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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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In our last issue we began our new *Perdix* column devoted to mathematics competitions and to the question of problem-solving skills. These may be developed by guided experience. Tricks learned in one context often turn out to be useful in another. A problem may be the same as, or very similar to, one that at first sight seems quite different. These are the matters that Perdix will discuss in each issue. His column begins this time on p.29 and discusses modular arithmetic.

THE FRONT COVER

This issue's front cover shows two views of a Raba Teasascope. See the article on p.2.

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RABA'S TEASASCOPE

Our front cover for this issue shows two views of a puzzle, related to Rubik's cube, designed by the French sculptor, Raoul Raba. Figure 1 shows the construction. Three overlapping circles, themselves dissected by circular arcs, are cut from a

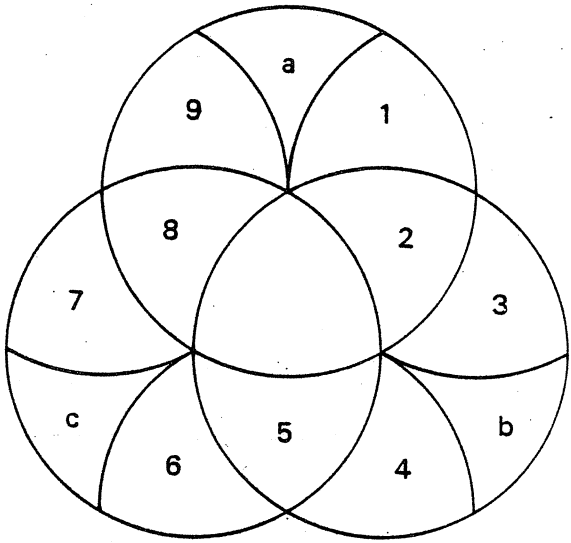

Figure 1

flat board as shown. The puzzle has fourteen pieces:

3 pointed cusp pieces, *a,b,c,*
9 shields or "Tasmanias" 1,2,...,9,
1 centrepiece,
the surround.

Each of the circles is free to rotate with respect to the rest of the puzzle. Call each circle after the cusp piece, *a*, *b* or *c*, contained in it. It will be found that there are two constraints on the motion:

1. The cusp pieces cannot leave their assigned circles;

2. A circle, if it is to turn, must contain the centrepiece.

The first of these constraints allows the notation introduced above to be applied unambiguously.

As with Rubik's cube, the object of the puzzle is to upset an initially ordered design and then to re-establish it. From any of the allowable disorderings (not all disorderings may be reached "legally"), we would like to get back, via some algorithm, to the originally ordered state. The analogy with the cube is strong (quite strong, as we shall see below), but it is easier to "cheat" with the teasascope by pulling it to pieces and re-assembling it correctly.

We may now use these same symbols a,b,c to apply to 60° clockwise rotations of the circles a,b,c respectively, and denote by $\bar{a},\bar{b},\bar{c}$, the inverses (anticlockwise rotations) respectively of these.

Suppose now that operation b is applied. The centrepiece now occupies the position of piece 2 which is common to circles a,b. By the second of our restrictions, these are the only circles that can now turn. Further application of b restricts any movement of circles other than b, unless circle b is rotated so far that operation $\bar{b}$ results. We thus neglect this possibility. Application of $\bar{b}$ immediately following application of b restores the status quo and is uninteresting. We thus consider $a,\bar{a}$.

If $\bar{a}$ is applied, only circle a can move and again a degeneration has occurred, so we concentrate on the final possibility: a. This restores the centrepiece to the centre and so we may proceed.

We have applied first b and then a, and we write this ab. (Note the reversal of order here.) We now have permuted the order of the pieces

$$1,2,3,b,4,5,8,9,a$$

by moving each into the position originally occupied by the next term in the sequence. This is referred to as an *elementary operation* for the puzzle, and is denoted C. (The bottom picture on the cover is the result of applying operation C to the top picture.) There are two other elementary operations: bc, i.e. A, and ca, or B. Each of these also has an inverse: $\bar{C} = \bar{b}\bar{a}$, etc. Strings of these elementary operations may now be envisaged as for the cube.

Raba called his device a *taquinoscope* from the French words *taquiner*, to tease, and *taquin*, meaning puzzle. This dual meaning does not translate precisely into English, but *teasascope* is probably the best approximation.

We learned of the teasascope through a sister journal *Le Petit Archimède*[†], with which *Function* has an exchange agreement.

[†]Available through the library of the Mathematical Association of Victoria.

They also sell teasascopes for FF50 and solution booklets (in French) for FF10 (FF50 ≅ A$10). Complete the order form below and remit to ADCS-Abonnement-B.P.0222-80002 AMIENS Cedex-France, with the money, to obtain your teasascope. The edition is limited to 1000 copies.

The teasascope is patented (French patents, Nos. 77-30347, 79-21130) but this does not prevent one from making one's own, which is what we did to get the photographs on the cover. The numbering differs from that of Figure 1 as we wanted to get a full clock-face. There are severe problems, however; the individual pieces must be very exactly cut if rotation as described is to be possible. Furthermore, a low friction material is necessary. (Ours, cut from strawboard, is very poor in this regard.)

The mathematics of the teasascope have been worked out by André Délédicq, who has published two articles on the subject. One is that available from *Le Petit Archimède* referred to above. It is co-authored with Raba and gives a full algorithm for solution. The other is a beautifully illustrated one in the French glossy *La Recherche* (Hargrave Library at Monash and the State Library of Victoria each have copies).

According to this latter one, Raba began by making a plane "cube", illustrated there, akin to but not equivalent to, Rubik's. More recently, he has produced a modified Rubik's cube precisely equivalent to a slightly simplified teasascope.

For more on this topic, you could consult his articles: Autour du cube de Rubik: une nouvelle génération de taquins, *La Recherche*, No.128, Dec.1981, p.1450; Le taquinoscope de Raba, *Le Petit Archimède* No.93-94, Sept.1983, p.23.

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BON DE COMMANDE

Mme ou M. :____

demeurant à *(adresse complète)*__________________________

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prie l'ADCS de bien vouloir lui faire parvenir

_____ PA-TAQUINOSCOPES pour la somme de _____ x 50 FF = _____ FF

et _____ étude mathématique du PA-Taquinoscope : _____ x 10 FF = _____ FF

THE FORGOTTEN ARTS OF ARITHMETIC: THE "LONG DIVISION" PROCESS FOR EXTRACTING SQUARE ROOTS

J.A. Deakin,
Shepparton College of TAFE

Calculate the square root of 823288249. Find the cube root of 445943744. A student's nightmare? Yet not so many years ago, such questions regularly appeared on secondary school arithmetic examination papers. The long division processes for extracting square and cube roots of numbers and algebraic expressions are arts which seem to have been forgotten by all except a few mathematicians of the old school.

The procedure for extracting the square root of a number or an algebraic expression is made to depend on the identity

$$\begin{aligned}(a + b)^2 &= a^2 + 2ab + b^2 \\ &= a^2 + b(2a + b) .\end{aligned}$$

To find the square root of the expression $a^2 + 2ab + b^2$, we note that the expression consists of the sum of two terms:

(i) a^2, the square of a, the first term of the root, and

(ii) the product of b and the expression $(2a + b)$, consisting of the second term of the root added to twice the first term of the root.

In order to extract the square root of the given expression, we simply reverse the process, and the work may be set out as follows.

	$a \quad + b$
a	$a^2 + 2ab + b^2$
	a^2
	$2ab + b^2$
$2a + b$	$\underline{2ab + b^2}$

To extract the square root of the expression $a^2 + 2ab + b^2$, we proceed as follows.

(1) Arrange the terms of the expression in descending powers of one pronumeral (a).

(2) The square root of a^2, a, is written down as the first term of the root, and its square is subtracted from the given expression.

(3) The first term of the remainder is divided by twice the first term of the root to give the second term of the root, b.

(4) The second term of the root is added to twice the term already found, to form the complete divisor, $2a + b$.

