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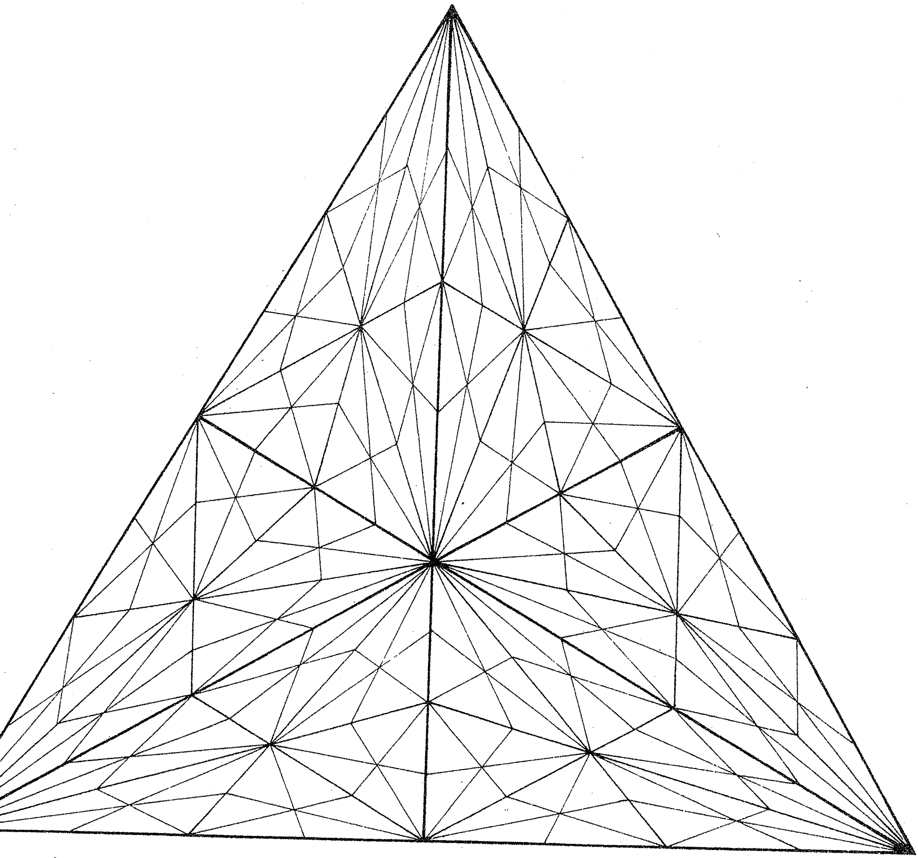

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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John Stillwell, a frequent contributor to *Function*, has allowed us to reproduce a diagram he drew to illustrate his recent book *Classical Topology & Combinatorial Group Theory*. Dr Stillwell not only wrote, but himself illustrated, this outstanding and important book. The book has been hailed by reviewers from round the world as providing a major new insight into a central mathematical subject and described as one of the great expository books of this century.

In our main article for this issue, Professor Morton shows how Newton's laws of motion may be applied in very simple ways to situations that, at first sight, appear very complicated. In his analysis of conveyor belts, he considers four separate situations. In each case, the motor must do twice as much work on the object being carried than we might at first believe. This result has important applications in the engineering design of such systems.

As this is our final issue for 1982, we take the opportunity to thank all our supporters and helpers - in particular our contributors. We look forward to your participation in 1983.

CONTENTS

THE FRONT COVER[†]

John Stillwell, Monash University

The cover shows three stages in the process of *barycentric subdivision* applied to an equilateral triangle. The barycentre, or centre of mass, of a triangle is found as the common point of its three medians. (The medians are lines joining the vertices to the midpoints of the opposite sides. It is a theorem that they all three pass through a common point.) In the process, the triangle is divided into six smaller triangles, and by continuing the process in each of them, the original figure becomes divided into arbitrarily small parts. The edges of a barycentric subdivision can be used to approximate curves on the original figure, and this is its main mathematical purpose, but the subdivision itself makes an attractive picture.

The barycentre, centroid or centre of mass of a triangle may be thought of in two ways. We may imagine it to be the centre of mass of a solid triangle cut from a uniform thickness of (say) sheet metal, or we may think of it as the centre of mass of three unit masses situated at the vertices. In this latter view, its coordinates are given by

$$\bar{x} = \frac{1}{3}(x_1 + x_2 + x_3)$$

$$\bar{y} = \frac{1}{3}(y_1 + y_2 + y_3),$$

where the vertices are (x_1,y_1), (x_2,y_2), (x_3,y_3).

We leave it as an exercise for the reader to show that this point lies $\frac{2}{3}$ of the way along each median. Once this is shown, it is straightforward to show that the medians intersect at a common point, namely $(\bar{x},\bar{y})$.

[†] This diagram appears as the Frontispiece to Chapter 0 of Dr Stillwell's recent book *Classical Topology and Combinatorial Group Theory* and appears in *Function* with permission. The book has received exceptionally good reviews from around the world. The publishers are Springer (New York).

WHAT'S THE DIFFERENCE?

John Mack, University of Sydney

§1. Simple models of population growth can be developed using two assumptions:

(i) the population size can be reasonably and effectively estimated by measuring it at regular intervals,

(ii) the change in population over such an interval depends on the population size at the beginning of the interval.

If the time interval has fixed length T (e.g., 1 year), if the initial population size is P_0 and if the population size after n time intervals has elapsed is P_n, then (ii) may be expressed as

$$P_n - P_{n-1} = f(P_{n-1}) \qquad (n \geqslant 1), \qquad (1)$$

where $f(P_{n-1})$ is a function of P_{n-1}. It is customary in discussions of this topic to write $f(P_{n-1})$ as kP_{n-1}, where the factor k is called the *specific growth rate*. (Note that k may depend on many quantities and need not be constant.) This equation may be rewritten as

$$P_n = (1 + k)P_{n-1} \qquad (n \geqslant 1). \qquad (2)$$

§2. If k is assumed constant, then (2) is easily solved for P_n in terms of k and P_0. For

$$P_1 = (1 + k)P_0, \quad P_2 = (1 + k)P_1 = (1 + k)^2P_0,$$

and generally

$$P_n = (1 + k)^nP_0.$$

Because n occurs as an exponent in this formula, the population is said to exhibit *exponential growth.* No matter what size P_0 is (it must be positive!), P_n has only three types of behaviour:

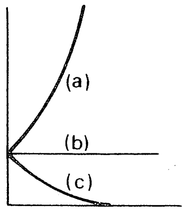
(a) if $k > 0$ (the birth rate exceeds the death rate), P_n grows ever larger, exceeding all bounds as n increases.

(b) if $k = 0$ (equal birth and death rates), P_n is constant at the value P_0.

(c) if $k < 0$ (the death rate exceeds the birth rate), P_n decreases and either tends to 0 (if $-1 < k < 0$) or $P_1 = 0$ (if $k = -1$). Note also that if $-1 < k < 0$, the population will have become extinct as soon as the calculated value of P_n is less than $\frac{1}{2}$, which will occur in a finite time.

These three cases are shown in the diagram.

Thus the case k = constant in (1) is easy and the case $k > 0$ is not very realistic. Populations do not continue to grow forever.



§3. A more realistic model might suppose that the value of k is not constant, but decreases as the population increases (i.e., the birth rate will fall or the death rate increase, or both, as population size grows and thus places greater competition on its members in the struggle for survival).

The simplest model in this case is to suppose

$$k = a - bP_{n-1}$$

where a, b are positive constants. This supposes that k depends linearly on the population size. Equation (2) in this case becomes

$$P_n = ((1 + a) - bP_{n-1})P_{n-1} , \qquad (3)$$

an equation called the *logistic difference equation.*

Given values for a, b and P_0, it is easy to calculate successive values $P_1, P_2, P_3, \ldots$ from (3). (This is very easy even if your calculator has just one memory.) What is very surprising is that the possible behaviour of P_n now depends quite remarkably on all three of a, b and P_0. To test this, use a calculator to calculate the first fifteen values of P_n in each of the following cases.

1. $a = 0{\cdot}5$, $b = 0{\cdot}002$ and

(i) $P_0 = 1000$, (ii) $P_0 = 3000$, (iii) $P_0 = 5000$,

(iv) $P_0 = 7000$, (v) $P_0 = 10\ 000$.

2. $a = 1 \cdot 4$, $b = 0 \cdot 0005$ and

(i) $P_0 = 1000$, (ii) $P_0 = 3000$, (iii) $P_0 = 4000$,

(iv) $P_0 = 5000$.

3. $a = 2 \cdot 5$, $b = 0 \cdot 001$ and

(i) $P_0 = 1000$, (ii) $P_0 = 2500$, (iii) $P_0 = 3000$,

(iv) $P_0 = 3500$.

