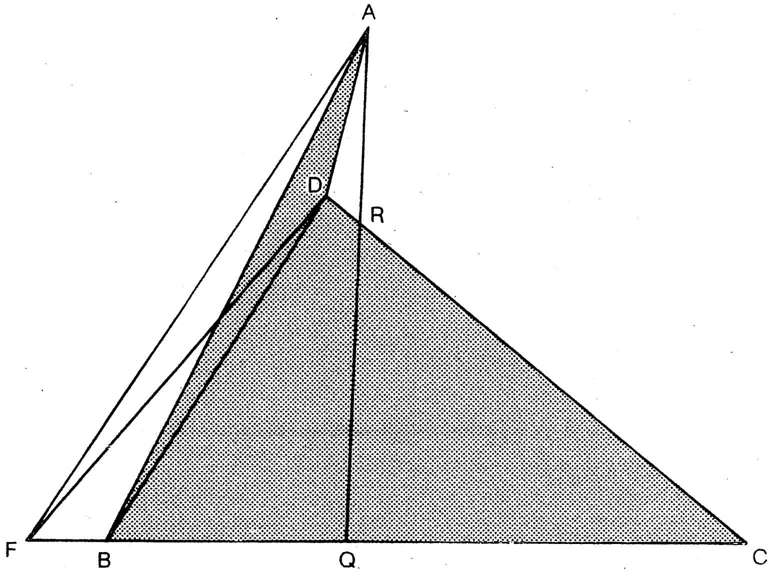
FUNCTION

Volume 11 Part 5 October 1987



A SCHOOL MATHEMATICS MAGAZINE
Published by Monash University

Reg. by Aust Post Publ. No. VBH0171

Function is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. *Function* is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

* * * * *

BUSINESS MANAGER: Mary Beal (03) 565-4445

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the institutions shown above.

The magazine is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $10.00*; single issues $2.00. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

* $5.00 for *bona fide* secondary or tertiary students.

* * * * *

Registered for posting as a periodical - "Category B"

ISSN 0313 - 6825

THE FRONT COVER

In this final issue for this year, we have an interesting article from John Burns and Laci Kovacs on the bisection of a quadrilateral. It is the first full account dealing with all the special cases (one is illustrated on the front cover) that, so far as we can discover, has been printed. There are many natural problems of a similar kind. In the following two, a construction for the bisecting lines, in general, cannot be given; but it can be proved that the required bisecting lines exist. The first problem is : if you have two pancakes in a plane, possibly overlapping each other, is it possible with a single straight line cut to divide each pancake into two equal parts. This is called the *pancake problem*. The second problem is the *ham sandwich* problem: if you have two slices of bread and a slice of ham between them can you, with a single plane cut, divide each piece of bread and also the piece of ham into two parts of equal volume? The answer is 'Yes' in each case. Food for thought while you eat your Xmas dinner!

* * * * *

CONTENTS

* * * * *

BISECTION OF A QUADRILATERAL BY A LINE THROUGH A VERTEX

J.C.Burns
Australian Defence Force Academy
Campbell, Australia 2600.

I.G.Kovacs
Australian National University
Canberra, Australia 2601.

One of the Hungarian mathematical competitions in 1978 included the following problem:

> Given a convex quadrilateral, construct a line through one of its vertices so as to cut the quadrilateral into two parts whose areas are equal.

This problem appears in textbooks on elementary Euclidean geometry (e.g. [1]) and, with the restriction that the quadrilateral be convex, is simple enough to solve.

However, as contestants in a mathematical competition are encouraged to generalize, one should inquire whether the solution can be adapted to deal with non-convex quadrilaterals. While some difficulty could be expected in ensuring that all cases are considered, it came as something of a surprise to us that in two cases a completely different approach was required.

J.C.Barton of the University of Melbourne has drawn our attention to [2] where, on page 77, under the general heading of "Area constructions by equivalent triangles or parallelograms" we find:

> The bisection of a triangle by a line drawn from a point in a side. The bisection of a quadrilateral by a line from a corner ...is a nice extension of this.

This reference to a "nice extension" may imply that the solution is straight-forward, as indeed it is when the quadrilateral is convex. On the other hand, the choice of the adjective "nice" rather than, say, "simple" or "routine", may have been intended as an indication that the extension is not without special interest of its own and as an invitation to the reader to pursue the matter further. If this was the case, the authors of the Report at least provided a clue, for, having referred as above to the bisection of the quadrilateral, they went on to remark that

> It is interesting to note that the obvious extension, the bisection of a triangle by a line through an external point, is specialist work, and hard at that, ...

