit of 2^n more likely to be 7 or 8?

Two readers supplied answers (a bit too long to publish): Colin A. Wratten of Bentleigh High School and Rod Worley of Monash (who incorporated his solution in a general article). The limit requested is given by

$$\lim_{N\to\infty} p_N(a) = \log_{10}(a + 1) - \log_{10}(a) = \log_{10}(1 + \frac{1}{a}). \qquad (*)$$

Mr Wratten notes that if 2 in the problem is replaced by any rational number other than a power of 10, the equation (*) still holds.

Equation (*) enables us to answer the second question. We have:

a	1	2	3	4	5	6	7	8	9	Total
$\lim_{N\to\infty} p_N(a)$	.301	.176	.125	.097	.079	.067	.058	.051	.046	1.000

Thus the first digit in 2^n is more likely to be 7 than 8, but 1 is the *most* likely digit.

Both readers gave full proofs of Equation (*), which we will not attempt here, although we will present a plausible argument. $\log(1 + \frac{1}{a})$ is a good approximation to $p_N(a)$ even for quite moderate values of N. Ralph Raimi, in an article published in *Scientific American* (Dec. 1969), gives the following table for $p_{99}(a)$, which you can reconstruct with a calculator.

a	1	2	3	4	5	6	7	8	9	Total
$p_{99}(a)$	.30	.17	.13	.10	.07	.07	.06	.05	.05	1.00

which approximates well the one above.

To justify Equation (*), replace, as Mr Wratten does, 2 by β and suppose that

$$\beta^n = d_1.d_2d_3...d_k \times 10^m \quad \text{for some } m.$$

Then

$$n \log \beta = m + \log d_1.d_2d_3...d_k.$$

(Here, and below, we assume that all logarithms are to the base 10.) m (the *exponent*) is of no interest as it does not affect the initial digit d_1.

First take a case for which $\log \beta$ is rational, e.g. $\beta = \sqrt{10}$, $\log \beta = \frac{1}{2}$. Successive powers of β begin with 1,3,1,3,..., and this "two-cycle" is readily seen to be characteristic of cases where $\log \beta$ has denominator 2. Similarly if the denominator is 3 a "three-cycle" results, etc.

If we lay out the interval $[0,1)$ on a logarithmic scale, we find that the points of such cycles are equidistant. A two-cycle has a point at the left-hand end 0, which corresponds to an initial digit 1 (as $\log 1 = 0$) and a point mid-way along, which corresponds to an initial digit 3, as $\log \sqrt{10} = \frac{1}{2}$, and $\sqrt{10} = 3.162...$. For larger denominators the same even sectioning of the interval persists.

Thus for $\log \beta = 0.301\ 029\ 995\ 1$ (an excellent approximation to $\log 2$), we would have a 10^{10}-cycle splitting the interval up into 10^{10} equal subintervals.

So far, we have given essentially Raimi's argument which differs only in that he wraps the interval around into a circle, and compares it to a calculating device, once widely used, the circular slide-rule. (See Ravi Phatarfod's article in this issue for another example of the trick of representing intervals by circles.)

The next step in the argument is the most difficult, although both our solvers negotiated it in all its gory detail. Assume (we prove it below) that $\log 2$ is irrational. Successive rational approximations to its true value give us cycles with more and more equal subintervals. It is plausible to assume that as we approach the actual value, the points separating these "smear" into a uniform distribution. (The actual proof is difficult because it only holds if the limit is irrational. As Dr Worley points out, other sequences can converge to *rational* numbers and the cycles suddenly collapse!)

Now the point representing 2 will be $\log 2$ of the way along this interval and will mark off $\log 2$ of its length. This much corresponds to initial digit 1. An initial digit of 2 is found in the sub-interval between the points representing 2 and 3, and this has length $\log 3 - \log 2$, etc. The lengths transform easily into probabilities because the distribution is uniform.

It remains to show that $\log 2$ is irrational. Assume otherwise. Then for some pair of integers a, b, $\log 2 = a/b$. That is to say $10^{a/b} = 2$ or $10^a = 2^b$. But the left-hand side is divisible by 5 and the right-hand side is not, so we reach a contradiction.

Equation (*) is known as *Benford's law* and it is widely found in other contexts. At first sight, it seems absurd that the house-numbers in *Who's Who* or the areas of the world's major rivers should obey it, but this is a fact. A practical use of the law occurs in the minimization of computer round-off error. When two numbers are multiplied, it is the leading digits that control the estimate of the propagated error.