From your results to the examples in 1. and 2., you may have guessed that it would always be the case that either P_n stabilises or else the population becomes extinct. The examples in 3. demonstrate that life wasn't meant to be easy!

§4. What *is* easy is the determination of the stable population size, should it exist. If P_n approaches a stable value P as n increases, then we may replace both P_n and P_{n-1} by P in (3) and obtain

$$P = ((1 + a) - bP)P.$$

Thus either $P = 0$ (extinction) or we may cancel P and obtain

$$1 = 1 + a - bP.$$

This gives

$$P = a/b.$$

Does this agree with the stable values of P_n you found in examples 1. and 2.?

We can still calculate P for the situation in example 3., obtaining either $P = 0$, or $P = 2 \cdot 5/0 \cdot 001 = 2500$. The fact that we can calculate P does not imply that it will be the eventual population size, as this example demonstrates.

§5. It is natural to replace assumption (i) of the introduction, which allows for measurement of size at discrete intervals of time, by the assumption

(i') The population size can be measured continously.

What is the appropriate form of (ii)? The population size $P(t)$ is a function of the continuous time variable t and is defined for $t \geqslant 0$, with initial size $P(0) = P_0$. We would also expect the change in population over the small time interval from time t to time $t + \Delta t$ to be proportional both to the population $P(t)$ and to the duration Δt of the interval. Thus we may express (ii) in the form

(ii') $P(0) = P_0$ and,

for $t > 0$, $P(t + \Delta t) - P(t) = kP(t)\Delta t$.

Dividing by Δt, we obtain $\frac{P(t + \Delta t) - P(t)}{\Delta t} = kP(t)$ $(t > 0)$. On the left hand side we have a "difference quotient". If we assume that k does not depend on Δt, we may let Δt tend to 0 and obtain the result that for $t > 0$,

$$\lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = kP(t).$$

The expression on the left is the derivative $\frac{dP(t)}{dt}$ of $P(t)$, and we have obtained the differential equation

$$\frac{dP(t)}{dt} = kP(t) \qquad t > 0 \tag{4}$$

with the initial condition $P(0) = P_0$.

§6. If we assume k is a constant, independent of t, equation (4) is easily solved using the exponential function and its solution is

$$P(t) = P(0)e^{kt} \; (t > 0).$$

Again, the behaviour of $P(t)$ as t increases depends only upon whether k is positive, zero or negative.

§7. If we suppose

$$k = a - bP(t),$$

where a, b are positive constants, (4) becomes the non-linear differential equation

$$\frac{dP(t)}{dt} = (a - bP(t))P(t),$$

which can also be written in the form

$$\frac{dP(t)}{dt} = b(P - P(t))P(t), \tag{5}$$

where $P = a/b$. This is called the *logistic differential equation*. This equation can be solved using partial fractions to yield the solution

$$P(t) = \frac{P}{1 + (P/P_0 - 1)e^{-at}} \tag{6}$$

Since $a > 0$, e^{-at} tends to zero as t increases. Thus $P(t)$ always tends to the stable value $P = a/b$ as t increases, irrespective of the positive values assigned P_0, a and b.

Moreover, there are essentially only three types of behaviour possible for the function $P(t)$, illustrated by the following examples.

4. $(P_0 < P)$ $P_0 = 100$, $P = 1000$, $a = 1$.

5. $(P_0 = P)$ $P_0 = P = 1000$, $a = 1$.

6. $(P_0 > P)$ $P_0 = 10\,000$, $P = 1000$, $a = 1$.

Using a scientific calculator (even less work if it's programmable) you may calculate values of $P(t)$ and sketch the graph of $P(t)$ against t in the above three cases.

Comparing the logistic difference equation with the logistic differential equation shows immediately that the former has the more versatile behaviour, because it has an extra parameter Are there real populations which exhibit some of the curious properties of the difference equation and which cannot be modelled by the differential equation? For example, are there species whose colonies exhibit a cyclic population pattern? Attention need not be confined to the life sciences when searching for examples. It may be that the supply-demand patterns for seasonal produce include examples of this type of behaviour.

Readers are invited to write to the Editor with information they have found on applications of the logistic difference equation.

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THE NEXT TERM IN THE SEQUENCE

What is the next term in this sequence?

1, 2, 4, 8, 16, ...

Most of us would unhesitatingly say "32". Here we show that a perfectly logical answer is 31.

Let n points be located around the circumference of a circle. Join each point to all of the others. How many regions are thus created? The answer for the typical case is given by the formula

$$\binom{n}{4} + \binom{n-1}{2} + n, \qquad (*)$$

or $\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$.

(For a proof of this result, see Ross Honsberger's *Mathematical 'ems*, Chapter 9.)

Successive substitution of $n = 1,2,3,4,5,6,7$, etc. yields he sequence

1, 2, 4, 8, 16, 31,57, ...

In certain special cases, collinearities may occur, so that he number of regions is less than that predicted by (*), but these re unusual.

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LIFE IN THE ROUND II†

M.A.B. Deakin, Monash University

In the last issue of *Function*, I wrote of spherical geometry and the ways in which it differed from the more familiar geometry of the plane. Here, I wish to extend that discussion to cover the case of *spherical trigonometry.* Much practical impetus for the study of spherical triangles comes from the need for a spherical trigonometry, particularly in navigation. Dr Moppert's article (*Function*, Vol.5, Part 5) on his sundial used two formulae from spherical trigonometry, and it is used in geodetic surveys (those over a sufficiently large area of the earth's surface to be affected by the earth's curvature).

Plane trigonometry begins by considering right-angled triangles and we begin this study in the same way. Figure 1 illustrates a right-angled triangle, that is to say a figure bounded by three great circular arcs, two of which meet at right angles. (By convention, spherical triangles are drawn with bowed sides.)

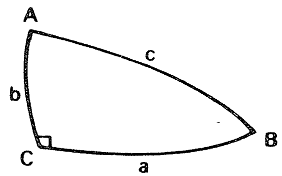

Figure 1

For such triangles, we may express the length (measured in *radians*) of the hypotenuse in terms of the lengths of the other two sides, by means of an analogue of Pythagoras' Theorem:

$$\cos c = \cos a \cos b \quad . \tag{1}$$

This may be proved by geometric constructions, but the easiest demonstration uses vectors and considers the scalar products of the vectors $\vec{OA}, \vec{OB}, \vec{OC}$ in pairs, where O is the centre of the sphere. If you are familiar with this technique, you could try to construct the proof for yourself; or see the appendix for a proof of the more general Equation (6).

Here it will be shown that Equation (1) reduces to Pythagoras' Theorem in a limiting case. Imagine a very large sphere, such as the earth, so that, over quite extensive parts

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†This article is a sequel to one appearing in the previous issue.

of its surface, it appears flat. A right-angled triangle drawn on such a locally almost flat region should, to an excellent approximation, obey Pythagoras' Theorem, although the *exact* formula is given by Equation (1).

But such a triangle has sides which (measured as angles) are very small (i.e. their lengths are short in comparison with the radius of the sphere). Now if θ is a very small angle, $\cos\theta$ is given, to a good approximation, by the formula

$$\cos\theta \approx 1 - \frac{\theta^2}{2} .$$

If we use this approximation three times in Equation (1), we find

$$1 - \frac{c^2}{2} \approx 1 - \frac{a^2}{2} - \frac{b^2}{2} + \frac{a^2b^2}{4} .$$

But a^2, b^2 are very small and so a^2b^2 is extremely small and we may neglect it. This gives, after some simplification,

$$c^2 = a^2 + b^2 ,$$

which is Pythagoras' Theorem.

You might like to investigate the case of a spherical triangle with two (or even three) right angles. (Yes, they do exist.) What does Equation (1) tell us about such triangles?

In plane trigonometry, we would have, for a figure analogous to Figure 1, formulae such as:

$$b = c\sin B$$
$$a = c\cos B$$
$$b = a\tan B .$$

In spherical trigonometry these are slightly altered. They become respectively

$$\sin b = \sin c \sin B \qquad (2)$$
$$\tan a = \tan c \cos B \qquad (3)$$
$$\tan b = \sin a \tan B. \qquad (4)$$

You can readily check that in the limiting case of very large radius, these formulae all reduce to the corresponding formulae of plane trigonometry.

It is not very difficult to prove Equations (2),(3),(4), but we will not do this here. It will appear later how this might be done.

Dr Moppert, in his article on the sundial (*Function*, Vol.5, Part 5) used Formulae (3),(4) to determine the position of the shadow at a given time and date.