It turns out that in order to provide a complete solution of the quadrilateral problem we need to be able to carry out the construction of a line through an arbitrary point to bisect a given triangle. Although a solution of this latter problem was evidently familiar to the authors of the Report, we have not been able to find one in print and the problem is discussed at some length in [3]. In the relevant part of [3], it is also required that a particular vertex of the given triangle lie in a triangular (rather than a quadrilateral) portion of the bisection, and that the bisector not go through this vertex. The number of such bisectors through a given point of the plane may be 0, 1, or 2, and [3] gives a construction for them.

Even if we assume the construction for bisecting the triangle, the original quadrilateral problem has more to it than appears at first sight and it seems worth while to offer a complete solution.

We shall begin with a construction which is adequate for a convex quadrilateral and then explore the possibilities of applying it to non-convex quadrilaterals. Let the quadrilateral be $ABCD$ with vertex C opposite vertex A and let A be the vertex through which the required line is to be drawn. In the first instance we distinguish four types of quadrilateral: (A) convex qudrilaterals and (B) non-convex quadrilaterals divided into three classes according to the position of the reflex angle relative to the vertex A: (B1) opposite A, (B2) at A, (B3) adjacent to A. Later it will be necessary to divide each of the classes (B2) and (B3) into three sub-classes.

We note first that if the diagonal AC bisects the diagonal BD, then AC bisects the quadrilateral. As shown in figure 1, this result holds for convex quadrilaterals and for non-convex quadrilaterals when the reflex angle is either opposite to A or at A. When the reflex angle is adjacent to A (B3), it is impossible for AC to bisect BD. In what follows, it will be assumed that the mid-point M of BD does not lie on AC.

The construction for the convex quadrilateral is of course well-known. It is set out now in a form which allows it to be used in other cases as well.

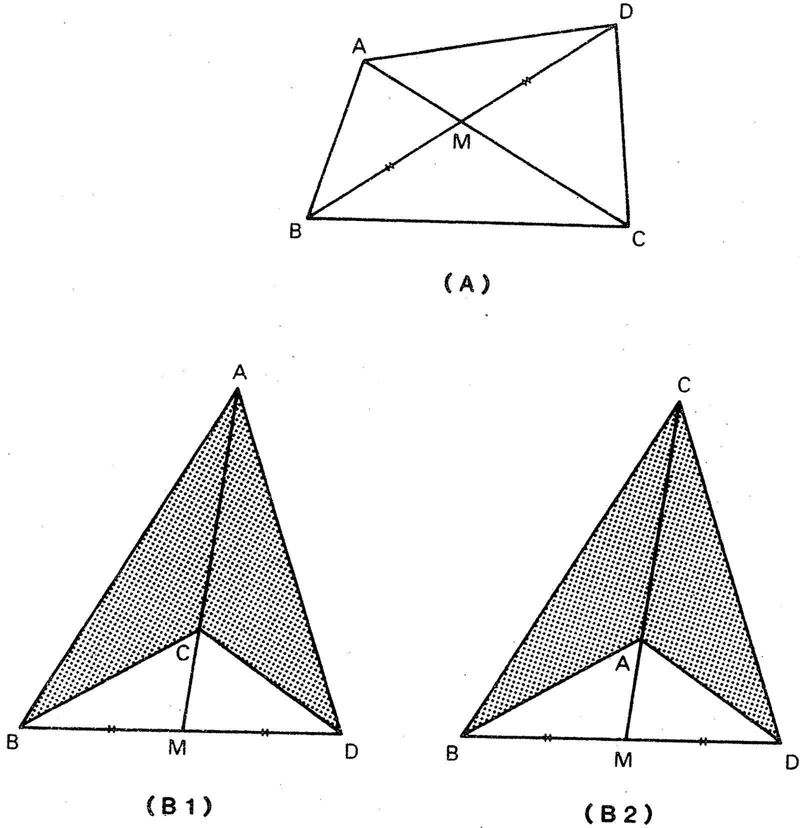


Figure 1.

Draw a line through the mid-point M of the diagonal BD parallel to the other diagonal AC to meet one (and, as M is not on AC, only one) of the segments BC, CD in a point P; and name the vertices in such a way that P lies on BC (in fact, strictly between B and C). Then the line AP is the required bisector provided the segment AP lies wholly within the quadrilateral $ABCD$ and is the only part of the line to do so. The construction is illustrated in figure 2 for cases (A), (B1) and (B2). Case (B2), in which the reflex angle is at the vertex A, is divided into three categories (B2a), (B2b), (B2c), according as BP is greater than, equal to, or less than BD' where D' is the point where DA cuts BC.