(5) The product of the second term b and the divisor $2a + b$ is subtracted from the first remainder.

By repeating steps 3,4,5, the square root of any algebraic expression may be found.

The rule may also be applied to extract the square root of ordinary numbers. Thus, to find the square root of 823288249, we proceed as follows.

	2 8 6 9 3	
2	8 23 28 82 49	
	4	
48	4 23	48 = 2(20) + 8 (cf. above: a=20, b=8.)
	3 84	
566	39 28	566 = 2(280) + 6
	33 96	
5729	5 32 82	5729 = 2(2860) + 9
	5 15 61	
57383	17 21 49	57383 = 2(28690) + 3
	17 21 49	

In numerical examples such as this, it is customary to omit the ciphers in the successive steps. The procedure may be summarized as follows.

(1) Divide the number into periods of 2 digits, from right to left, as shown.

(2) Below the first period on the left, write the largest square that is equal to it, or less (4), and write its square root (2) as the first digit of the root.

(3) Subtract the square from the first period, then bring down the next period to make the first remainder (423).

(4) In a second column to the right, double the quotient so far as it goes, adding a zero (40).

(5) Now estimate the digit which must be added to this number so that when the total is multiplied by the same digit, the product will be the largest possible number equal to or less than the first remainder (8). Write this digit as the second digit of the quotient.

(6) Write the new product, 8×48, under the first remainder, subtract, and bring down the next period to form the second remainder.

(7) Repeat steps 4,5,6 until all the periods are exhausted.

The process can be used to extract the square roots of numbers which are not perfect squares, and of decimal fractions, to any desired number of significant figures. It will be instructive for the reader to use this process to find the square root of $(1 + x)$, and to compare the result with the binomial series expansion for $(1 + x)^{\frac{1}{2}}$; under what conditions do these series converge?

In a similar way, an algorithm for extracting cube roots can be constructed depending on the identity

$$\begin{aligned}(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ &= a^3 + b(3a^2 + 3ab + b^2) .\end{aligned}$$

Can you work out the steps in it?

It is of interest that the procedures for extracting square roots and cube roots are both easier to apply to algebraic expressions than to numerical examples. In simple cases, the square roots and cube roots of algebraic expressions can be found by direct factorization; however, the above algorithms provide a systematic procedure for use when the factors of the given expression are not readily found by inspection. Also, with a little practice, the reader should be able to find mentally the square root of any perfect square less than 10 000 and the cube root of any perfect cube less than 1 000 000; this skill was expected of students in the author's own junior secondary school days.

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THE LAW OF AVERAGES

The most important questions of life are, for the most part, really only problems of probability.

P.S. Laplace, 1812.
Théorie Analytique des Probabilités

GAUSS:

THE MATHEMATICAL MOZART

Trevor Halsall, Ursula College, A.N.U.

Many consider Johann Karl Friedrich Gauss to be one of the three greatest mathematicians of all time. He is classed with Archimedes and Newton. His work encompassed all branches of mathematics - an extraordinary feat in itself. Although crowned as the "Prince of Mathematicians" Gauss had a very humble background indeed.

His father, a bricklayer, wanted him to follow in a similar career. Gauss, however, started to show his genius early. At the age of three, he is said to have spotted a mistake in his father's calculations and promptly stated the correct answer. His schooling was dotted with such achievements, but on a much grander scale. As a twelve year old, he was already questioning the universality of Euclidean geometry. Fortunately, an assistant at the school, Johann Bartels, had a similar passion for mathematics. The two studied together. Through Bartels, Gauss came to the attention of Duke Ferdinand of Brunswick. From then on, he was educated at the expense of the Duke.

Like Mozart, Gauss showed his genius at an early age and continued to display it throughout his life. Like Mozart too he was versatile, conversant with the entire body of the mathematics of his day and indeed greatly extending it. Gauss and Mozart, each in his own field, reached the very highest levels of genius, enriching the world of today. The two men were approximate contemporaries, Mozart being 21 years the senior. However, Mozart's early death in 1791 occurred before any of Gauss's major achievements.

At fifteen, Gauss entered Caroline College in Brunswick. He was gifted at languages and literature as well as mathematics. A year later, the concept of a non-Euclidean geometry was introduced to him.

By 1795, Gauss had invented the method of "least squares". It has direct applications in surveying, astronomical calculations and making predictions based on a large number of statistics. Essentially, it involves finding the best curve to fit data points.

Gauss entered the University of Göttingen in 1795. It was here, at nineteen years of age, that he gave the first proof of the law of quadratic reciprocity (a result that shed light on the division of a square number by a prime leaving a prime remainder). He went further and gave a total of six different proofs. In 1796, Gauss decided that his career lay in mathematics (he had been considering philology!).

While at university, he constructed a regular polygon of seventeen sides using only a straightedge and compass. This was an unsolved problem left by the Ancient Greeks. He showed that only polygons of certain numbers (related to the *Fermat numbers*, primes of the form $2^{2^n} + 1$) of sides could be constructed by classical methods.† Gauss made many more discoveries in the field of number theory before returning to Brunswick in 1798. Here he earned a modest living by giving private tuition. He disliked teaching and so had few students.

Gauss began a small diary in 1796, which he continued until 1814. It contains 146 very concise statements relating some of his discoveries. Most are remarkable in themselves although they were never published in his lifetime. All traces of how he arrived at them were destroyed. It took several decades for mathematicians to provide their own proofs. Many of Gauss's contemporaries would communicate a result to him only to find that he had reached the same conclusion many years before.

Gauss is reported to have said that "...he undertook his scientific works only in response to the deepest promptings of his nature, and it was a wholly secondary consideration to him whether they were ever published for the instruction of others."†† Many other important and extensive papers were discovered unpublished after his death. "...He could have advanced mathematics by a half-century or more if he had chosen to make his knowledge available during his lifetime."§

Gauss held steadfast to his personal motto "Few, but ripe". There were untold numbers of brilliant, original ideas in his head but he chose carefully. He released them to the world only when he had found an entirely rigorous proof. Despite his attention to detail, Gauss still managed to publish over 155 papers.

In 1799, he proved the fundamental theorem of algebra. Simply stated, it says that every algebraic equation in one unknown has a solution of the form $a + bi$ (where $i^2 = -1$; and a, b are real numbers). For this, Gauss received his doctor's

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† See Courant and Robbins, *What is Mathematics?*

†† Bell, E.T.: *Men of Mathematics* (Penguin, 1953) p.251.

§ Siedel, F. & J.M.: *Pioneers in Science* (Houghton Mifflin, 1968) p.78.

degree in absentia from the University of Helmstedt. He came to the notice of many fine mathematicians with the publication of *Disquisitiones Arithmeticae* in 1801. In the same year, he proved the fundamental theorem of arithmetic: that every natural number can be represented as the product of primes in only one way.

After calculating the orbit of a new planet, Ceres, Gauss gained international fame. He declined an offer of a Professorship in St Petersburg in 1803. In 1806, when his sponsor, the Duke, died fighting against Napoleon, Gauss had to find an alternative means of support. His work on Ceres led to an appointment as director of the Göttingen Observatory and Professor of Astronomy in 1807. The wages were minimal but adequate for Gauss's needs. He remained in these posts until his death. In fact, he only once slept away from the Observatory - due to a scientific congress in Berlin.

Gauss's first wife, Johanne, died in 1809 after the birth of their third child. They had been married for four years. It is said that their first son, Joseph, inherited his father's skill for mental calculations. Still grief-stricken, Gauss wed Minna Waldeck and they had a further two sons and a daughter. Minna died young and only one of Gauss's six children survived him.

After many years involved in astronomical discoveries, Gauss returned briefly to pure mathematics. In 1811, he developed the theory of analytic functions of a complex variable but never published it. The following year saw him publish a masterpiece on the hypergeometric series. Logarithms, trigonometric functions and the general binomial theorem are just some of the special cases of this series.