Equations (1),(2),(3),(4) and three other equations analogous to Formulae (2), (3) and (4) but using angle A instead of angle B allows us to compute the values of any three of A, B, a, b, c provided that the other two are known. It is not, however, possible in all cases to give merely arbitrary values to the two "known" quantities - they must be consistent. For example, if A is known, a given value of B must satisfy

$$A - \frac{\pi}{2} < B < \frac{3\pi}{2} - A$$

if real values of a, b, c are to result.

Further, if (say) a, A are given, two possible solutions emerge. The two triangles so described form a lune $ABA'C$ as shown in Figure 2. Clearly these both satisfy the data.

This, apart from some calculational short-cuts which are somewhat obsolete in this age of computing machinery, exhausts the topic of right-angled spherical triangles.

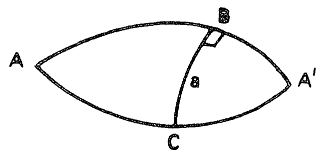

Figure 2

Let us now turn to the more general case.

We began the earlier argument with Pythagoras' Theorem, which is not true for general triangles. In plane trigonometry, Pythagoras' Theorem is replaced by the *cosine rule*

$$c^2 = a^2 + b^2 - 2ab \cos C . \qquad (5)$$

There is a cosine rule for spherical triangles also. This goes

$$\cos c = \cos a \cos b + \sin a \sin b \cos C . \qquad (6)$$

This may be proved by the same method as that indicated as a means of proving Equation (1). An indication of the proof appears as an appendix to this article.

For sufficiently small triangles, Equation (6) reduces to the familiar cosine rule (5), but this you can check for yourself.

In the general triangle, we can use the principle of duality to obtain a new valid formula directly from Equation (6). It reads

$$\cos C = \cos A \cos B + \sin A \sin B \cos c . \qquad (7)$$

Two problems are now left to the reader.

(a) Can you give the corresponding formula for plane trigonometry?

(b) We did not use the principle of duality in the right-angled case. Why not?

Just as spherical trigonometry has its cosine rule, so it also has its sine rule, which is more symmetrical than its plane counterpart,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \tag{8}$$

This may be proved entirely from Equation (6) which (with work) allows for the deduction of all formulae for spherical triangles. There are three processes involved:

(a) permuting the sides (and, consistently with this, the angles),

(b) using the principle of duality,

(c) combining the formulae so obtained.

Formulae (1),(2),(3),(4) are specialisations of general cases, after the simplification $C = \frac{\pi}{2}$ is employed.

Captain Noble, in his article (*Function*, Vol.4, Part 1) on great circle navigation, made reference to the use of spherical trigonometry in that area. Figure 3 is adapted from this article. Suppose a ship or aircraft to be at A and to be headed for B. We know:

(a) the latitude of A,
(b) the latitude of B,
(c) the longitudes of A and B.

We thus, in the spherical triangle ABC, can determine:

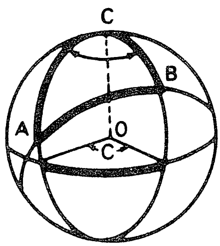

Figure 3

(a) CA, i.e. b, from the first piece of data (by subtraction from $\frac{\pi}{2}$),

(b) CB, i.e. a, from the second piece of data (similarly),

(c) the angle C, as the difference in the longitudes.

We need to know the "heading" of the vessel - i.e. angle A. Here is one way to do this. Use Equation (6) to determine the distance, AB (i.e. c), yet to be travelled (a handy thing to know in any case). Then use Equation (8) to complete the calculation. (In practice, a somewhat different method is used, but this need not detain us here.)

As the vessel travels toward B, this heading changes. It follows that the calculation must be carried out repeatedly on the journey. Modern ships and aircraft have on-board computers which could do this, but increasingly they rely on electronic and inertial systems, which are not part of our story here.

Because of its practical utility, spherical trigonometry has a long history. It has been traced back to two Greek mathematicians: *Hipparchus of Rhodes* (c. 150 B.C.) and *Menelaus of Alexandria* (c. 100 A.D.). The latter is credited by some authors with a version of Equation (8). The more basic Equation (6) was probably first discovered by the Islamic mathematician *al-Battānī* (also known as *Albategnius*), who lived in the late 9th century in what is now Iraq. (It has also, however, been credited to *Regiomontanus*, a 15th century Italian mathematician.) Many of the practical aspects of computation were settled by *Napier*, the discoverer of logarithms, and published in 1614. A rigorous proof of the basic Equation (6) was first given by *Euler* in 1753.

Appendix

Figure 4 shows the triangles OCB, OCA with OB extended to D where CD is tangent to the arc CB, and OA similarly extended to E.

$$OB = OC = OA = R.$$

Now consider vectors $\vec{OA}, \vec{OB}, \vec{OC}$. We have from the figure

$\vec{CD} = \vec{OB} \sec a - \vec{OC}$ and

$\vec{CE} = \vec{OA} \sec b - \vec{OC}$. But

$|\vec{CD}| = R \tan a$, $|\vec{CE}| = R \tan b$.

Thus

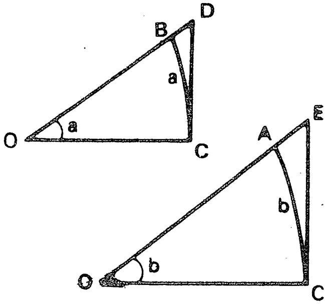

Figure 4

$$R^2 \tan a \tan b \cos C = (\vec{OB} \sec a - \vec{OC}).(\vec{OA} \sec b - \vec{OC}) \qquad (9)$$

since C is defined as the angle between the two tangents CD and CE. We now simplify Equation (9), making use of the equations $\vec{OB}.\vec{OA} = R^2 \cos c$, $\vec{OC}.\vec{OB} = R^2 \cos a$, $\vec{OA}.\vec{OC} = R^2 \cos b$. After some quite elementary algebra, Equation (6) results.

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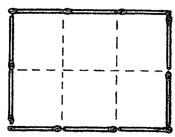
MATCH TRICK NO. 18

To the right, we show the Bryant and May Match Trick No. 18, supplied to us courtesy of the Wilkinson Match Company. Their answer and some mathematical speculation appears on page 23.

MATCH TRICK No 18



Using 12 matches, can you form a quadrilateral with the same area as this rectangle?

THE DYNAMICS OF CONVEYOR BELTS

B.R. Morton, Monash University

As you check in at the air terminal, your bag is placed on a horizontal moving belt which carries it off for loading, and you expect to see it next at your destination, where perhaps it may appear on an inclined belt from a lower level. Conveyor belts have the great advantage of continuous operation over varied topography and are used to carry items on assembly lines, people on escalators and moving pavements, and for bulk transport of finely divided or crushed solids such as sand, gravel, sugar, coal and ores. They are dynamically interesting and provide an excellent illustration of the methods of dynamics.

A package on a moving horizontal belt.

The simplest case is that of a package or bag of mass m placed gently on a horizontal belt which we may take to be driven at constant speed v. When the bag is placed on the belt it will generally have no horizontal momentum, and will therefore slip until the frictional force between belt and bag has accelerated it to belt velocity. As our law of dry friction we shall assume that during sliding a constant tangential force of magnitude $\mu \times$ (normal reaction) acts to accelerate the bag, but that once the bag has achieved belt velocity it experiences no further frictional force on a horizontal belt in uniform motion.

Suppose that x is the horizontal displacement of the bag from its initial position relative to a coordinate system fixed in the building. Figure 1(ii) shows an "exploded" view with bag and belt drawn separately and the contact represented by the normal reaction N and frictional force F. As there is no vertical acceleration of the bag, $N - mg = 0$, and the equation for horizontal motion of the bag is

$$m\ddot{x} = \mu m g \ ,$$

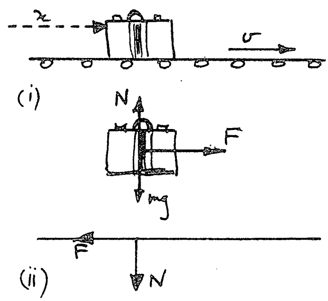


Figure 1

where the frictional force $F = \mu N = \mu mg$ is constant while the bag slides, but zero when sliding ceases. Thus the bag moves with constant acceleration μg during sliding, and as it has displacement $x = 0$ and velocity $\dot{x} = 0$ at the initial instant $t = 0$ we obtain

$$\dot{x} = \mu g t$$

and

$$x = \tfrac{1}{2}\mu g t^2 .$$

Thus the bag reaches belt speed v at time $t = v/\mu g$ after travelling a distance $v^2/2\mu g$, during which time the belt travels a distance $vt = v^2/\mu g$. For example, if $v = 1\ ms^{-1}$, $\mu = 0{\cdot}5$ and $g = 10ms^{-2}$, the bag accelerates to belt speed in time $0{\cdot}2s$ over a distance $0{\cdot}1m$.