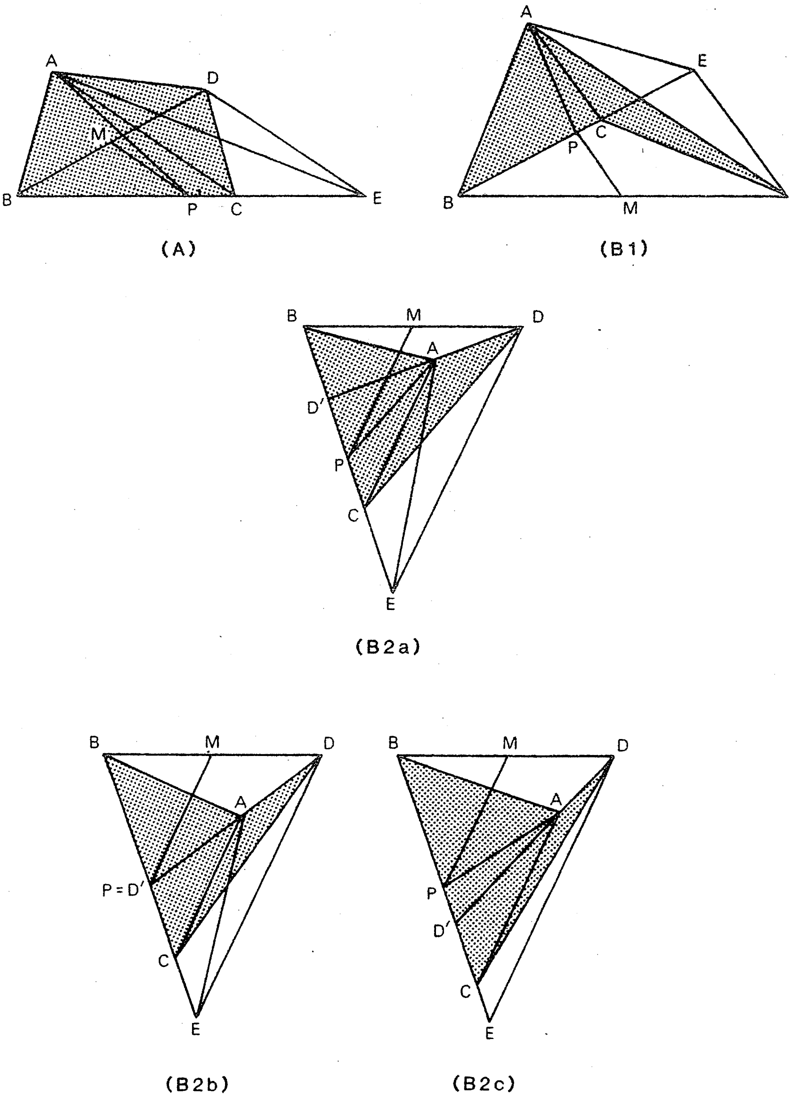


Figure 2

In all these cases, the construction produces a segment AP which lies wholly within the quadrilateral and, as we now show, divides the quadrilateral into two parts of equal area. Let the line through D parallel to AC meet the side BC produced beyond C in E. Since MP and DE are parallel and M is the mid-point of BD, $BP = PE$ and it follows that area ABP = area APE. Also area ACD = area ACE for the altitudes corresponding to the common side AC of these triangles are equal, being the distance between the parallel lines DE, AC. In all cases (A), (B1), (B2*a*), (B2*b*), (B2*c*) in figure 2 we now have

$$\begin{aligned} \text{area } ABP &= \text{area } APE \\ &= \text{area } APC + \text{area } ACE \\ &= \text{area } APC + \text{area } ACD \\ &= \text{area } APCD . \end{aligned}$$

Since area $ABCD$ = area ABP + area $APCD$, the segment AP bisects the quadrilateral.

It will be noted that in case (B2c), the distinction between the line AP and the segment AP becomes important. In the other four cases in figure 2, the only part of the line AP to lie within the quadrilateral is the segment AP so the construction has produced the required line through A. This is not so in (B2c) where the segment AP bisects the quadrilateral but the line evidently does not. Further constructon is needed in this case.