From 1821 to 1848, Gauss ventured into the field of geodesy. He was scientific adviser to the Hanoverian and Danish governments for an extensive geodetic survey. From his investigations into certain types of curved surfaces, Gauss began the branch of mathematics called differential geometry. He devised a heliograph, which uses the sun's rays as straight lines to mark the earth's surface. This allowed more precise trigonometrical measurements of the planet's shape to be calculated. Gauss made numerous original contributions to the theories of surfaces and conformal mapping (that is, preserving angles (at, for example, road intersections) on a two-dimensional map). The latter has important applications in electrostatics, hydrodynamics and aerodynamics.

Gauss concentrated on mathematical physics between 1830 and 1840. In 1832, he introduced a logical set of units for measuring magnetic phenomena. Working with W.E. Weber, he invented the declination instrument and the bifilar magnetometer. In the following year, they devised an electromagnetic telegraph system which sent messages one and one quarter miles. Samuel F.B. Morse patented his telegraph four years later. Gauss's instrument could have made him a fortune if he had been interested in developing it commercially. However, "he was not inspired by the prospect of practical applications, for he sought truth for its own sake, finding his reward and pleasure in the success of his efforts alone."†

Terrestrial magnetism was another area in which Gauss 'specialised'. He instituted the first observatory designed specifically for work in that field. His calculations concerning the location of the magnetic poles were amazingly accurate. It was Gauss who pointed out that the units of quantities such as density and energy could be expressed in terms of a few fundamental units (for example, length, mass and time).

The endeavours of the 'Prince of Mathematicians' were both extensive and ahead of his time. His work helped to establish the branch of pure mathematics. He was the first mathematician to pay due attention to the question of the convergence of infinite series. He made many and deep contributions to number theory (e.g. the theory of Fermat numbers), to geometry, topology and the theory of optical instruments. His work on the capillary action of a fluid led eventually to the principle of the conservation of energy. He worked out theories of perturbations that helped towards the discovery of the planet Neptune. His methods for astronomical calculations are still in use today.

As can be surmised, his intense concentration resulted in reduced contact with humanity. He did find time to read the classics of European literature and keep up with world politics in all the newspapers. His hobbies included foreign languages and the new sciences such as botany and mineralogy. A fascination with numbers led to a large collection of numerical records. They included such oddities as the length of lives of famous men in days. As regards his disposition, it is generally agreed that, "Gauss was deeply religious, aristocratic in bearing, and conservative".†

Sartorius von Walterhausen wrote, "As he was in his youth, so he remained through his old age to his dying day, the unaffectedly simple Gauss. A small study, a little work table with a green cover, a standing-desk painted white, a narrow sopha (*sic*) and, after his seventieth year, an arm chair, a shaded lamp, an unheated bedroom, plain food, a dressing gown and a velvet cap, these were so becomingly all his needs".††

Even in old age, his agile mind constantly searched for knowledge. He taught himself Russian at the age of sixty-two. Within two years, he was writing and speaking the language fluently. At sixty-eight, Gauss completed the huge task of reorganizing the Fund for Widows and Children of Professors. Two years later, he gave his fourth distinct proof of the fundamental

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†*Encyclopaedia Brittanica-Macropaedia* (Volume seven, 1979) p.967.

††Bell, E.T.: *Men of Mathematics* (Penguin, 1953) p.269.

theorem of algebra. The city of Göttingen made him an honorary citizen.

Early in 1855, Gauss began suffering from an enlarged heart and shortness of breath. The symptoms of dropsy appeared. He passed away peacefully in February at seventy-eight years of age. Coins were struck in his honour and a statue was raised.

His name lives on in many scientific laws and theorems. Magnetic flux density has the gauss as its unit. The 1001st planetoid discovered was named Gaussia. Most of all, he is remembered as the last man to contribute significantly to all branches of mathematics.

It is unlikely that his achievements will every be surpassed.

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NOT SO SURPRISING AFTER ALL!

Eddington once told me that information about a new (newly visible, not necessarily unknown) comet was received by an observatory in misprinted form; they looked at the place indicated (no doubt sweeping a square degree or so) and saw a new comet. (Entertaining and striking as this is, the adverse chance can hardly be put at more than a few times 10^6.)

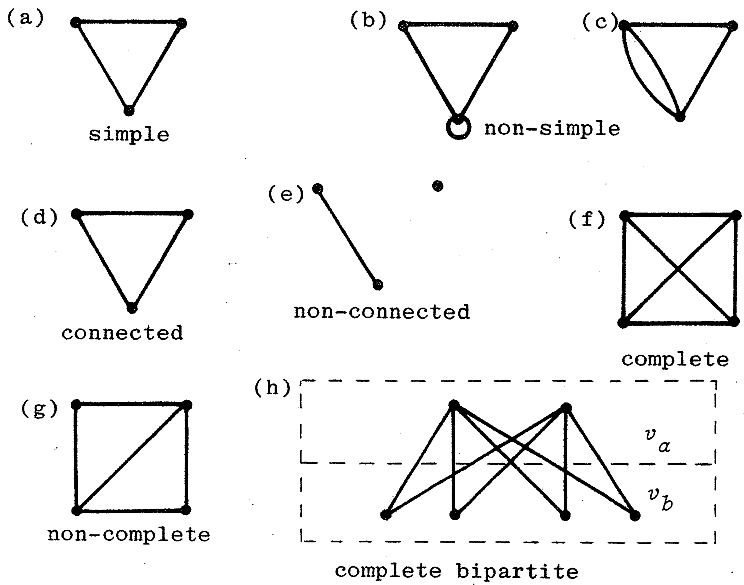
J.E. Littlewood, *A Mathematician's Miscellany*, 1953.

FAMILY RELATIONSHIPS AND GRAPHS

Jacqueline Wong,

Student, Monash University

In graph theory, a graph is considered to be a set of vertices (usually drawn as dots or small circles) and an associated set of edges (drawn as lines). Each edge must begin at a vertex and end at a vertex (either the same vertex or a different one); a vertex may or may not be attached to the rest of the graph by edges. Graphs are characterised in various ways. A simple graph is one in which there are no loops (edges which begin and end at the same vertex) and no more than one edge between each pair of vertices; a connected graph is one which cannot be divided into two isolated parts. A complete graph has every vertex of the graph connected to every other vertex; a complete bipartite graph consists of two disjoint sets of vertices, v_a and v_b : each vertex of v_b being connected to all vertices of v_a, and each vertex of v_a to all vertices of v_b. We will see an example of this below.

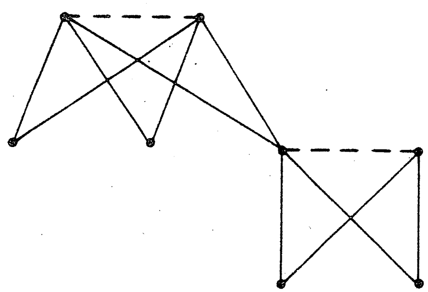


Note that some features of the graph (such as the arrangement of the vertices and the shape and length of the edges) are not important; if two edges are shown in a graph diagram as crossing at a point which is not a vertex, they are treated as if they did not touch.

A graph is essentially an abstract diagram of the relationships between a set of points (vertices), but it may represent a great variety of physical or mathematical concepts - for example, a map of towns and their connecting roads, an electrical network, a chess problem. Here, I consider the relationships between members of a family and how they can be represented by a graph diagram. In general the graphs of family relationships are simple, usually connected, and can be split up into a series of bipartite (mostly connected) subgraphs.

To consider family relationships as graphs let people be vertices, and lines of descent (parent to child) the edges. (I have put in "marriage" relationships as dashed lines - these are not really part of the graph, but serve to tie parts of it together e.g., to include someone who has married into a family but left no descendants.)

We have to impose a restriction on the arrangement of the vertices - they must be so ordered that we can see whether or not people are in the same generation, and, more importantly, whether we are travelling "up" or "down" the family tree as we traverse the edges. (It would be possible to label lines of descent with arrows pointing from parent to child, in which case we would speak of going "back" up the tree or "forward" down the tree). For clarity, I will put vertices representing people of the same generation in a horizontal line, those of one generation earlier in a row above, those of a generation later in a row below.