While the bag is being accelerated to belt speed the drive motor must work through the belt at the uniform additional rate

$$Fv = \mu mgv ,$$

since the force on the belt is equal and opposite to that on the bag and must be matched by a corresponding additional torque from the motor if the belt speed is to remain constant. We have assumed that the bearings are well lubricated and that the downwards reaction N on the belt does not cause any additional bearing resistance. Hence the total additional work done by the motor in accelerating the bag is

$$W = Fvt = \mu mgv \frac{v}{\mu g} = mv^2 ,$$

since the frictional force is constant. The direct rate of working on the bag at time t, however, is only

$$F\dot{x} = \mu mg\ \mu g t$$

and the total work done on the bag in the time $v/\mu g$ of acceleration

$$\begin{aligned} W_b &= \text{(mean rate of working)} \times \text{(time)} \\ &= \text{constant force} \times \text{mean velocity} \times \text{time} \\ &= \mu mg \cdot \frac{v}{2} \cdot \frac{v}{\mu g} \\ &= \tfrac{1}{2}mv^2 , \end{aligned}$$

since in this case the rate of working is proportional to the time elapsed. The work W_b is precisely that needed to produce the final kinetic energy of the bag. The difference between the work done by the belt and that done on the bag arises because in frictionally resisted sliding motion, work is done also against the frictional forces at the rate

$$F(v - \dot{x}) = \mu mg(v - \dot{x}) = \mu mg(v - \mu g t) ,$$

contributing a total

$$W_f = W - W_b = \tfrac{1}{2}mv^2 .$$

In any motion involving frictional forces there is always some dissipation of energy which is converted to heat that cannot be recovered directly as mechanical energy, and the loss of half the energy or the wastage of half the work done in this case is a fairly common result. (Friction does, however, allow you to warm your hands by rubbing them vigorously.)

Should we, in these energy conscious days, attempt to increase the efficiency of our luggage belts by preventing any slip? This might be done by bolting a series of bars across the belt and instructing the baggage handlers to place each piece firmly against a bar, when each will be set impulsively in motion with an instantaneous impulse (that is, the limiting case of a very large force acting for a very short time to produce a finite momentum increase) of magnitude mv thereby doing work $(mv)^2/2m = \tfrac{1}{2}mv^2$. The disadvantage is that a very large force of mean magnitude mv/τ is called into action for the brief time τ during which the bar pushes into your bag. For example, a bag of mass 15 kg accelerated impulsively in 0·01s experiences a mean force of $1500N$ in contrast to the frictional force $\mu mg = 75N$ experienced by the sliding bag. Thus the sliding bag takes twenty times as long to accelerate but suffers much less stress, and even your eggs are likely to survive (provided that the bag is not thumped down onto the belt)!

A package on a moving inclined belt.

On a belt inclined to the horizontal at angle α, a proportion $mg \cos\alpha$ only of the weight of the bag acts normally towards the belt. Hence, as there is no acceleration normal to the belt,

$$N = mg \cos\alpha .$$

While the bag slides, the friction $F = \mu mg \cos\alpha$ is partly offset by the component of bag weight $mg \sin\alpha$ acting down the plane; and when sliding ceases the friction force decreases to $mg \sin\alpha$ (which must in this case be less than $\mu mg \cos\alpha$) and the bag moves with the belt.

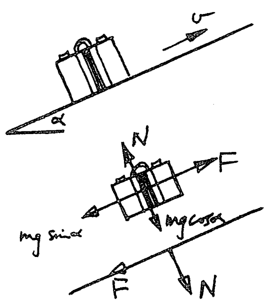


Figure 2

While sliding continues, the tangential component of the equation of motion,

$$m\ddot{x} = \mu mg \cos\alpha - mg \sin\alpha ,$$

shows that the bag moves with constant acceleration $(\mu \cos\alpha - \sin\alpha)g$; and a bag placed gently on the belt at $t = 0$ has subsequent velocity

$$\dot{x} = gt(\mu \cos\alpha - \sin\alpha)$$

and displacement $\quad x = \tfrac{1}{2}gt^2(\mu\cos\alpha - \sin\alpha)$,

x being measured along the inclined belt from the point at which the bag is placed initially. In this case the bag takes time $v\{g(\mu\cos\alpha - \sin\alpha)\}^{-1}$ to accelerate to belt speed over the slant distance $v^2\{2g(\mu\cos\alpha - \sin\alpha)\}^{-1}$. As the inclination of the belt is increased, the driving force on the bag decreases and both time and distance required for acceleration increase, until in the limiting case in which the belt is inclined at the "angle of friction", $\alpha = \mathrm{Tan}^{-1}\mu$, the bag is no longer accelerated and the belt is unable to carry it upwards. Note that this result does not depend at all on the mass of the bag (because both frictional force and weight are proportional to mass). At belt inclination 18° , our bag (with $\mu = 0{\cdot}5$, etc. - see earlier) takes $0{\cdot}6s$ to accelerate to belt speed over a distance $0{\cdot}3m$; and at 27° it slides slowly down the belt (tan 27° > 0·5).

While the bag is sliding the belt does additional work at the rate $\mu\, mg(\cos\alpha)v$, and work is done on the bag at the rate

$$\mu\, mg(\cos\alpha)\dot{x} = \mu m g^2(\cos\alpha)(\mu\cos\alpha - \sin\alpha)t \ .$$

The total work done on the bag as it is accelerated to speed v is

$$\begin{aligned} W_b &= (\text{mean rate of working}) \times (\text{time}) \\ &= \mu m g^2(\cos\alpha)(\mu\cos\alpha - \sin\alpha)\frac{v^2}{2g^2(\mu\cos\alpha - \sin\alpha)^2} \\ &= \frac{\mu m(\cos\alpha)v^2}{2(\mu\cos\alpha - \sin\alpha)} \\ &= \tfrac{1}{2}mv^2 + mg(\sin\alpha)\frac{v^2}{2g(\mu\cos\alpha - \sin\alpha)} \\ &= \tfrac{1}{2}mv^2 + mgh \ , \end{aligned}$$

where h is the vertical distance through which the bag is lifted during acceleration. The total work done by the belt in this time is

$$W = \mu mg\cos\alpha\, v\,\frac{v}{g(\mu\cos\alpha - \sin\alpha)} = \frac{\mu m\cos\alpha v^2}{\mu\cos\alpha - \sin\alpha} = 2W_b \ ,$$

and again half of the work done by the belt goes jointly to the increase in kinetic energy $(\tfrac{1}{2}mv^2)$ and gravitational potential energy (mgh) of the bag and half is done against friction. When the bag has ceased to slide the frictional force called into play is just enough to balance the downslope component of weight, $mg\sin\alpha$, and the rate of working on the bag is then

$$(mg\sin\alpha)v = mg(v\sin\alpha) \ ,$$

which is just the rate of increase of potential energy of the bag. No further work is done against friction in this phase of the motion as the bag no longer slides on the belt.

The control volume method.

Suppose that we wish to carry out a similar calculation for the transport of dry sand fed at steady rate $\dot{m}$ from a hopper *H* onto a horizontal belt *AB* driven at uniform speed *v*. In place of the single airline bag in the previous problem we have now a steady stream of grains of sand which fall to the belt with negligible horizontal momentum and are drawn into motion by an unknown blend of frictional forces exerted by the belt and action-reaction forces between sand grains, finally moving in equilibrium at belt velocity. Viewed in terms of sand grains as particles this is clearly a far more difficult problem, but just as we have previously ignored the particular contents of the airline bag taking into account only its mass, we now seek a corresponding way of determining the bulk motion of sand without having to take into account the detailed pattern of force and acceleration of individual sand grains.

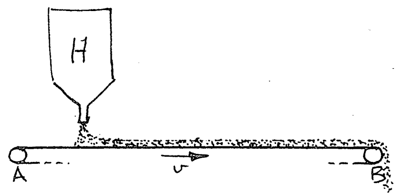


Figure 3

The equation of motion for the airline bag involved:

(i) the gross mass of bag and contents;

(ii) the resultant force comprising the vector sum of

(a) forces acting throughout the interior of the bag and its contents termed *body forces*, in this case the weight, and

(b) forces acting across the outer surface of the bag termed *contact forces*, in this case the normal reaction of the belt and tangential friction.

There is no effect of action-reaction forces between items packed within the bag, because these act in equal and opposite pairs which contribute nothing to the resultant.

We can respecify the equation of motion for the bag in terms of its enveloping outer surface, the *bag surface*:

(mass contained within bag surface) × (acceleration)

= (resultant of body forces acting on material contained within bag surface plus contact forces acting across it).