For more on this, see either Raimi's article or one by J.W. Sanders in our sister journal *Parabola*, Vol.17, No.1.

This is the only solution we publish in this issue. Our problems vary greatly in difficulty from this standard down to one solved by a nine-year old. We would like to see many more letters from you about our problems. Here are some more. Again they vary from the fairly simple to the quite difficult.

PROBLEM 5.3.1

An *Armstrong number of order* m is an m-digit number the mth powers of whose digits add up to the number itself. Thus 153 is an Armstrong number of order 3 as

$$1^3 + 5^3 + 3^3 = 153,$$

and each of the digits 1, 2, ..., 9 is an Armstrong number of order 1.

Prove: (a) There are no Armstrong numbers of order 2;

(b) There are only finitely many Armstrong numbers.

PROBLEM 5.3.2 (From a Soviet problem book, and submitted independently by Colin A. Wratten and Y.-T. Yu.)

Is $2222^{5555} + 5555^{2222}$ divisible by 7? (*Don't* try to use a calculator on this one!)

PROBLEM 5.3.3 (Also submitted by Colin A. Wratten.)

Prove

$$\sqrt{2 + \sqrt{2}} + \sqrt{2 - \sqrt{2}} < 2\sqrt{2}$$

and its generalisation

$$\sqrt[n]{n + \sqrt[n]{n}} + \sqrt[n]{n - \sqrt[n]{n}} < 2\sqrt[n]{n} \text{ for } n \geqslant 2.$$

PROBLEM 5.3.4

Let $S(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$.

Show that $S(n)$ is not an integer if $n > 1$.

PROBLEM 5.3.5 (Also from a Soviet problem book and submitted independently by Colin A. Wratten and Y.-T. Yu.)

Find *all* integer pairs whose sum and product are equal.

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BOOK REVIEW

Ian Stewart and John Jaworski (Eds). *Seven Years of MANIFOLD: 1968-1980.* (Shiva Publ. Ltd, 4 Church Lane, Nantwich, Cheshire, England.) U.K. Price £5. Australian Price approx. $15†.

Q: Why is a mouse when it spins?
A: The higher the fewer.

With such zany humour, John Jaworski, Ian Stewart, Ramesh Kapadia, *Cosgrove*, Tim Poston, Jozef Plojhar, Eve laChyl and a number of other onymous and pseudonymous editors, contributors and supporters put out the distinctive student journal MANIFOLD from the University of Warwick between 1968 and 1980, when the last MANIFOLD appeared.

The jokes and the mathematics came thick and fast. Poston's very simple gyro composed of two point mouses (and some connecting rods) led to a witty and elegant account of tops and gyroscopes.

MANIFOLD required an active, mentally alert involvement. *Seven Years* ... follows this tradition. It collects typical MANIFOLD articles from the entire period 1968-1980. As J^2 + *INS* remark, it is "in no sense a *best* of MANIFOLD"; rather it gives a "concentrated dose of the 'spirit of MANIFOLD'". That spirit was a jaunty, buccaneering, rollicking, iconoclastic, high-powered, plagiaristic, highly original one.

The mathematics is challenging - the reader is assumed to be interested. The aim is to introduce him to the sub-culture of Pure Mathematics; the method (to mix metaphors, and why not?) is to throw him in at the deep end and bury him under a rapid fire of stimulating if inane profundities.

Function has long had an exchange agreement with MANIFOLD - an arrangement under which we can pinch each other's material. We did it once (*Cosgrove* turned up in Volume 3, Part 5), do it this issue (p.21), and will do it again. Basically, however, our audiences are rather different. You graduate from *Function* to MANIFOLD. We hope that after a dose of *Function* (a much more sedate, almost a family, magazine) and other mathematical experiences, readers will appreciate the $[(7+\sqrt{1+48p})/2]$ - colour theorem, the instructions on how to knit a Klein bottle (though the result seems to need darning), the 15 new ways to catch a lion, and the mating dance of the Alexander horned spheres.

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† Six copies are to arrive at the Monash University bookshop.

The editors are broad-minded, and so include some lateral thinking exercises (stolen) from an Engineering prospectus. (Could you use a suction hovercraft on a ceiling?). However, they do tell of the fellow, who, walking through a university campus and seeing a sign saying CONTROL ENGINEERING, agreed, being a very pure mathematician.