Note that each person should have 2 edges going "up" the graph (connecting to his parents); each person has as many edges going "down" the graph as they have children. "Marriage" lines are the only horizontal lines in the graph.

Then a path of length 2 (i.e. containing 3 vertices) with all steps down goes from grandparent to grandchild, and if the steps are all up, grandchild to grandparent. A path of length 3 with 1 step up then 2 down connects aunt or uncle to nephew or niece, and with 2 steps up then 1 down, nephew or niece to aunt or uncle. If all steps are down, the path connects great grandparent to great grandchild and vice versa if all steps are up.

Paths of length 4 connect great great grandparents to great great grandchildren or vice versa and there are also these possibilities:

1 step up then 3 down: great {uncle, aunt} to grand {niece, nephew}

3 steps up then 1 down: grand {nephew, niece} to great {aunt, uncle}

2 steps up then 2 down: the two vertices represent first cousins.

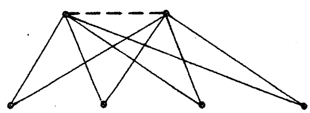
As to other cousins, English family relationships have a rather confusing nomenclature for the more distant cousins, but by this method we can define them fairly accurately. If the path lengths up and down are equal, the two people are 1st, 2nd, 3rd cousins, etc. if the path lengths are of 2, 3 and 4 steps respectively. If the path lengths (up and down) are unequal, the shorter path decides if people are 1st, 2nd etc. cousin (by the same rule as above) while the difference between the path lengths tells how many times "removed" they are.

E.g. (a) 2 up, 3 down - 1st cousin, once removed,
(b) 4 up, 6 down - 3rd cousin, twice removed.

(Notice that the "removed" number is really an indication of difference in generation level - one of the cousins in (b) belongs to a generation two steps earlier than the other.)

The "nuclear family" forms a complete bipartite graph of the form $U_{2,n}$ where n is the number of children:

e.g.

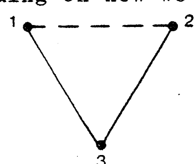


$U_{2,4}$.

It should be possible, for any given family, to isolate from the total graph a subset of the form of the above - i.e. a family "tree" is composed of interlocking family groups.

Tracing the relationship between any two individuals in the graph consists in counting the path length between them, noting (a) the number of edges "up" and "down" (b) how the path changes from "up" to "down" (c) in some cases, the number of paths joining them.

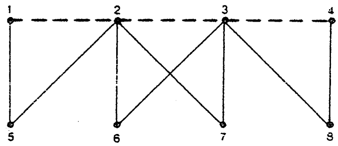
Parents and children are connected by a path length of 1, going up or down depending on how we view the relationship. In this graph,



1 is parent of 3 (1 step down), and 3 is child of 1 (1 step up).

Sisters and brothers are connected by paths length of 2, one step up and one down. Full siblings will each be connected by two different paths, half-sisters and brothers by only one. Step-brothers and sisters are only connected by descent lines if you go up, then down then up then down (or across the marriage lines). If you have to go *down* then *up* at any stage, no blood relationship exists.

For example this graph



shows a family with two divorced parents each with one child marrying and producing two further children. 2 and 3 are the parents, 1 and 4 their divorced spouses; 5 and 8 the children of the previous marriages, 6 and 7 the children of the present marriage.

6 and 7 are full siblings (2 different paths of length 2: 6-2-7, 6-3-7), while

5 and 6 are half-brothers (one path of length 2: 5-2-6).

5 and 8 are step-siblings (no path of length 2).

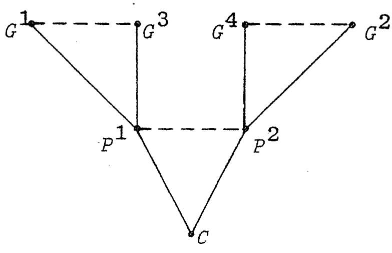
The only ways to get from 5 to 8 are

5 up 2 down 6 up 3 down 8
5 - 2 - 7 - 3 - 8
5 - 2 - 3 - 8 .

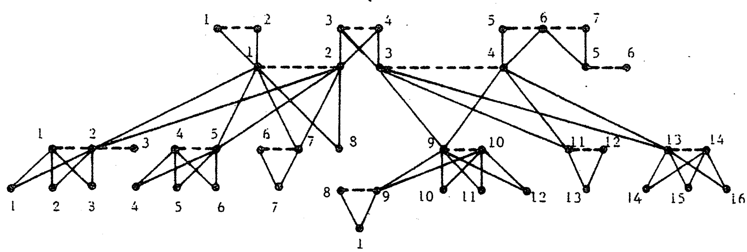
The fact that you have to change direction from "down" to "up", or else go across the marriage line, means that 5 and 8 are not related "by blood", but only "by marriage".

(Similarly, a child's two sets of grandparents are not related by blood, because to reach one from the other, you must go down then up:

$G^1 - P^1 - C - P^2 - G^2$.)



Here is a more complicated example.



II2 and II3 are siblings, as are III2, III5, III7 and III8; III9, III11 and III13 and IV9, IV10, IV11 and IV12 (etc.).

II4 and II5 are half brothers (and the relationships between the descendants of II4 and II5 should probably be designated "half" relationships e.g. half-uncle, half-cousin etc., but this is usually ignored, and certainly, from the point of view of nearness of relationship, becomes less and less meaningful as you go down the generations).

I3 and I4 are the common grandparents of III2, III5, III7 and III8 and of III9, III11 and III13, I1 and I2 are grandparents of III2, III5, III7 and III8 but not of III9, III11 and III13.

II2 is III9's aunt; II1 can be described as III9's "uncle by marriage" - descriptions such as "my cousin's father" or something of the sort are ambiguous and not usual in English.

III9 and III5 are first cousins.
IV9 and III5 are first cousins once removed.
V1 and III5 are first cousins twice removed.

Note that III3 is not related to the family except by marriage. He could only be described as "my daughter's first husband" or something of the sort.

In English usage, except for the case of aunt, niece, uncle, and nephew, it is not necessary to specify the gender of the people concerned in order to specify their relationship. (Gender-specific terms are common, of course, but there are "non-sexist" terms available, e.g. grandparent, sibling.)

In some other cultural systems, the sex of the two people concerned is very important, as may be whether they are related through a female or a male. This is true of Chinese family nomenclature; as well, one needs to know which of the two people is older or younger, and sometimes where they rank in their own generation. For instance, there are different terms for: an uncle who is one's father's older brother; an uncle who is one's father's younger brother; an uncle who is one's mother's older brother, etc.

This makes it much more difficult to reduce the family relationship to a graph. One can write different symbols for male and female e.g. ○ for female □ for male, as is common in most ways of writing family trees; but to include information about relative ages one would have to introduce some sort of horizontal stratification into the generations, e.g. older persons are always found to the right of younger persons - but as cousin's families tend to overlap a great deal in age, this would produce a very complicated and confusing graph.

Another complication sometimes arises where there is (between cousins) marriage across the generations, e.g. between IV6 and V1 in the example given above. Offspring of such a marriage are related to the other members of the family in two different ways, which can only be accurately specified by giving both relationships - "through the mother" and "through the father" [like J.R.R. Tolkien's characters Frodo and Bilbo Baggins, who were "first and second cousins, once removed either way, as the saying is"].

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

MATHEMATICS APPLIED

I am convinced that the future progress of chemistry as an exact science depends very much indeed upon its alliance with mathematics.

A. FRANKLAND

In mathematics we find the primitive source of rationality; and to mathematics must the biologists resort for means to carry on their researches.

A. COMTE

In the near future, mathematics will play an important part in medicine; already there are increasing indications that physiology, descriptive anatomy, pathology and therapeutics cannot escape mathematical legitimation.