More generally we define a *control surface* as any notional surface enclosing a control volume of material in motion. Where the control surface moves with its contents it may envelop a normal mechanical body, but where matter crosses the control surface we have a new situation in which the Newtonian equation of motion must be applied afresh. In particular, in the case of sand transport on a moving belt, we may consider a control surface fixed in space and represented by its intersection *CDEF* with the plane of the page. The end faces *FC* and *DE* are normal to the direction of belt motion and large enough to intersect the whole sand stream, and upper and lower faces are parallel to the belt with *CD* on the belt face and *EF* above the upper surface of the sand. In steady motion there can be no accumulation of sand, the mass *flux* (or flow per unit time) is $\dot{m}$ across each of *FC* and *DE*, and the mass of sand contained in the control volume is constant; moreover, the centre of mass of the contained sand is stationary and its bulk acceleration zero. There is, however, a stream of sand entering across *FC* with velocity v_C and momentum flux $\dot{m}v_C$ and leaving across *DE* with velocity v_D and momentum flux $\dot{m}v_D$. We can view this situation in two ways: (i) that the body and contact force throughout and over the control surface are together generating momentum at the rate $\dot{m}v_D - \dot{m}v_C = \dot{m}(v_D - v_C)$, or (ii) that the sand stream crossing *FC* brings momentum into the control volume at the rate $\dot{m}v_C$ and is therefore dynamically equivalent to a force $\dot{m}v_C$ acting across the section *FC* of control surface or to a contact force $\dot{m}v_C$ across *FC*, and similarly $\dot{m}v_D$ across *DE*. (Note that this is a direct interpretation of the equation of motion in the momentum form $d(mv)/dt = F$.)

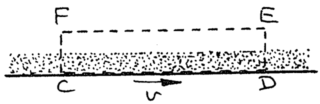


Figure 4

The transport of sand on a horizontal conveyer belt.

We can now consider the transport of sand by a horizontal belt driven at constant speed v. Choose the fixed control surface *CDEF* so that: (i) sand enters through the upper surface sketched as *EF* at constant rate $\dot{m}$; (ii) no sand crosses the end surface *FC*; (iii) *DE* is chosen so that the sand has come to equilibrium with uniform belt velocity v before it leaves the control volume; and (iv) the lower surface *CD* lies along the the belt. We are not greatly concerned with the vertical component of motion, beyond noting that the belt rollers must carry both the weight of sand on the belt and the force $\dot{m}\sqrt{2gh}$

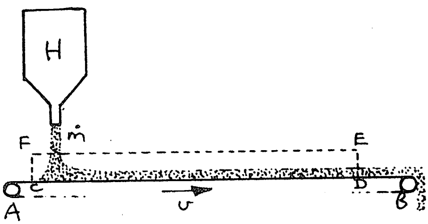


Figure 5

corresponding with the momentum flux of sand falling through height h from the hopper. The kinetic energy of the falling sand is likely all to be dissipated on impact with the belt, and the sand may be assumed to start its motion on the belt with neither horizontal momentum nor kinetic energy.

We assume therefore that sand enters the control volume with mass flux $\dot{m}$, zero horizontal momentum flux and zero kinetic energy flux; and that it leaves at D with horizontal momentum flux $\dot{m}v$ and kinetic energy flux $\frac{1}{2}\dot{m}v^2$. The rate of generation of kinetic energy in sand on the belt is $\frac{1}{2}\dot{m}v^2$. There is no horizontal component of body force within the control volume, and the only horizontal contact force over the control surface is the undetermined friction force which acts to accelerate sand until it reaches belt speed.

The rate of working by the belt on the sand

= (gross force exerted by the belt) × (velocity of the belt)

= (gross force exerted on the sand) × (velocity of the belt)

= (gross rate of increase of momentum of sand) × (velocity of belt)

= (momentum flux out of control volume at D − momentum flux into control volume at C) × (velocity of belt)

= $\dot{m}v \times v$

= rate of generation of kinetic energy in the sand + rate of dissipation of energy.

Thus half of the work done goes to increase the kinetic energy of the sand and half is dissipated. Note the tremendous advantage gained from the use of the control volume argument. Provided that we are willing to forgo a detailed knowledge of the way in which the sand is accelerated to belt velocity, we need only a knowledge of the momentum fluxes past C and D and we can substitute for the unknown distribution of friction.

Loading sand into a cement mixer on an inclined belt.

In this case we wish to find the power required to drive an inclined moving belt at steady speed v as it carries a uniform stream of sand from a hopper upwards through height h to a cement mixer. We shall again assume that the sand from the hopper reaches the belt with negligible momentum and kinetic energy, and that it is accelerated to belt

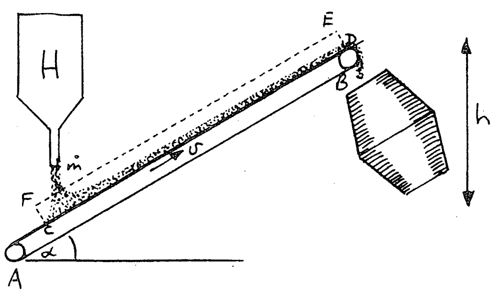


Figure 6

velocity v before being discharged into the mixer. Thus sand enters the control volume to the right of FC at the steady rate $\dot{m}$ without kinetic energy or momentum, and leaves past D at rate $\dot{m}$ with kinetic energy flux $\frac{1}{2}\dot{m}v^2$, rate of increase of potential energy $\dot{m}gh$ and momentum flux $\dot{m}v$ along the belt.

As before, the rate of working by the belt is the product of the force exerted by the belt on the sand and the velocity of the belt. In this case, however, the force exerted by the belt on the sand is not the only tangential force acting on the sand, as the weight of the sand divides into components in the ratio $g\cos\alpha$ normal to the belt and $g\sin\alpha$ down the belt. Moreover, although we know the flux of sand $\dot{m}$ past each section of the belt, we do not know the actual mass (or weight) of sand on the belt. Suppose that the mass of sand per unit length of belt is $\sigma(x)$ and the velocity of sand $u(x)$, increasing from $u(0) = 0$ where the sand falls onto the belt to v at G and thereafter remaining v to the top of the belt. Then the flux of mass past any point $P(x)$ is $\sigma u = \dot{m}$, and it follows that $\sigma = \dot{m}/u$ decreases with increasing distance to $\dot{m}/v$ at G and thereafter remains uniform.

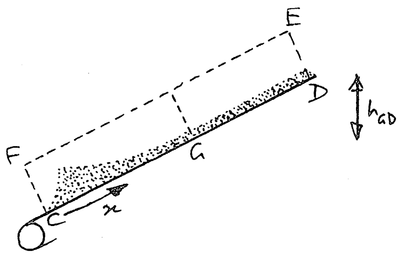


Figure 7

The sand on GD moves in equilibrium at belt speed v with the downslope component of weight per unit length,

$$-g\sigma \sin\alpha = -g(\dot{m}/v)\sin\alpha ,$$

exactly balanced by the upslope friction $g(\dot{m}/v)\sin\alpha$ of belt on sand. In this section the rate of working by the belt per unit length

$$= \text{(force exerted by belt)} \times \text{(velocity of belt)}$$

$$= g(\dot{m}/v)(\sin\alpha)v$$

$$= \dot{m}g\sin\alpha ,$$

and the gross rate of working by the section GD of belt is

$$\dot{m}gx_{GD}\sin\alpha = \dot{m}gh_{GD} ,$$

which is precisely the rate at which gravitational potential energy is generated in this section; there is no dissipation.

The sand on CG is being drawn into motion, and although we know that gravitational potential energy is being generated at the rate $\dot{m}gh_{CG}$, kinetic energy at rate $\frac{1}{2}\dot{m}v^2$ and belt-wise momentum at rate $\dot{m}v$, we do not know the rate of dissipation. We might reasonably assume, however, from the earlier solutions involving dry friction that half the work done by the belt between C and G is lost by dissipation, in which case

$$
\begin{aligned}
F_{\text{belt}}v &= (F_{CG} + F_{GD})v = F_{CG}v + F_{GD}v \\
&= 2(\tfrac{1}{2}\dot{m}v^2 + \dot{m}gh_{CG}) + \dot{m}gh_{GD} \\
&= \dot{m}v^2 + \dot{m}gh_{CG} + \dot{m}gh_{CD} \\
&= (\dot{m}v + \dot{m}v^{-1}gx_{CG}\sin\alpha + \dot{m}v^{-1}gx_{CD}\sin\alpha)v,
\end{aligned}
$$

and the force of the belt on the sand is $\dot{m}v + (\dot{m}g\sin\alpha/v)(x_{CG} + x_{CD})$.

The distribution $\sigma(x)$ of sand on the belt may be found using a more detailed and difficult argument. When this is known, we are enabled to calculate directly the dissipation and to verify our guess that half the energy is lost.