MANIFOLD is now dead; God save MANIFOLD. Cosgrove died back in 1969, but the obituary made too much of this and he's little the worse for the episode. Meanwhile we have this volume, which despite its big value total of 2294 pages (well, if you count the right way) omits much. Even Poston's mouses - apart from the chap who turns up three times on the cover. We'll just have to have a sequel - called, perhaps, *The Same Seven Years of MANIFOLD: 1968-1980.*

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BARBEROUS MATHEMATICS

The cartoon opposite (from *Manifold 15*) is based on a paradox due to Bertrand Russell. It is well described in this excerpt from *The Argus Students' Practical Notebook No.5*.

"It seems that [a] baron told the barber [of his village] that everyone in the village MUST be clean shaven, and that the barber MUST shave everyone who did not shave himself. And the baron threatened death if the barber did not obey these orders.

Of course the barber was pleased to be given such a monopoly and accepted the terms gratefully. But when he had shaved all those who did not shave themselves he noted that his own whiskers were beginning to sprout.

He was just about to begin shaving himself when the baron and the executioner entered his shop. Then, in terror, he realised that if he shaved himself he would be shaving someone who shaved himself, and this would be against his contract. But if he did not shave himself he'd break his contract if he did not shave himself!

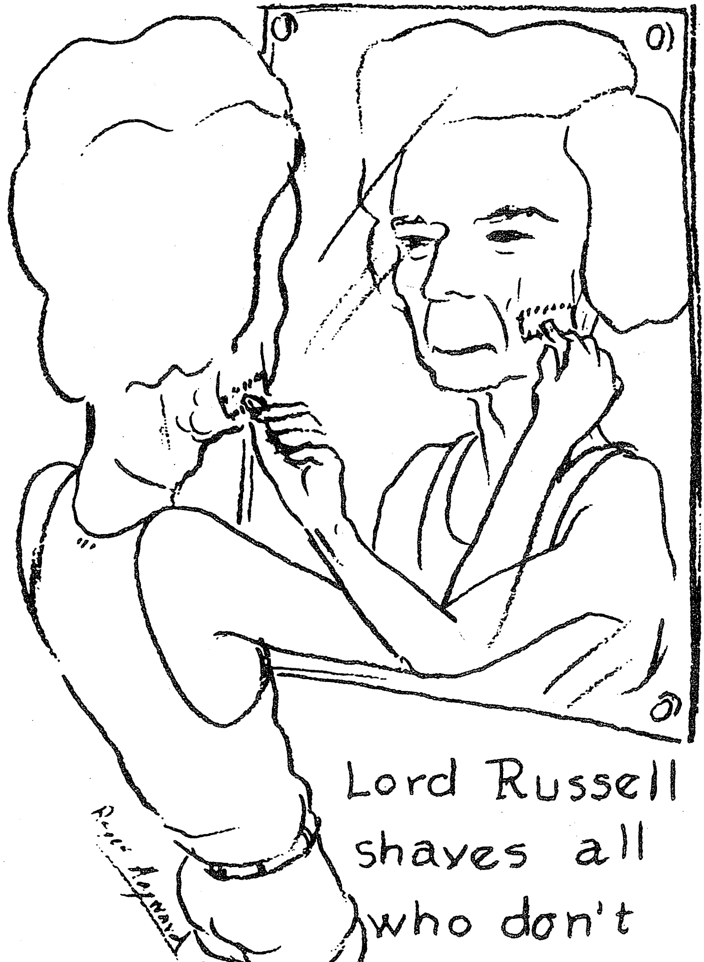
Here the problem is to help the barber decide how to resolve the difficulty. What do you suggest?"

The name *Occam* is a reference to the mediaeval philosopher William of Occam (1285-1349) best remembered for a methodological principle called *Occam's Razor*. This is stated (in Latin): "Entia non sunt multiplicanda praeter necessitatem". A rough translation runs "Don't complicate things needlessly". Nowadays we take it to refer to the principle that the simpler of two explanations is the preferable.

In preparing this note, the chief editor consulted *Webster's New World Dictionary of the American Language: College Edition* (1957) and found himself plunged into a capturing DO-loop of remarkable simplicity. The entry reads: Occam, William of *see* Occam, William of. He still doesn't know if this is a deliberate joke or not!

OCCAM'S BARBER

ROGER HAYWARD



YOU CAN'T ADD A SCALAR TO A VECTOR?

Student: I seem to have got stuck on this problem, sir.

Teacher: You stupid twit! How many times do I have to say it? *You can't add a scalar to a vector!*

A common enough conversation, one would have thought. Let's see what lies behind it. Matters are not as simple as the teacher would have us believe.