M. DESSOIR

The permeation of biology by mathematics is only beginning, but unless the history of science is an inadequate guide, it will continue. Mathematics may very often help in proving the obvious, but the obvious is worth proving when this can be done.

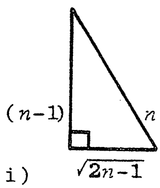
J.B.S. HALDANE

LETTERS TO THE EDITOR

NEWTON'S FIRST THEOREM

The square root of every odd number that is not a perfect square generates (in a way I will make clear) at least one infinite set of Pythagorean triads such that for every triad in a given set there is the same mathematical relationship between a pair of corresponding sides in that triangle, and such that that same mathematical relationship is unique to that particular set, and hence to the square root of that particular odd number.

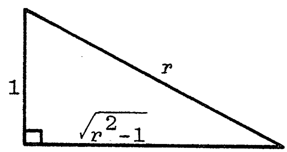
Every odd number can be written in the form $(2n \pm 1)$, where n is a whole number. For $\sqrt{3}$ and $\sqrt{5}$ the relationship is that the shortest side and the hypotenuse are approximately in the ratio $(n - 1):n$. If that were the precise ratio, then $\sqrt{3}$ and $\sqrt{5}$ would not be irrational. The $\sqrt{3}$ triads are a different set from the $\sqrt{5}$ triads, because the ratios are not the same. For $\sqrt{3}$ the ratio is approximately 1:2, while for $\sqrt{5}$ it is approximately 2:3.



(i)

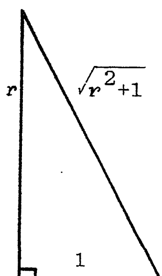
For the square roots of the other odd numbers the relationship is that the second shortest side and the hypotenuse are approximately in the ratio $(n - 1):n$, provided that that odd number is not a perfect square.

Does the theorem hold for even numbers? Yes, if and only if that even number is not a perfect square, and is of the form $r^2 \pm 1$, where r is an integer.



(ii)

The square root of an even number with the above properties generates only one infinite set of Pythagorean triads. If the square root is of the form $\sqrt{r^2 - 1}$, it has as its mathematical relationship that the shortest side and the hypotenuse are approximately in the ratio $1:r$.



If the square root is of the form $\sqrt{r^2 + 1}$, then the only set has as its mathematical relationship that the sides which form the right angle, that is the two shortest sides, are approximately in the ratio $1:r$.

The square root of an odd number which is not a perfect square and which gives the form $\sqrt{r^2 \pm 1}$ where r is an integer generates not one but two infinite sets of Pythagorean triads, with the mathematical relationships that appertain to each set being distinctly different, apart from the case of $\sqrt{3}$, which is a special case for the following reason.

If n in diagram (i) and r in diagram (ii) are such that $n = r = 2$, the triangles are identical. For every higher value of n and r the two triangles in diagrams (i) and (ii) are quite different triangles. That is why the two sets of triads generated form a single set with the relationship between the hypotenuse h and the shortest side s being

$$h = 2s \mp 1 .$$

If the hypotenuse was precisely double the shortest side, and there was not this discrepancy of just one unit, then $\sqrt{3}$ would not be irrational.

I will now make clear just in what way the Pythagorean triads are generated by generating them for the square root of three.

Let $x^2 = 3$. Subtract 4 from both sides of the equation $x^2 - 4 = -1$. The left-hand side is the difference of 2 squares.

$$(x - 2)(x + 2) = -1 .$$

$$x - 2 = \frac{-1}{x + 2} = \frac{-1}{4 + \{(x + 2) - 4\}}$$

$$= \frac{-1}{4 + (x - 2)}$$

$$= \frac{-1}{4 - \cfrac{1}{4 + (x - 2)}} \quad \text{(substituting for } (x-2) \text{ from the left-and side)}$$

$$= \cfrac{-1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \ldots}}}} \quad \ldots \text{ ad infinitum.}$$

If the rth rational approximation to this infinite fraction is $-\frac{n}{d}$, then the $(r+1)$th approximation is $-\cfrac{1}{4 - \frac{n}{d}} = \cfrac{-1}{\frac{(4d-n)}{d}} = \frac{d}{4d-n}$.

The first approximation $\frac{-n_1}{d_1}$ is obviously $-\frac{1}{4}$.

$$-\frac{n_2}{d_2} = -\frac{4}{16-1} = -\frac{4}{15}\,; \quad -\frac{n_2}{d_2} = -\frac{15}{56}\,; \quad -\frac{n_3}{d_3} = -\frac{56}{209}$$ and so on.

Now write each fraction as a parameter pair (p,m), giving (1,4); (4,15); (15,56); (56;209); and so on. Use the well-known formulae for generating a Pythagorean triad from a pair of integers (p,m) that do not have any common factor. That is $(m^2 + p^2)$; $(m^2 - p^2)$; and $2mp$. (1,4) gives the triangle (8,15,17). Now $17 = (2 \times 8) + 1$, and the triads obey the general law $h = 2s + 1$, when h is the hypotenuse and s is the shortest side.

Observe that $|x - 2| = |\sqrt{3} - 2| < 1$. My routine works only when the modulus of the infinite fraction is less than one, and it was not necessary to go negative in order to achieve that. What can we do to obtain the other infinite set of triads?

The above fractions form an infinite sequence in which the fractions tend to the limit $(\sqrt{3} - 2)$ as $r \to \infty$. To obtain the infinite sequence in which the fractions tend to the limit $\sqrt{3}$ as r tends to infinity, it is only necessary to add 2 to each of the above fractions. Thus $-\frac{1}{4} + 2 = \frac{7}{4}$; $-\frac{4}{15} + 2 = \frac{26}{15}$; and so on. Taking (4,7) as the first parameter pair (p,m), generate a triad using formulae $m^2 + p^2$; $m^2 - p^2$; and $2mp$.

$49 + 16 = 65$; $49 - 16 = 33$; and $2 \times 7 \times 4 = 56$.

Observe that $65 = (2 \times 33) - 1$, and the triads obey the general law $h = 2s - 1$, where h is the hypotenuse and s is the shortest side.

These two sets are different, and yet they may also be viewed as one set with one missing triad in one case, the (3,4,5) triad. The mathematical relationship for this one set is $h = 2s \mp 1$, where in each triad the hypotenuse is double the shortest side alternately minus or plus one, being minus one for the first and missing (3,4,5) triangle.

The most significant thing about this missing 3,4,5 triangle is that $3^2 + 4^2 = 5^2 = 25$, and I have discovered the second most significant thing about it, that the *perimeter* $3 + 4 + 5 = 12$.

For just as the Newton sequence for $\sqrt{3}$ is 1,4,15,56,... taking four times the previous term and subtracting the term before that to get the next term in the sequence, the sequence for $\sqrt{2}$ is 1,2,5,12,... taking twice the previous term and adding the term before that to obtain the next in the sequence.

What are the next four terms in the sequence?

(i) $(2 \times 12) + 5 = 29$; (ii) $(2 \times 29) + 12 = 70$;
(iii) $(2 \times 70) + 29 = 169$; (v) $(2 \times 169) + 70 = 408$.

If the next triangle in the set generated by $\sqrt{2}$ after the 3,4,5 triangle has a *perimeter* of 70 and a hypotenuse of 29, we shall discover that this gives a 20, 21, 29 right-angled triangle, for $20^2 + 21^2 = 29^2$.

And if the next triangle in this sequence after that has a *perimeter* of 408 and a hypotenuse of 169, we shall discover that this gives a 119,120,169 right-angled triangle, for $119^2 + 120^2 = 169^2$.

What is the mathematical relationship that is true for a pair of corresponding sides in every one of these 3 triangles - (3,4,5); (20,21,29) and (119,120,169)? For every triangle in this infinite sequence you will discover that the two shortest sides forming the right angle differ by only one unit.

S.J. Newton,
348A Bourke Street,
Darlinghurst, N.S.W.