Newton's equation of motion is *the* basis for all mechanics, and can be applied far more widely than to particles and bodies if we understand its implications.

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JUMPER'S CHANCES REVISITED

Function, Vol.6, Part 3 commented on a newspaper controversy concerning mortality of horses in jumps races. More recent figures have been supplied by Mr. Barker of the RSPCA (*Age* letters, 11/9/82). The improved description of these makes it clear that Peter Singer did (contrary to our suggestion) interpret the figures correctly.

According to the latest figures, a horse, starting in a jumps race, faces a 1 in 95 chance of death. (Mr. Barker's earlier figure was 1 in 88.) This means that its chance of death in a 50-race career is $1-(94/95)^{50}$ or 41.1%.

One may still query the figure of 1 in 95, as Mr. Barker remarks of these 1982 figures that they are the worst ever - in other words, they are atypically high. Nonetheless, even if the true figure were much lower, e.g. 1 in 200, the calculation gives what Paul Sheahan, in the course of the newspaper debate, called a "distressingly high" figure. For our example the chance of death in 50 races is 22.2%.

It was not *Function*'s intention to comment on the matter of whether such races should be abolished. We do, however, argue that discussion of such controversial matters requires accurate data and correct statistical technique.

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LETTERS TO THE EDITOR

TWIN PRIMES

Colin Wright's letter on Twin Primes in the April edition prompted us to modify an existing computer program in order to produce consecutive twin primes. There is in fact a multitude of these which is not surprising when it is considered that all prime numbers are of the form $6n + 1$ or $6n - 1$. It is interesting to note that 4 consecutive twin primes exist in the range $1 \leqslant N < 10\,000$, i.e. 9419, 9421; 9431, 9433; 9437, 9439; 9461, 9463.

Here is our programme.

```
READY
LISTNH
00001  INPUT "IF LIST OF PRIMES REQUIRED, TYPE L, IF TWIN PRIMES
 REQUIRED TYPE T";A$
00002  IF A$="L" THEN 5
00003  INPUT "HOW MANY CONSECUTIVE TWIN PRIMES ARE TO BE
 DISPLAYED?";T
00005  DIM A(100000)
00010  INPUT "INPUT LOWER AND UPPER BOUNDS OF SEARCH";L,H
00020  IF L<=2 THEN 25 ELSE 30
00025  A(1) = 2\ A(2) = 3\ M = 2
00030  IF L = 3 THEN 33 ELSE 40
00033  A(1) = 3\ M = 1
00040  TL=(L+1)/6
00050  TH=(H+1)/6
00060  FOR N=INT(TL)+1 TO TH
00070  K=6*N-1
00080  J=K+2
00090  S=SQR(K)
00100  FOR I=2 TO S
00110  F=K/I
00120  IF F=INT(F) THEN 150
00130  NEXT I
00135  IF K<L THEN 190
00136  M=M+1
00137  A(M)=K
00150  IF K=J THEN 190
00160  K=J
00170  IF J>H THEN 999
00180  GO TO 90
00190  NEXT N
00191  IF A$="L" THEN 200
00192  GO TO 210
00200  FOR I=1 TO M\ PRINT A(I);\ NEXT I
00205  GO TO 999
00210  FOR I=1 TO M
00215  FOR J=1 TO 2*T STEP 2
00220  IF A(I+J)-A(I+J-1)=2 THEN 225 ELSE 230
00225  NEXT J
00227  FOR J=1 TO 2*T\ PRINT A(I+J-1);\ NEXT J
```

```
00228  PRINT
00230  NEXT I
00999  END
```

David Shaw and Year 11 students,
Geelong West Technical School.

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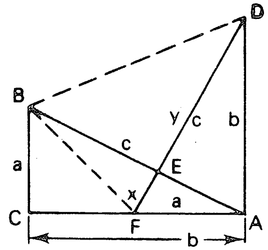
PYTHAGORAS AGAIN

I have discovered over 70 original proofs of Pythagoras' Theorem. This is one of the shortest.

Let ΔABC be right-angled at C, and let $AC = b$, $BC = a$. Let F lie between A, C and let $AF = a$. Let $AD \perp AC$, and $AD = b$. Join BD, BF, DF and let DF cut AB in E. Let $DE = y$, $FE = x$.



As ΔDFA and ΔABC are congruent $x + y = c$. But

Area of $\Delta ABF = \frac{1}{2}a^2$,

Area of $\Delta ABD = \frac{1}{2}b^2$. But

Area of $\Delta ABF = \frac{1}{2}cx$,

Area of $\Delta ABD = \frac{1}{2}cy$.

Thus $\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}cx + \frac{1}{2}cy = \frac{1}{2}c^2$, and the theorem is proved.

Garnet J. Greenbury,
Taringa, Queensland.

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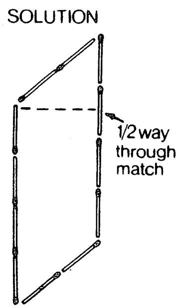
SOLUTION TO MATCH TRICK NO.18

At right, we display the solution as provided by the Wilkinson Match Company. A question arises as to whether or not it is unique. A 3-4-5 triangle has an area of 6 and a perimeter of 12. Could the right-angle be decreased slightly and the hypotenuse "bowed out" to give another solution? We suspect not, but have no proof of this.



PROBLEM SECTION

We give here solutions to problems posed earlier in the year, beginning with one that proved unexpectedly difficult.

PARTIAL SOLUTION TO PROBLEM 6.1.1.

Mr B.W. Harridge of Melbourne High School wrote to tell us of the following problem posed to students at the MAV Mathematics Camp at Glenaladale this year.

"Represent each of the integers from 1 to 100, using the digits of 1982 *in that order*, and any of the following mathematical operations:

$+$, $-$, $\div$ (or $/$), $\times$, raising to a power

$\sqrt{\ }$, !, brackets, decimal points.

Adjacent digits such as 98 are understood to be decimal numbers (ninety-eight here).

For example $23 = 19 + 8/2$

$51 = 1 + (\sqrt{9})! \times 8 + 2$.

There was one number which defied all our efforts, namely 52."

We left it to *Function* readers either to find a representation of 52 in the required form or to prove that such a form does not exist.

Ilana Bush (Year 12, Mt Scopus College) wrote to let us know that the problem can be solved if further numerals are used

$$52 = \{(1 \times 9) + 8^{2/3}\} \times 2^2 ,$$

and Don Kerr (Cavendish Road State H.S., Queensland) sent five separate solutions with the ordering condition relaxed, e.g.

$$52 = (\cdot 9 \times 8 - 2) \div \cdot 1.$$

We couldn't do it (nor could we do 53, 93, 94) and a proof of impossibility seemed tediously long for a relatively uninteresting problem, so we decided to "cheat" and use Don Kerr's solution above in this form:

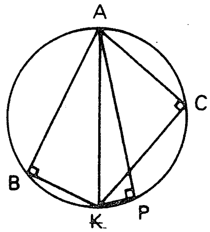
$$52 = \frac{\cdot 9 \times 8 - 2}{\cdot 1} .$$

Our Finnish counterpart *Funktio* also published this problem. If they have an answer, we will let you know.

SOLUTION TO PROBLEM 6.2.2.

This problem, from the 1982 Australian Mathematical Olympiad, read as follows.

ABC is a triangle and the internal bisector of the angle *A* meets the circumcircle of *ABC* at *P* (as well as at *A*). *Q* and *R* are similarly defined in relation to *B* and *C* respectively. Prove that $AP + BQ + CR > AB + BC + CA$.



Refer to the diagram at right. We will show that $AB + AC < 2AP$; *AK* is a diameter. Thus angles *ABK*, *ACK* and *APK* are all right angles. Let $\angle KAP = \alpha$, and $\angle BAP = \angle CAP = \theta$. Then, if r is the radius of the circumcircle,

$AB = 2r\cos(\theta - \alpha)$, $AC = 2r\cos(\theta + \alpha)$, $AP = 2r\cos\alpha$. We now have

$$AB + AC = 2r\{\cos(\theta - \alpha) + \cos(\theta + \alpha)\}$$

$$= 2r\{2\cos\theta\cos\alpha\} = 2AP\cos\theta$$

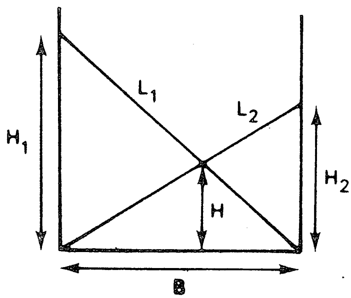
$$< 2AP.$$

Similar inequalities hold for *BQ*, *CR* and the result follows.

We have drawn the case in which the triangle straddles *AK*. If it lies to one side, some details differ, but the result still follows.