Two-dimensional space may be represented by means of number pairs. We later learn that such number pairs may be thought of as complex numbers. Thus the complex numbers and their algebra provide a natural tool for the investigation of plane geometry (which does prompt the remark that we rarely use them for this purpose!). However, our world of experience is three-dimensional, so that an extension of the complex numbers is needed if we are to have an algebra applicable to it. The obvious thing to do is to try to form an algebra of number triples $[a,b,c]$ or $a + bi + cj$.

We would like the triples $[a,b,0]$ to obey complex algebra, so we choose $i^2 = -1$. Now investigate ij - set it equal to $a + bi + cj$. Then

$$i(ij) = i(a + bi + cj),$$

and assuming $i(ij) = (ii)j$ we find

$$-j = ai - b + cij \qquad (*)$$

or

$$ij = \frac{b}{c} - \frac{a}{c}i - \frac{1}{c}j \text{ , unless } c = 0.$$

But

$$ij = a + bi + cj,$$

so that, equating coefficients of j, we find $c^2 = -1$, which is no help, as we are looking for real coefficients. (Besides, even if we give up this requirement, we only find $ij = ij$, which isn't much use either.)

Assume now that $c = 0$. Then (*) states that $j = b - ai$, so that j lies in the plane and we have a two-dimensional situation.

Thus we reach an impasse. In fact, it may be shown that if x,y,z represent triples of numbers such as $[a,b,c]$ no algebra of such triples can exist if we make the two natural requirements (where 0 stands for $[0,0,0]$):

(i) $xy = 0$ implies $x = 0$ or $y = 0$;
(ii) $x(yz) = (xy)z$.

Indeed, we might expect that any usable algebra of the real world would obey these requirements.

This was the situation that confronted the mathematician Hamilton when he crossed Brougham Bridge, Dublin, one evening in 1843, while walking with his wife. But then:

> "[Quaternions] started into life, or light, full grown ... . That is to say, I then and there felt the galvanic circuit of thought close, and the sparks that fell from it were the fundamental equations between i, j, k; *exactly such* as I have used them ever since. I pulled out, on the spot, a pocketbook, which still exists, and made an entry, on which, *at that very moment*, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come. But then it is fair to say that this was because I felt a *problem* to have been at that moment *solved*, an intellectual *want relieved*, which had *haunted* me for at least *fifteen* years before."

This account is Hamilton's own (as given in Crowe's *A History of Vector Analysis*). Bell (in *Men of Mathematics*) has Hamilton pulling out a pocket-*knife* and carving the basic table on the stone of the bridge, but this story (like much else in Bell, see p. 27) would seem to be apocryphal. In any case, a plaque now stands on the bridge and gives the multiplication properties of quaternions.

The problem is solved by admitting four dimensions, rather than restricting consideration to the impossible three. There are three square roots of -1, called i, j, k, so that

$$i^2 = j^2 = k^2 = -1.$$

Furthermore

$$ij = k, \; jk = i, \; ki = j,$$

which formulae are neatly summarisable as $ijk = -1$, the result given on the plaque.

The algebra has one peculiar feature: if the order of the terms in a product is reversed, the value may alter. E.g. $ij = -ji$. This is readily proved:

$$-ji = (ii)ji = i(ij)i = iki = i(ki) = ij.$$

You may care to explore matters further yourself.

Hamilton would have been aware of algebras that do not allow automatic reversal of products (so-called non-commutative algebras), so that his insight was essentially the admission of the extra dimension.

Hamilton, and later another mathematician, Tait, devoted much energy to the development of quaternions. The endeavour, indeed, occupied much of the rest of Hamilton's life. For Tait, the four-dimensionality of the result became the key to an intellectual development in which abstract results came to be seen as more important than the needs of the practical man on which they were based.

The general quaternion, $a + bi + cj + dk$, was seen as composed of two parts (cf. the real and imaginary parts of a complex number): the real or *scalar* part a, and the *vector* part $bi + cj + dk$. This latter, being three-dimensional, was of most interest for applications.

On this view, a vector is a quaternion of the special type $[0,b,c,d]$ or $bi + cj + dk$. If two such are multiplied together, there results:

$$[0,b_1,c_1,d_1]\,.\,[0,b_2,c_2,d_2] =$$
$$[-b_1b_2 - c_1c_2 - d_1d_2,\ c_1d_2 - c_2d_1,\ d_1b_2 - d_2b_1,\ b_1c_2 - b_2c_1],$$

a full quaternion, so that multiplication of pure vectors does not result in a pure vector product. (Much the same as multiplication of pure imaginary numbers yields a real number.)