[*Pressure of space has forced us to omit many other interesting details supplied by Mr Newton, following his earlier letter (*Function*, Vol.8, Part 1). Newton's First Theorem generalises a similar result, referred to in* Vol.5, Part *2 as Cohen's First Theorem. Eds.*]

PROBLEMS AND CORRECTIONS

Two problems, not numbered as such, were posed on p.3 of *Function, Vol.*8, *Part* 1. Here are the solutions.

First, let $x = 1{\cdot}2345678901234567890\ldots$. Clearly $10^{10}x - x = 12345678900$, and so

$$x = \frac{12345678900}{10^{10} - 1} = \frac{1371742100}{1111111111} .$$

This fraction cannot be simplified, since

$$1371742100 = 2^2 \times 5^2 \times 3607 \times 3803$$

and $$1111111111 = 11 \times 41 \times 271 \times 9091 .$$

Secondly, to prove Colin Fox's Theorem 1, note that

$$\frac{\tan\theta_{n+1}}{\tan\theta_n} = \sqrt{\frac{\sec^2\theta_{n+1} - 1}{\sec^2\theta_n - 1}} .$$

As $n \to \infty$, this will behave like $(\sec\theta_{n+1})/(\sec\theta_n)$, i.e. $(\cos\theta_n)/(\cos\theta_{n+1})$.

But $\cos\theta_n = \cos(90 - 10^{-n})° = \sin(10^{-n})°$, and similarly $\cos\theta_{n+1} = \sin(10^{-n-1})°$. But, for such small angles, we may approximate the sines as

$$\sin(10^{-n})° \cong \left(\frac{\pi}{180}\right) \times 10^{-n} , \quad \text{etc.}$$

and from this it follows that

$$\lim_{n\to\infty} \frac{\tan\theta_{n+1}}{\tan\theta_n} = 10 , \qquad \text{as stated.}$$

On p.25, a minor correction is needed. $(a + b + c)^2$ should read $(a + b + c)^n$. It would be possible to take an n-dimensional analogue to give the kth "layer" of the sum to n terms of the kth power. These coefficients are given by $k!/(a_1!a_2!\ldots a_n!)$, where the numbers a_i sum to k and each is non-negative.

Finally, a word on the problem of finding 100 consecutive composite numbers. The smallest such set begins with 370262 and continues for 111 consecutive integers. I found this by leaving a computer searching for a couple of days.

J. Ennis
Year 12, M.G.S.

[*For the last paragraph, refer to earlier accounts in* Function, *Vol.7, Parts 3,4,5. The improvement given here is enormous. Eds.*]

MORE ON THE PAPERMOBILE

I would like to draw attention to the article by Jean-Pierre Declercq entitled "A 'Papermobile' to Multiply Polynomials" in the August, 1982, issue of *Function*.

Since, for example,

$$x^3 + 9x^2 + 8x + 4 = 1984 \quad \text{if} \quad x = 10 ,$$

the "papermobile" method may be used to multiply base 10 numbers.

Example: 1984 × 123 .

Write 1 9 8 4 on a sheet of paper. Reverse the digits of 123 and write 3 2 1 on the papermobile, a narrow strip of paper. Place the papermobile as shown

```
        1 9 8 4
    [3 2 1]
        1
```

and multiply the adjacent digits, i.e. 1 × 1 = 1 . Then move the papermobile to the right one space at a time and calculate the sum of the products of adjacent digits. The final stage is shown.

```
1   9   8   4
           [3   2   1]
1  11  29  47  32  12
```

Now, "reallocate" the place values, i.e. 12 = 10 + 2, 33 = 30 + 3, 50 = 50 + 0 and so on, giving the product

2 4 4 0 3 2

The method can, of course, be applied to numbers in other bases.

My students (at all levels) have enjoyed using the "paper-mobile" algorithm.

David Shaw
Geelong West Technical School

OOPS!

MATHEMATICS TALENT QUEST, 1984

The Mathematical Association of Victoria with the help of 18 sponsors (including Monash University) organises an annual Mathematics Talent Quest. First held in 1982, it grew in 1983 to 1800 entries representing involvement of almost 3000 students from 20 schools.

Section 3, involving students in Years 10, 11, 12, and Section 4 (for computer entries) are those most likely to interest readers of *Function*. Entries can be from individuals, groups of 2, 3 or 4 students, or classes (at least 50% of the class participating).

You should ask your teacher for information on how to enter and for his help in processing your entry, as all entries must be sent through the school the entrant attends.

Oh dear! And after such a hooray-feminist article, too!

I.D. Rae
Monash University

[Function *is properly chastened. Eds.*]

HAMILTON THE STONE-CARVER

In *Function* Volume 5, part 3, page 27, you quote from Sir William Rowan Hamilton (*via* Crowe's *A History of Vector Analysis*) some comments he recorded about his discovery of quaternions. You then add:

"Bell (in *Men of Mathematics*) has Hamilton pulling out a pocket-knife and carving the basic table on the stone of the bridge, but this story (like much else in Bell, see p.27) would seem to be apocryphal."

The following is a quotation from a letter by Hamilton to his son Archibald.

"In October, 1843, having recently returned from a meeting of the British Association in Cork, the desire to discover the laws of the multiplication of triplets regained with me a certain strength and earnestness, which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above-cited month, on my coming down to breakfast, your brother William Edwin and yourself used to ask me, 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head, 'No, I can only add and subtract them.' But on the 16th day of the same month - which happened to be a Monday and a Council day of the Royal Irish Academy - I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps been driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should ever be allowed to live long enough distinctly to communicate the discovery. I pulled out on the spot a pocket-book, which still exists, and made an entry there and then. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k;

$$i^2 = j^2 = k^2 = ijk = -1 ,$$

which contains the solution of the Problem, but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (October 16th, 1843), which records the fact that I then asked for and obtained leave to read a paper on Quaternions, at the First General Meeting of the Session: which reading took place accordingly on Monday the 13th of November following."†

This extract from a letter of Hamilton disposes of the doubt you raise about Bell's story. One wonders whether the denigrating comment "like much else in Bell" that you throw in, has any better justification.

We offer for your consideration the following comment of E.T. Bell (Preface, *The Development of Mathematics*, 2nd Ed. McGraw-Hill, New York, 1945).

"It has, unhappily, been necessary in writing the book to consider many things besides the masterpieces of mathematics. Rising from a protracted and not always pleasant session with the works of bickering historians, scholarly pedants, and contentious mathematicians, often savagely contradicting or meanly disparaging one another, I pass on, for what it may be worth, the principal thing I have learned to appreciate as never before. It is contained in Buddha's last injunction to his followers:

Believe nothing on hearsay. Do not believe in traditions because they are old, or in anything on the mere authority of myself or any other teacher.

Bamford Gordon, 7 Burnside Ave. Hamilton

†From Sir W.R. Hamilton to the Rev. Archibald H. Hamilton, August 5, 1865. See R.P. Graves, Life of Hamilton, Volume II, p.434, Arno Press, New York, 1975.

PROBLEM SECTION

The problems in *Vol*.8, *Part* 1 brought a most gratifying response. Here are the solutions.

SOLUTION TO PROBLEM 8.1.1.

The problem read:

How many permutations are there of the digits 1,2,3,...,8 in which none of the patterns 12,34,56,78 appear?

Jonathan Ennis (Year 12, M.G.S.) writes:

"There are 8! permutations of 8 digits. Of these, 7! will contain the pattern 12 and similarly for the other patterns. But some of the $4 \times 7!$ permutations just considered contain two such patterns, so we have overcounted by $\binom{4}{2} \times 6!$. But again some of the permutations may contain three such patterns, so we have undercounted by $\binom{4}{3} \times 5!$. Finally, 4! permutations will contain all four patterns.

Thus the total number of allowable permutations is $8! - \binom{4}{1} \times 7! + \binom{4}{2} \times 6! - \binom{4}{3} \times 5! + 4!$, which works out to be 24 024.

The problem may easily be generalised to the case of $2m$ digits and the pairs $12, 34, \ldots, (2m-1)2m$."