SOLUTION TO PROBLEM 6.2.4.

This is the infamous "crossed ladders" problem.



The ladders diagrammed at right are of lengths L_1, L_2 ($L_1 > L_2$) respectively. They cross at a point whose distance above the baseline is H. What is the distance between their feet?

In the version submitted to us, the following figures were given: $L_1 = 3m$, $L_2 = 2m$, $H = 0{\cdot}8m$.

Two solutions were received, one from J. Ennis (Year 10, M.C.E.G.S.), the other from David Shaw and his students at Geelong West Technical School). We publish a composite of these replies.

Let B, H_1, H_2 be as shown in the diagram. Then

$$B^2 + H_1^2 = L_1^2 \qquad (1)$$

$$B^2 + H_2^2 = L_2^2 \quad , \qquad (2)$$

and, by similar triangles, we may readily deduce that

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{H} \quad . \qquad (3)$$

Now write $h_1 = H_1/H$, $h_2 = H_2/H$, $\ell_1 = L_1/H$, $\ell_2 = L_2/H$.

Then $h_1 + h_2 = h_1 h_2 = S$ (say) from Equation (3)

and $h_1^2 - h_2^2 = \ell_1^2 - \ell_2^2 = \lambda^2$ (say) from Equations (1), (2).

But $(h_1^2 - h_2^2)^2 = S^2(S^2 - 4S) = \lambda^4$.

Thus

$$S^4 - 4S^3 - \lambda^4 = 0. \qquad (4)$$

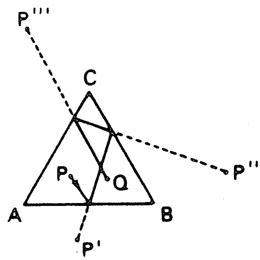
This is a quartic in S, whose solution is $S = 4{\cdot}619$, in the numerical case given. This then yields the solution, after rather more work, $B = 1{\cdot}62m$. For other numerical values, Equation (4) works out rather more easily.

SOLUTION TO PROBLEM 6.2.5.

Let ABC be any triangle and P, Q two distinct points inside it. Find the shortest path from P to Q subject to the condition that the path must hit *each* side of the triangle.

J. Ennis also solved this problem. He writes:

"The shortest path fulfilling the conditions will be that of a beam of light travelling so that its angle of incidence equals its angle of reflection, as shown in the diagram. Reflect the path from P to the line AB in AB as shown, then reflect the path from P' to BC in BC and finally reflect the path from P'' to AC in AC. The three reflections take P to P'''. The shortest distance from P''' to Q is a straight line and thus the path may be found by reversing the reflections, as shown."



SOLUTION TO PROBLEMS 6.3.1.

A student lives on the bus route between the university and his girl friend's house. The buses run regularly, but the student gets up at quite random hours. When he leaves home, he takes the first bus that comes, irrespective of its direction of travel. On average he visits his girl friend twice as often as he goes to the university. Explain how this can be.

Another solution from J. Ennis. He writes:

"Call the time between buses going in a specific direction x minutes. The bus timetable is arranged so that the university bus will pass the student's stop $\frac{1}{3}x$ minutes after the last bus going the other way (to his girlfriend's house) passed the same point. If the student arrives at the stop after the university bus passes but before the next bus passes the other way, a period of $\frac{2}{3}x$ minutes, he will end up going to his girlfriend's house. If, on the other hand, he arrives at the stop after the bus to his girlfriend's house has passed, but before the next university bus, a period of $\frac{1}{3}x$ minutes, he will go to the university. If he keeps quite arbitrary hours, then he is twice as likely to get to the stop in the period of $\frac{2}{3}x$ minutes between buses then in the period of $\frac{1}{3}x$ minutes, and thus twice as likely to go to his girlfriend's house rather than to university."

SOLUTION TO PROBLEM 6.3.2.

This fascinating problem is of great practical importance, odd though this may sound. It read, in our version, as follows.

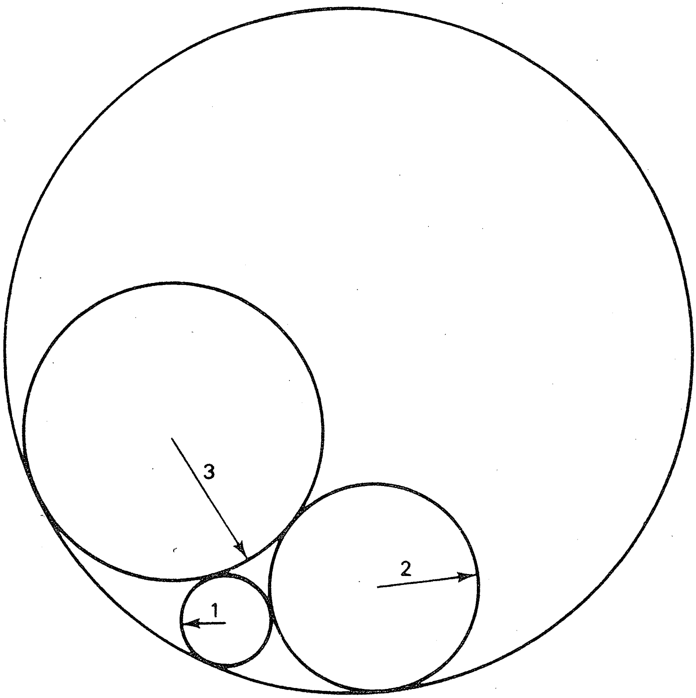
Al Capone is holed up and wishes to communicate with his confederate Squizzy Taylor, who is similarly disadvantaged. Al writes his message and seals it in a strong box which he padlocks. He is forced to entrust this to a crafty but unscrupulous courier, who on no account is to see the message. This means that Al has to keep the key, of which no copy exists. How does Squizzy read the message?

Secrecy is assured by a rather elaborate process. Al padlocks the strongbox and sends it to Squizzy. Squizzy padlocks it again and returns it to Al. Al removes his own padlock and sends the box back to Squizzy who can now open it.

The practical applications lie in coding theory. For "padlocking" read "coding" and for "unlocking" read "decoding". This system allows coded messages to be sent publicly without the key to the code being relinquished.

SOLUTION TO PROBLEM 6.3.3.

This problem (submitted by J. Ennis) had three circles, each tangent to the other three, as shown in the diagram overleaf.



We wanted the radius of the large circle.

John Barton, Drummond Street, Carlton, sent two solutions. This is one. Refer to the figure opposite.

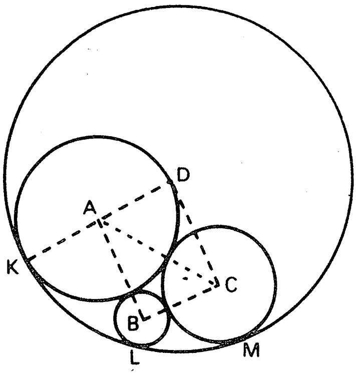
"Since $AB = 4$, $BC = 3$, $CA = 5$, $\measuredangle ABC$ is a right angle. Complete the rectangle $ABCD$.

Join DA and produce to K on circle "3",

join DB and produce to L on circle "1",

join DC and produce to M on circle "2".

Then $DK = DL = DM = 6$ and a circle centre D, radius 6, passes through KLM and since these lines are diameters of the respective circles, the large circle touches all three at K, L, M."



Mutually tangent circles have a fascinating property, first discovered in 1936 by Frederick Soddy, better known as the Nobel Laureate who discovered isotopes. It was first announced (in verse!) in the journal *Nature*. This excerpt gives the main theorem - the full verse is entitled *The Kiss Precise*.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends,
Is half the square of their sum.

This means that, in our example, if r is the radius of the large circle,

$$2\left(\frac{1}{r^2} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) = \left(\frac{1}{r} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)^2 .$$

This has two solutions $r = -6$, $r = \frac{6}{23}$. The second gives the radius of a small circle nested among the main three. At first sight, we might be tempted to neglect the first, but the minus sign (as the verse remarks) merely shows that the large circle is touched (kissed) on its inside.

For more on Soddy's Theorem see *Scientific American*, May 1968, or (for a proof) H.S.M. Coxeter's *Introduction to Geometry*.

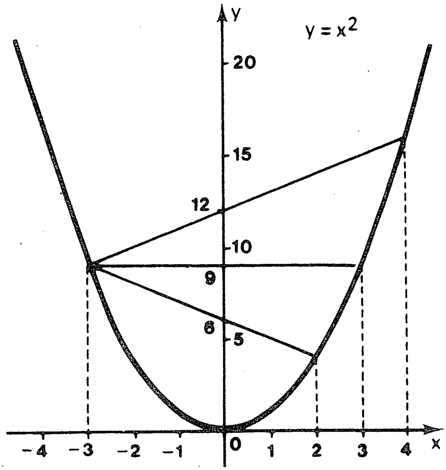
SOLUTION TO PROBLEM 6.3.4.