Two physicists, Gibbs and Heaviside, cut this Gordian knot by defining two products of the vectors (b,c,d), as they called them. There was the scalar product:

$$b_1b_2 + c_1c_2 + d_1d_2$$

(note the omission of the minus sign), and the vector product:

$$(c_1d_2 - c_2d_1,\ d_1b_2 - d_2b_1,\ b_1c_2 - b_2c_1).$$

A roaring controversy ensued. Hamilton and Tait felt that an essential algebraic insight was being lost, that mathematical rigour was flying out the window, and that mere amateurs were attempting to take over the reins of advanced mathematics. Gibbs and Heaviside, for their part, saw their opponents as deliberate obscurantists. For them, vectors and scalars were quite distinct species, and even though you might mathematically justify their being added together, it showed lack of physical sense to do so. The eminent physicist Maxwell weighed in on their side with the grumble that he didn't see why he should yoke an ass to an ox to plough the furrow of Physics.

The participants indulged in marvellous invective, which makes splendid reading even today. Thus Heaviside:

> "'Quaternion' was, I think, defined by an American school-girl to be an 'ancient religious ceremony'. This was, however, a complete mistake. The ancients - unlike Professor Tait - knew not, and did not worship Quaternions ... . A quaternion is neither a scalar, nor a vector, but a sort of combination of both. It has no physical representatives, but is a highly abstract mathematical concept."

Nowadays, vectors are quite mathematically respectable, and it takes some work to understand what all the fuss was about. The dispute was very largely over questions of notation, but other more metaphysical matters did keep intervening. The whole matter seems somehow so *passé*, so Victorian, that we tend to forget how recent it is. One of the lesser and later participants, C.E. Weatherburn, a professor at the University of Western Australia, died as recently as 1974.

The completeness of the victory won by Gibbs and Heaviside may be judged from the conversation at the start of this article and its dogmatic conclusion: *You can't add a scalar to a vector*!

For Hamilton and Tait, you not only could, but you *had to*.

Oddly enough, the Minkowski 4-vectors of Special Relativity are close relatives of the quaternions, and even Maxwell's equations, whose formulation in vector terms so helped the anti-quaternionists, are expressible, using a slightly artificial device, as a single quaternionic equation. [*M.D.*]

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MATHEMATICAL SWIFTIES (AGAIN!)

The "Tom Swiftie" is a form of humour that had a vogue in the U.S. in the early sixties ("By the way, don't make the same mistake as I did with that big electric fan", remarked Tom offhandedly). In Volume 3, Part 3, we gave some background and a number of examples of a variant entitled the "Mathematical Swiftie". Some more examples followed in Volume 3, Part 4.

These were taken up by K.D. Fryer who edits the *Ontario Secondary School Mathematics Bulletin*. Professor Fryer provided twenty more examples, such as these.

"The coefficients of $ax^2 + bx + c$ are not numerical", said Tom literally.

"The number is divisible by 2", Tom stated evenly.

"You should know the shape of $y = |x|$ in the neighbourhood of the origin", declared Tom squarely.

"I'm no good at drawing three-dimensional diagrams", said Tom flatly.

"Only a matrix with the same number of rows and columns can have an inverse", Tom stated squarely. "But isn't it strange that not every such matrix will have an inverse", Jane replied degenerately. "It most certainly will not if its determinant has that value", Tom concluded naughtily.

"I'll have to change all the coefficients if you multiply by -1", noted Tom resignedly.

Some more examples were provided by another editor of the Ontario Bulletin, Steve Brown. Here are two samples.

"One, three, four, five, thixth, theven, ...", Tom counted toothlessly.

"The distribution of test scores has two modes", Tom noted abnormally.

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AUSTRALIAN MATHEMATICAL OLYMPIAD 1981

For a number of years now a group of countries has got together each year for what is known as the International Mathematical Olympiad. This consists essentially of submitting at most eight students from each country to the tortures of two four-hour mathematical examinations on consecutive days. Traditionally there are only a few questions on each paper and equally traditionally, the questions are extremely challenging. Most of us who manage to get one of these questions solved in a reasonable amount of time are more than satisfied. But the best students at the IMO solve more than one question.

This year for the first time Australia will send a team of eight when the IMO is held in Washington, D.C., in July. The team was chosen on the strength of their performance in the Australian Mathematical Olympiad examination which also consisted of two four-hour papers which students sat for in April this year.