SOLUTION TO PROBLEM 8.1.2.

Ten people form the queue at a bank. The first has a bank balance of one cent, while the tenth has a little over \$5 million. The accounts of the others are each computed by adding ten elevenths of the account of the person ahead to one eleventh of the account of the person behind. Can the sixth person afford to buy a new car?

Let T_n be the number of dollars in the nth person's account. Then

$$T_{n-1} = (10/11).T_{n-2} + (1/11).T_n ,$$

$$\text{i.e.} \quad T_n = -10.T_{n-2} + 11.T_{n-1} .$$

David Halprin (P.O. Box 23, Carlton North) now solved this equation to find

$$T_n = A + 10^n B ,$$

where A, B are constants. But $T_1 = 0{\cdot}01$ and T_{10} is a little over 5 000 000 . So

$$A + 10B = 0{\cdot}01$$
$$A + 10^{10}B \cong 5 \times 10^{6} .$$

These relations give $A \cong 0{\cdot}01$, $B \cong 5 \times 10^{-4}$, which may then be checked. Then $T_6 \cong 10^6 B \cong 500$. This will not at today's prices buy a new car.

Jonathan Ennis, who also solved the problem by a slightly different method, using approximations earlier in the calculation, remarked that T_6 would not pay for a new car, but the account's owner might try holding up the teller!

SOLUTION TO PROBLEM 8.1.3.

We asked for the value of

$$\left(9 + 4\sqrt{5}\right)^{1/3} + \left(9 - 4\sqrt{5}\right)^{1/3} .$$

David Halprin and Jonathan Ennis both reasoned as follows.

If $S = \left(a + b\right)^{1/3} + \left(a - b\right)^{1/3}$,

then $$S^3 = a + b + a - b + 3.\left(a + b\right)^{2/3}.\left(a - b\right)^{1/3} + 3.\ a + b^{\ 1/3}.\ a - b^{\ 2/3} .$$

Therefore $$S^3 = 2.a + 3.\left(a^2 - b^2\right)^{1/3}.\left(\left(a + b\right)^{1/3} + \left(a - b\right)^{1/3}\right)$$
$$= 2.a + 3.S.\left(a^2 - b^2\right)^{1/3} .$$

In this problem $a = 9$ and $b = 4.\sqrt{5}$, $a^2 - b^2 = 1$, and so $S^3 - 3S - 18 = 0$. Thus $(S - 3)(S^2 + 3S + 6) = 0$, which gives $S = 3$ as the only real root.

A similar approach was used by R.P. Hale of Deakin University and it was most likely this method of attack that the editors of *Mathematical Spectrum* had in mind when they proposed the problem. There is a formula for solving cubics and, in the case of the cubic reached above, it gives the expression in the problem. However, we got three other answers also.

Devon Cook, editor of *Scientific Australian*, put $\left(a + b\sqrt{5}\right)^3 = 9 + 4\sqrt{5}$ to get two simultaneous cubics

$$a^3 + 15ab^2 = 9, \quad 3a^2b + 5b^3 = 4.$$

He then put $a = mb$ and found

$$4m^3 - 27m^2 + 60m - 45 = 0.$$

This equation has one real root: $m = 3$. This enabled a,b to be found and so the problem was solved.

Leigh Thompson (RMB 5110, Bairnsdale) had yet another approach. Having been working with the Golden Ratio $\frac{1}{2}(1 \pm \sqrt{5})$ and investigating its successive powers, he noticed that

$$\left\{\tfrac{1}{2}(1 \pm \sqrt{5})\right\}^6 = 9 \pm 4\sqrt{5} .$$

This meant that

$$\left(9 \pm 4\sqrt{5}\right)^{1/3} = \left\{\tfrac{1}{2}(1 \pm \sqrt{5})\right\}^2 = \tfrac{1}{2}(3 \pm \sqrt{5}) ,$$

and so the problem fell out!

Ricardo Montebon (Zamboanga College, Philippines) had yet another approach. Setting

$$\left(9 \pm 4\sqrt{5}\right)^{1/3} = x \pm \sqrt{y} , \qquad (1)$$

he multiplied to find

$$x^2 - y = \left(81 - 80\right)^{1/3} = 1 \qquad (2)$$

(by the difference of two squares in each side). He also cubed Equation (1) to find

$$9 + 4\sqrt{5} = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}$$

and equated the rational parts to get

$$9 = x^3 + 3xy . \qquad (3)$$

He then solved Equations (2), (3) by eliminating y and using trial and error to find $x = \frac{3}{2}$, $y = \frac{5}{4}$, giving now the result as above.

SOLUTION TO PROBLEM 8.1.4.

111 players enter a tennis tournament. How many games must be played to determine the winner?

As such tournaments always adopt the convention that a player who loses one game is eliminated, the winner is the only player to have sustained no losses, while all other players have sustained precisely one. Thus 110 matches have been played.

This solution was submitted independently by Jonathan Ennis and by David Dyte (Year 10, Scotch College).

We hope our readers can do as well with this set of problems.

PROBLEM 8.3.1 (Submitted by D.R. Kaprekar.)

In the year 1949, a man turned 67. His four sons turned 37, 31, 29, 23 respectively. All five reached prime age in a prime year. It was the golden year for that family. When was or will be their next golden year?

PROBLEM 8.3.2 (Submitted by Garnet J. Greenbury.)

Are there Pythagorean triangles whose perimeters equal their areas?

PROBLEM 8.3.3 (Submitted by Jonathan Ennis.)

Toss a fair coin 100 times and keep a tally of progressive numbers of heads and tails. How many times (on average) will the lead change from one to the other?

PROBLEM 8.3.4 (Submitted by Jonathan Ennis.)

Find a cuboid such that its sides and the diagonals of its faces all have integral lengths. Alternatively, prove no such cuboid exists.

PROBLEM 8.3.5 (From *The Mathematical Gazette*.)

In a common type of logic-puzzle, we are confronted with two categories of person: those who always lie and those who always tell the truth. A traveller reached a land in which the inhabitants all fell into two such classes and, seeing a house, he wished to ascertain whether it was an inn where he could spend the night. Approaching two people, he asked the first, but received a cryptic reply, insufficient to give him his answer. He addressed exactly the same question to the second person and received exactly the same reply. He then knew the house to be an inn.

What was the cryptic reply?

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PERDIX

The 1984 International Mathematical Olympiad will take place from July 2 to July 10 in Prague. Australia will send a team of six.

The Australian government will provide no help towards costs. In 1983 Algeria, Austria, Brazil, Bulgaria, Columbia, Cuba, Czechoslovakia, East Germany, Finland, France, Great Britain, Hungary, Italy, Kuwait, Luxembourg, The Netherlands, Poland, Romania, Sweden, U.S.S.R., U.S.A., Vietnam, West Germany, and Yugoslavia each provided full financial support for correspondence instruction, regular training sessions, final training camps for up to four weeks' duration together with all travel costs for team and officials. Except for three countries, namely Belgium, Morocco, and Spain, where I have been unable to find out what government support was given, I find that every country except Australia substantially supported its team. DO WHAT YOU CAN TO URGE AUSTRALIAN GOVERNMENT SUPPORT FOR ITS MATHEMATICAL OLYMPIAD TEAM.

Support for the International Mathematical Olympiads is not just support for a game. Although only a handful are selected for the final team the process of finding the right team involves extra training in mathematics being given, or offered, virtually to all high school students. The consequent great interest generated in mathematics directly helps all Australian technological and scientific development. Australia has made a magnificent start in the Olympiads. We must make certain that the sustaining interest in mathematics continues and that Australia has the best team to compete for it internationally. There must be no possibility that anyone is excluded from the Australian team because they cannot personally pay all the costs involved. URGE THE AUSTRALIAN GOVERNMENT TO SUPPORT THE AUSTRALIAN MATHEMATICAL OLYMPIAD TEAM.