This problem, submitted by Garnet J. Greenbury of Taringa, Queensland, considered a property of the parabola $y = x^2$. (See the diagram.) We have:

$3 \times 4 = 12$

$3 \times 3 = 9$

$3 \times 2 = 6.$



Why is this so?

J. Ennis solved and generalised this problem. We quote his interesting analysis in full.

"Consider a point on the curve anywhere other than at (-3,9). The co-ordinates of the point are (x, x^2). Now consider a line joining the two points. It will have a gradient of

$\frac{x^2 - 9}{x + 3} = \frac{(x + 3)(x - 3)}{x + 3} = (x - 3)$. It follows that $(x - 3)x + b = x^2 = y$, where b is the y-intercept. Solving the equation for b: $(x - 3)x + b = x^2$. $\therefore x^2 - 3x + b = x^2$. $\therefore b = 3x$. Thus the value of the y-intercept will be three times the value of the x co-ordinate of the point chosen.

This can be generalised as follows: If we take non-coincident points on the curve (x_0, x_0^2) and (x_1, x_1^2), then the line joining them has gradient $\frac{x_0^2 - x_1^2}{x_0 - x_1} = x_0 + x_1$, and $b = x_0^2 - x_0(x_0 + x_1) = -x_0x_1$, where b is the y intercept.

If we consider higher powers of x, we arrive at the following:

Take two points (non-coincident), (x_0, x_0^n) and (x_1, x_1^n) which lie on the curve $y = x^n$. The gradient of the line joining the two points is $\frac{x_0^n - x_1^n}{x_0 - x_1}$ which equals $\Sigma x^\alpha y^\beta$ where α and β pass through all the possible integral non-negative values satisfying the condition $\alpha + \beta = n - 1$. If we call the gradient a, then $b = x_0^n - x_0 a = -x_0 x_1(\Sigma x^\alpha y^\beta)$ where b is the y-intercept, and α and β pass through all the possible integral non-negative values satisfying the condition $\alpha + \beta = n - 2$."

SOLUTION TO PROBLEM 6.3.5.

For this problem, which consisted of several parts, see Professor Crossley's article in *Function*, Vol.6, Part 3. Essentially we asked of Figures 1, 2 for proofs of the following facts.

(1) In Figure 1 below, prove $AD = BD + CD$ (ΔABC is equilateral).

(2) In Figure 2 below, prove AD is proportional to $BD + CD$.

J. Ennis (Year 10, M.C.E.G.S.) solved both problems. His solutions are given below.

$\measuredangle ABC = 60°$ (ABC is equilateral)

$\therefore$ $\measuredangle ADC = 60°$.

Find a point E on AD such that $\measuredangle CED = 60°$ i.e. that $CE = CD = DE$.

$AC = BC$

$EC = CD$

$\measuredangle DCB = \measuredangle ECA$

($\measuredangle ECD = \measuredangle ACB = 60°$).

$\therefore$ Triangles AEC and BDC are congruent.

$\therefore$ $BD = AE$.

$AD = AE + ED$

$\therefore$ $AD = BD + CD$.

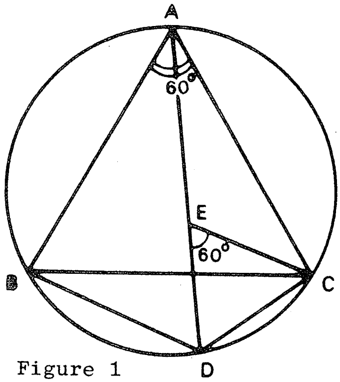


Figure 1

Find a point E such that $\measuredangle DEC = \measuredangle BAC$.

$\measuredangle ADC = \measuredangle ABC$.

$\therefore$ Δ's DEC and BAC are similar.

$\therefore$ $\dfrac{BC}{AC} = \dfrac{DC}{EC}$

$\measuredangle BCD = \measuredangle ACE$.

$\therefore$ Δ's ACE and BCD are similar.

$\therefore$ $\dfrac{BD}{AE} = \dfrac{BC}{AC} = \dfrac{CD}{ED}$.

$AD = AE + ED$

$\therefore$ $AD = BD\left(\dfrac{AC}{BC}\right) + CD\left(\dfrac{AC}{BC}\right)$.

$\therefore$ AD is proportional to $BD + CD$.

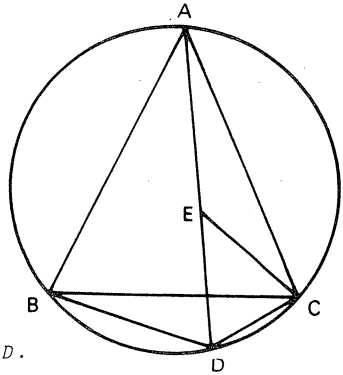


Figure 2

A different solution, using trigonometry, rather than a construction, came from Colin Fox, a teacher at Scotch College and another, different again, from John Barton, Drummond Street, Carlton.

None of our solvers addressed themselves to the third exercise of Professor Crossley's article, which was to show that Ptolemy's Theorem (*Function*, Vol.5, Part 3; Vol.6, Part 3) is the equivalent of the trigonometric result

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi .$$

To see this, consider Figure 3. Ptolemy's Theorem states that

$$AC.BD = AB.CD + BC.DA.$$

Consider the case in which AD is a diameter. Then angles ABD, ACD are right angles. The result then follows fairly easily, where

$$\measuredangle CDA = \theta, \quad \measuredangle BDA = \phi.$$

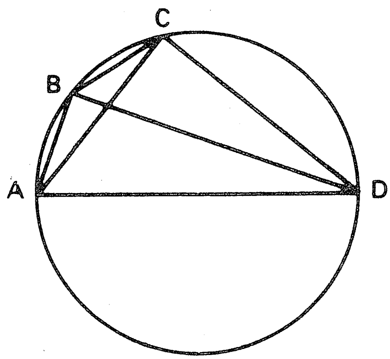


Figure 3

SOLUTION TO PROBLEM 6.4.1.

On a piece of paper are N statements. Statement n (where $1 \leqslant n \leqslant N$) reads: "There are exactly n incorrect statements on this page." Which statement(s), if any, are true? What if the word "incorrect" in each statement were replaced by the word "correct"?

If the word is "incorrect", $n = N - 1$ gives a true statement.

If the word is "correct", $n = 1$ is true if ungrammatical; if by "correct" we also imply correct grammar, there is no solution.

SOLUTION TO PROBLEM 6.4.2.

Prove that if P_n is the product of n consecutive integers then $n!$ divides P_n.

The argument opposite was submitted by Clayton Smith, Year 10, Lilydale High School.

Let $P_n = (N - n + 1)(N - n + 2)\dots N$.

Then $\frac{P_n}{n!} = \binom{N}{n}$ and this is the number of ways in which n objects can be chosen from N. This number is clearly integral and the result follows.

Two new problems for the long vacation.

PROBLEM 6.5.1.

N men each toss a cent. If $N - 1$ cents agree and the Nth does not, the Nth takes all the money. If this does not happen the money jackpots. What, on average, does the eventual winner gain?

PROBLEM 6.5.2.

Compute the smallest positive root of $x = \tan x$.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

REPORT FROM BUDAPEST

In *Function*, Vol.6, Part 3, we gave news of our team in the 1982 International Mathematical Olympiad held in Budapest. We now have news as to the results. We print below an excerpt from the official report.

"The 23rd IMO was held in Budapest, Hungary from 5 July to 14 July 1982. Thirty countries were represented at the Olympiad, all but one by the maximum team of four. Of the 119 competitors, seven were girls and of these three came from Tunisia.

The competition comprised two 4½ hour examinations held on 9 and 10 July, each paper having three questions. There were 7 marks for each question, a total of 42 for the two papers. Marks were awarded to each question on a scale which emphasised the importance of complete answers. For example, an answer which gave only half the correct solution was likely to receive only 1 mark and, at the other end of the scale, an answer with only a minor blemish was likely to lose at least 1 mark.

As is customary, first, second and third prizes were awarded to approximately 1/12, 2/12 and 3/12 of the students respectively, the mark ranges for the three levels being, on this occasion, 42-37, 36-30 and 29-21. The 10 first prizes went to Austria, United States, France and Viet Nam with one each and to East and West Germany and Russia with two each. In all, 23 countries shared the prizes, Australia being one of these with one third prize (which went to David Chalmers of South Australia) and, it should be added, another competitor (Alan Blair of NSW) just 1 mark below the cut-off."

The 1983 Olympiad will be held in France. It is expected that Australia will be host nation in 1988.