Pushing back one stage further, students from all over Australia were invited to participate in the AMO as a result of their performance in State competitions such as the Melbourne University Mathematics Competition, and/or in the national Wales Competition.

Students identified as high flyers in these competitions were first invited to solve monthly sets of problems and discuss them with professional mathematicians. Then the best and keenest of these were asked if they would like to try the AMO examination.

So now they are eight. Of these, five are from New South Wales, two from Queensland and one from Victoria, whose lone representative is Ken Ross, a Year 11 student at Mount Scopus Memorial College. Congratulations and good luck from *Function*!

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THE MAGIC OF THE EAST

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TWO UPDATES

We like to keep our readers abreast of modern developments and so we update two stories run in earlier issues. Our first takes us back to Volume 1.

Problem 1.4.3 (as we would, in present notation, describe it) asked for a characterisation of functions such that $\{u(x)v(x)\}' = u'(x)v'(x)$. Two school students successfully solved the problem and a third correctly pointed that that E.T. Bell (in *Men of Mathematics*) claims that Leibniz (one of the founders of calculus) at one time thought this to be a universal rule.

The editors were sceptical, but the story *is* in Bell - so we ran it. Although Bell makes fun reading, he is unreliable, and here, as elsewhere (see p.23), he goofed. One editor (N.C.) looked into the matter, and writes:

"In a Leibniz manuscript of November 1675 he poses the question as to whether

$$d(uv) = (du)(dv)$$

and answers negatively by noting that

$$d(x^2) = (x + dx)^2 - x^2 = 2xdx + (dx)^2$$
$$= 2xdx$$

(ignoring the higher order infinitesimal) while

$$dxdx = (x + dx - x)(x + dx - x) = (dx)^2.$$

Here he is still searching for the correct product rule.

In a manuscript of July 1677 he states and 'proves' the product rule correctly."

More recently (*Function*, Vol.5, Part 1) we reported on some medical speculation concerning the other founder of calculus. It was hypothesized that Sir Isaac Newton suffered from mercury poisoning. A recent article in *Scientific American* (Jan. 1981) disputes this. The more recent claim is that Newton's illness was psychiatric in nature.

Here your editors adopt a neutral stance. Medical historians argue endlessly over other such cases (Charles Darwin, Henry VIII, the Philistines, etc., etc.). Doctors, we note, do disagree over the diagnosis of living patients with all the resources of 20th-century medicine at their disposal! [*M.D.*]

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NEW MATH IN THE THIRD WORLD

Depressingly many third world countries rushed their schools into trendy "New Maths" syllabuses quite unsuited to their needs. One of these, Kenya, abandoned the experiment recently. The move seems to have been a popular one as this excerpt from the *East African Standard* indicates. (The entire article is reprinted in *The Age*, 24.4.81.)

"Hats off to His Excellency President Daniel arap Moi! Hats off to the timely demise of new mathematics from the entire syllabi of standards I-III in all Kenya's primary schools and from form I in high schools with effect from 1 January 1981.

His Excellency, the President, a former schoolmaster who has always been aware of the fact that all the world's renowned mathematicians, in their heyday, had no desire whatsoever to complicate mathematics to the extent of making the grasping of the subject impossible for the youth of the Third World in order to catch up with the trend of "science teaching" and its intricacies, has saved this newly emergent nation from shame and embarrassment.

Parents - and their affected children - from all corners of the Republic, have been expressing genuine gratitude to President Moi for having ordered that, with immediate effect, the teaching of new mathematics should be scrapped from the syllabi.

When the new maths were being introduced in the Kenyan teaching system just over a decade ago, it was claimed that our schooling youth would be "brought into line" with those of the world's most developed nations. As mathematics was the common language used by shipbuilders, computer manufacturers and programmers, designers and manufacturers of modern armaments and other destructive devices, it was then genuinely suggested that the Kenyan youth should not be left lagging behind instead of "dashing forth to catch up with the marvels of the year 2000".

Hence we see the need to go on importing skilled personnel from abroad."

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A VIENNESE PUZZLE

Let a_0 be an odd number (say 3). Multiply it by 3 and add 1 (to get 10). Now divide out by b_0, the maximum power of two that will divide this new answer (here $b_0 = 2$). Call the result a_1 (here 5). Repeat the process to get a_2 ($3 \times 5 + 1 = 16$, $b_1 = 16$, $a_2 = 1$), and so on. The general formula is

$$a_{n+1} = \frac{3a_n + 1}{b_n} . \qquad (*)$$

Dr Hans Lausch passed this on to *Function*. He learned of it at the University of Vienna. The question is: Does the process always result (ultimately) in a string of ones? As far as we can determine, the problem is unresolved, although it seems likely that the answer is "yes". Clearly the number 1 generates itself by the formula (*) ($b_0 = b_1 = \ldots = 4$), and also (fairly clearly) no other number can be self-generating. We might envisage, however, some sequence of a_n increasing without limit, or locking into a cycle.