* * * *

If you wish to try to get into the Australian Olympiad team enquire at your school. In each State there are arrangements for selecting possible members of the team. It is important to practice problem solving and the training sessions set up in various centres in each State provide you with the opportunity to practice and also offer you guidance to improve your skills. If you cannot get the information you need from your school, contact the State organiser in your State. Here are their names and addresses.

New South Wales	Mr G.R. Ball, Department of Pure Mathematics, Building SO7, Sydney University, SYDNEY, 2006.
Tasmania	Mr J. Kelly, Mathematics Resources Centre, 2 Edward Street, Glebe, 7000.
Victoria	Mrs Judith Downes, 46 Hill Road, North Balwyn, 3104.
South Australia/ Northern Territory	Mr V. Treilibs, Mathematics Project Team, Wattle Park Teachers Centre, 424 Kensington Road, Wattle Park, 5066.
Western Australia	Dr Phillip Schultz, Department of Mathematics, University of Western Australia, Nedlands, 6009.
Queensland	Dr N.H. Williams, Department of Mathematics, University of Queensland, St Lucia, 4067.
Australian Capital Territory	Dr R.A. Bryce, Department of Pure Mathematics, A.N.U., P.O. Box 4, Canberra, 2600.

* * * *

The International Commission for Mathematical Instruction organises an International Congress on Mathematical Education (ICME) every four years. This year (1984) it is being held for the first time in Australia and will take place in Adelaide from 24 August to 30 August. Over 2000 mathematicians and teachers of mathematics are expected to attend.

At ICME 1984 there will be six separate sessions devoted to discussing mathematical competitions. The sessions are entitled: The Creation of Competition Questions; National Mathematics Competitions; A Kaleidoscope of Competitions; The International Mathematical Olympiad; Why do 1 in 55 Australians enter a Mathematics Competition? [Quarter of a million took part in such competitions in 1983.]; World Federation of National Mathematics Competitions.

Also at ICME 1984 will be a poster display of journals (*Function* will be there) having problem solving sections for school students. Some video films giving information about some successful competitions will be shown.

If you need further information please contact Mr P.J.O'Halloran, Chief Organiser, Competitions ICME-5, Canberra College of Advanced Education, P.O. Box 1, Belconnen, A.C.T., 2616.

* * * *

Solving problems

Olympiad problems are chosen so that they will stretch the abilities of the contestants. Thus many of the Olympiad problems are difficult. However, what makes a problem difficult? After you have solved a problem, or after you have been shown a solution, a problem that was found difficult often seems simple, and you could kick yourself for not seeing how to do it sooner.

How should one set about trying to solve a difficult problem? Clearly knowledge helps. The more mathematical results, i.e. facts, that you know which are related to the questions posed by a problem, the more likely you are to solve the problem.

We now consider a problem concerned with divisibility of integers.

PROBLEM (First International Olympiad, 1959, problem 1)

Prove that the fraction $\dfrac{21n+4}{14n+3}$ *is irreducible for every natural number* n.

Note first that a *natural* number is the same thing as a positive integer†. Note secondly that a fraction is *irreducible* if its numerator and denominator have no common factors other than 1 (we may clearly restrict ourselves to positive integers). We now understand the problem.

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†Nowadays 0 is often counted as a natural number. Not in 1959. However in this problem it makes no difference: when $n = 0$, $\dfrac{21n+4}{14n+3} = \dfrac{4}{3}$ is irreducible.

As the next step it is often useful to carry out some experiments, i.e. look at some special cases. These special cases may give you a feel for the situation and, by noting some common feature, may lead to a general argument.

Experiments

n	$(21n+4)/(14n+3)$	$(21n+4)-(14n+3)$
1	25/17	8
2	46/31	15
3	67/45	22
4	88/59	29

Two integers that have highest common factor 1 are said to be *co-prime*, or to be *prime to each other*. The above experiments clearly show that $(21n+4)/(14n+3)$ is irreducible when $n = 1,2,3,$ and 4, for 25 and 17 are co-prime, 46 and 31 are co-prime, etc. Can we glean any other information?

Well, $25 - 17 = 8$ is prime to each of 25 and 17; similarly $46 - 31 = 15$ is prime to each of 46 and 31. Is this generally true? Yes. We have

(*) *Let ℓ and m be co-prime positive integers, with $\ell > m$. Then $\ell - m$ and ℓ are co-prime, as are $\ell - m$ and m.*

Proof. Suppose that the positive integer d divides $\ell - m$ and ℓ. Thus, there are positive integers k and h say, such that

$$\ell = kd \tag{1}$$
$$\ell - m = hd . \tag{2}$$

Substituting from (1) in (2) gives

$$kd - m = hd$$
i.e.
$$m = (k-h)d .$$

Hence d divides m. Thus d divides each of the co-prime integers ℓ and m. Hence $d = 1$. This shows that $\ell - m$ and ℓ are co-prime.

Similarly, $\ell - m$ and m are co-prime.

<u>Exercise 1</u>. Show that the following result holds.

(**) *Let ℓ and m be co-prime positive integers. Then $\ell + m$ and ℓ are co-prime, as are $\ell + m$ and m.*

<u>Exercise 2</u>. Results (*) and (**) may be generalised. Show that

(***) *Let ℓ and m be positive integers with highest common factor $d\,(>0)$. Then the highest common factor of each of the pairs $\ell - m$ and ℓ; $\ell - m$ and m; $\ell + m$ and ℓ; $\ell + m$ and m is also d.*

As a corollary to (**) we have:

(****) *Let ℓ and m be positive integers such that $\ell > m$ and $\ell - m$ and m are co-prime. Then ℓ and m are co-prime.*

Proof. By (**), since $\ell - m$ and m are co-prime so also are $(\ell - m) + m$ and m, i.e. ℓ and m are co-prime.

It is a good idea when finding small results, such as (*) here, that seem relevant to the solution of a problem, to note at the same time other closely related results such as (**), (***) and (****), here. At this stage of solving the problem, you do not know which facts may be relevant: collect facts as you go along.

We have enough facts (together with some that are not necessary) to solve our problem. For, by (*) and (****), $(21n + 4)/(14n + 3)$ is irreducible, i.e. $21n + 4$ and $14n + 3$ are co-prime if and only if $(21n + 4) - (14n + 3)$ and $14n + 3$ are co-prime, i.e. if and only if $7n + 1$ and $14n + 3$ are co-prime.

But, if d divides $7n + 1$ and $14n + 3$ then d divides $14n + 3 - 2(7n + 1)$ (=1), and so d divides 1. Hence $d = 1$ and $7n + 1$ and $14n + 3$ are always co-prime.

Hence $(21n + 4)/(14n + 3)$ is always irreducible. Q.E.D.

Knowledge of some facts about divisibility was necessary to solve Problem 1. Problems concerned with divisibility have frequently turned up in the International Mathematical Olympiad tests and other competitions. This is perhaps not surprising, because the idea of divisibility is a simple one and divisibility is at the heart of a major area of study in mathematics, the area called the *theory of numbers*. In the same area is the next problem.

PROBLEM 2 (Sixth International Olympiad, 1964, problem 1)

(a) *Find all positive integers* n *for which* $2^n - 1$ *is divisible by* 7.

(b) *Prove that there is no positive integer* n *for which* $2^n + 1$ *is divisible by* 7.

Send to Perdix your solutions to the problem - or your attempts at a solution. Hint: if n, m, k and r are integers and $m = 7k + r$, then mn and rn give the same remainder on division by 7.

PROBLEM 3 (Twelfth International Olympiad, 1970, problem 4)

Find the set of all positive integers n *with the property that the set* $\{n, n+1, n+2, n+3, n+4, n+5\}$ *can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.*

Note that a set A is *partitioned* into two sets when it is divided into two subsets such that each member of A belongs to one of the two subsets and no member of A belongs to both the subsets.

PROBLEM 4. *Let* m *be an odd positive integer. Show that there is a positive integer* k *such that* m *divides* $2^k - 1$.

Turn to Perdix's column next issue for a discussion of solutions to some of these and further problems.