We took some small odd numbers and explored their behaviour. 1,3 have been dealt with above, and so, by implication, has 5. From $a_0 = 7$, we find:

$$7 \to 11 \to 17 \to 13 \to 5 \to 1 \to 1 \to 1, \text{ etc.}$$

And following on in this way, we can check the next calculations, omitting a_0 occurring in previous calculations.

$9 \to 7$, etc.

$15 \to 23 \to 35 \to 53 \to 5$, etc. ($b_3 = 32$)

$19 \to 29 \to 11$, etc.

$21 \to 1$ ($b_0 = 64$)

$25 \to 19$, etc.

$27 \to 41 \to 31 \to 47 \to 71 \to 107 \to 161 \to 121 \to 91 \to 137$

$\to 103 \to 155 \to 233 \to 175 \to 263 \to 395 \to 593 \to 445$

$\to 167 \to 251 \to 377 \to 283 \to 425 \to 319 \to 479 \to 719$

$\to 1079 \to 1619 \to 2429 \to 911 \to 1367 \to 2051 \to 3077$

$\to 577 \to 433 \to 325 \to 61 \to 23$, etc.

That last one looked like getting away for a while, didn't it? We leave the next cases (33,37,39, etc.) as an exercise for the reader. It is instructive to investigate calculations such as these in binary arithmetic.

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SHUTTLE NUMBERS

The following extracts from an article in *The Guardian* 13.4.1981 give some basic data on the space shuttle. The full article is entitled "Reaching for the sky" and is by Harold Jackson. We have not metricated his figures - sorry!

"The vehicle was supposed to be self-contained at first, but that idea has long gone. Its basic weight is 67 tons in its stockinged feet. To get this into the air needed an engine so large that there wasn't room to carry the fuel for it. So they added an external fuel tank weighing 35 tons empty (and which now gets thrown away in space after eight minutes' use). That in turn meant two booster rockets to help it into orbit - 86 tons apiece - though they float back on parachutes to be re-used. The object of this exercise is to propel this 274-ton mass vertically to a speed of 17 500 mph within 8½ minutes.

So up goes the weight again with 500 tons of solid propellant for each booster and 100 tons of liquid fuel for the Shuttle's own main engines. The vehicle itself is capable of carrying 29 tons of scientific experiments, space station modules, military satellites, or whatever. The grand total now soaking in gravity on the launch pad is 1403 tons.

To generate the thrust needed to get this lot on its way, the fuel has to be burned at the rate of 300 gallons a second in the main engines and more than 900 gallons a second in the boosters. The resultant blast exerts 5 750 000 lbs of pressure against the concrete below - over 2500 tons pushing against 1400. So there isn't that majestically slow rise characteristic of the Saturn launches in the Apollo programme. The Shuttle comes off its ramp like a flea off a rabbit.

Within seven seconds it rises more than 500 feet and, a minute later, it is at more than eight miles. By the time its fuel is expended and it has cast off both its boosters and the fuel tank, it is 73 miles up and 1400 miles down range. It will not win any awards for economy - it will have averaged one mile to every 179 gallons.

Once it is actually in orbit, of course, the ride is free. It can spin round the Earth as the whim of the ground controllers and the patience of the crew dictates. The operation of the vehicle has a childlike simplicity in many ways. Of course, it is equipped with some pretty elaborate gadgetry but, coming down to basics, the crew will be able to launch a new satellite simply by tossing it out of the window. (If you have a satellite you want in orbit, you can hire space in some of the flights after July 1985. The going rate is now $2650 a cubic foot.)"

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A THEOREM ON IRRATIONAL NUMBERS

This rather nice theorem yields many irrationalities straight away and may be proved by elementary means.

THEOREM. *Let* x *satisfy the equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0 \qquad (*)$$

where $a_1, a_2, \dots, a_n$ *are integers. Then* x *is either integral or irrational.*

PROOF. Suppose the theorem to be false. Then x will be rational, that is to say $x = p/q$ for some integers p, q. We assume that p/q is in its lowest terms, i.e. p, q have no common divisor other than 1. We also assume $q > 1$. We now show that these assumptions lead to a contradiction, and so prove the theorem.

Substitute p/q in place of x in Equation (*). Then

$$p^n = -q(a_1 p^{n-1} + a_2 p^{n-2} q + \ldots + a_n q^{n-1})$$

after multiplying through by q^n). Now suppose r is a prime divisor of q. It follows that r divides p^n. But this in turn means that r divides p. So r is the common divisor of p,q, namely 1. So $q = 1$, since every prime divisor of it is 1, contrary to our assumption.

It follows immediately that $\sqrt{2}$ is irrational for clearly $1 < \sqrt{2} < 2$, so that $\sqrt{2}$ is not integral. Similarly for $\sqrt{3}$, $\sqrt{5}$, etc. In fact we have the

COROLLARY. *If* N *is a positive integer, which is not of the form* m^n *(*m *being an integer), then* $\sqrt[n]{N}$ *is irrational.*

The theorem does not, however, tell us about the numbers e, π which are irrational but are also non-algebraic - i.e. cannot be roots of polynomial equations. An unsolved and difficult problem is whether $e + \pi$ is rational or irrational.

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MONKEY BUSINESS

"This law is extremely simple and intuitively evident, though rationally undemonstrable: *Events with a sufficiently small probability never occur;* or at least, we must act, in all circumstances, as if they were *impossible.*

A classical example of such impossible events is that of *the miracle of the typing monkeys,* which may be given the following form: A typist who knows no other language than French has been kept in solitary confinement with her machine and white paper; she amuses herself by typing haphazardly and, at the end of six months, she is found to have written, without a single error, the complete works of Shakespeare in their English text and the complete works of Goethe in their German text. Such is the sort of event which, though its *impossibility* may not be rationally demonstrable, is, however, so unlikely that no sensible person will hesitate to declare it *actually impossible.* If someone affirmed having observed such an event we would be sure that he is deceiving us or has himself been the victim of a fraud."

Probabilities and Life,
Émile Borel (1943).

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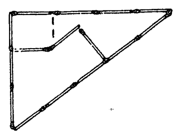
MATCH TRICK NO.17 (CONT.)

The solution to the match trick shown on p.15 is illustrated at right. The unit square at the top left has been transferred from the left-hand region to the right-hand one, while the converse transfer has occurred for a region, also a unit square, above the centre of the hypotenuse. Thus the total area is still exactly halved.

SOLUTION



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UNCONSCIOUS MATHEMATICAL CREATIVITY

The French mathematician *Hadamard*, towards the end of his life, researched and wrote on the mental processes involved in mathematical creativity and research. He published the results in a fascinating book titled *The Psychology of Invention in the Mathematical Field*. It is Hadamard's contention that much such discovery (or invention) is the product of our sub-conscious mind. This is now widely believed by mathematicians and psychologists alike.

In this excerpt from his book, he tells an interesting story which demonstrates that a creative insight may be the indisputable product of a mind which remains quite unconscious of the result. According to Hadamard, the observation is

> "... reported by the prominent American mathematician, Leonard Eugene Dickson, who can positively assert its accuracy. His mother and her sister, who, at school, were rivals in geometry, had spent a long and futile evening over a certain problem. During the night, his mother dreamed of it and began developing the solution in a loud and clear voice; her sister, hearing that, arose and took notes. On the following morning in class, she happened to have the right solution which Dickson's mother failed to know."

Do any of our readers know of comparable cases?

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GIRLS AND MATHEMATICS

A conference entitled "Expanding your Horizons in Mathematics and Science" addressed to girls in secondary school (years 7 to 12) will be held at Burwood State College on Saturday, July 11. It aims

* To encourage the interest of girls in mathematics and science
* To foster in girls an awareness of career opportunities in fields related to mathematics and science
* To provide students and teachers with an opportunity to meet women working in a variety of such occupations.

For further information, contact the conference organizer: Dr Susie Groves, Burwood State College, 221 Burwood Highway, Burwood, 3125.

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MONASH OPEN DAY

Monash University will hold its Open Day on Saturday, August 1. The Mathematics department will run talks, films, displays (including the popular Foucault pendulum) and will have counsellors available for discussion with prospective students, parents and teachers.

Also of interest will be computing displays.

These activities will all be located in the Mathematical Sciences building near the North-West corner of the campus.

If the day is sunny, the analemmic sundial on the North wall of the Union building will provide another focus of mathematical interest. Its accuracy is very good and its principles of construction are noteworthy.

There may also be a Rubik cube competition.