We note first that because area ABP = (1/2) area $ABCD$ > (1/2) area ABC, it follows that $BP > (1/2)BC$ so $BP > CP$. In figure 3, we choose Y on BC so $CY = BP$. Then $CY > CP$; thus the order of points on BC is $BYPD'C$.

Draw the line through A parallel to BC to cut CD in X; choose Z so that X is the mid-point of CZ. Three subcases arise according as CZ is greater than, equal to or less than CD; we picture in figure 3 only two as $Z = D$ can be handled as a degenerate version of, say, the first. Even for these two sub-cases, the arguments will not branch for a while.

The triangle CYZ has area double that of CYX which, because $CY = BP$ and AX is parallel to BC, has area equal to that of ABP and so to half that of the quadrilateral.

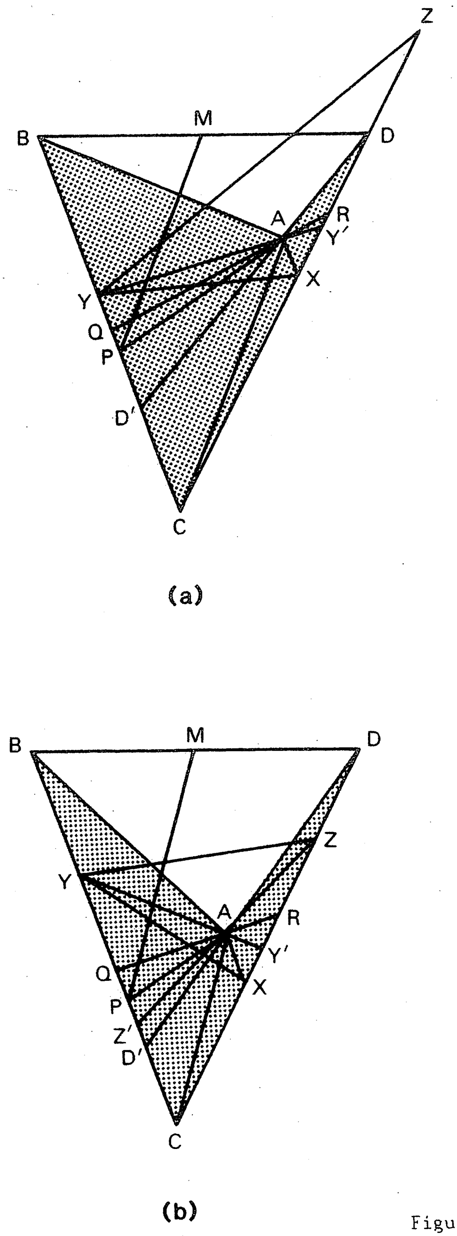


Figure 3.

We shall prove that there exists a line QAR, with Q on the segment YD' and R on the segment CD, which bisects the area of triangle CYZ. Then the area of CQR is half that of CYZ or of the quadrilateral so the line also bisects the quadrilateral; on the other hand, as a bisector of CYZ through A, the line (or perhaps two such lines) can be constructed by the method of [3].

The first point is to observe that A always lies inside the triangle CYZ, else this triangle would be a proper part of the quadrilateral in spite of their areas being equal. Let Y' be the intersection of YA and CZ; then Y' lies on the segment CZ, and also between C and D because D' is between C and Y. Moreover, area $CYY' >$ area $CYA =$ area BPA (because $CY = BP$) $= (1/2)$ area $ABCD$; thus area $CYY' > (1/2)$ area CYZ. Similarly, area $CDD' <$ area $PCDA$ (because D' lies between P and C) $= (1/2)$ area $ABCD$; so area $CDD' < (1/2)$ area CYZ.

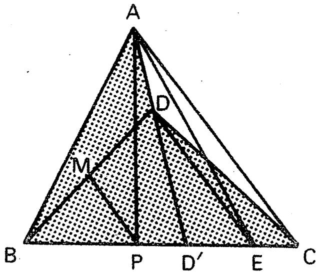
Continuity now guarantees the existence of a point Q between Y and D' such that QA cuts CD in R between Y' and D and area $CQR = (1/2)$ area CYZ. When $CZ \geq CD$ (as in figure 3*a*), we have R on CZ so QAR bisects both the quadrilateral and triangle CYZ and our aim has been achieved.

It remains to consider the case in which $CZ < CD$ (figure 3*b*). We define Z' as the point where ZA cuts BC. Because $CZ < CD$ and A lies inside triangle CYZ, Z' lies between D' and Y, and Y' therefore lies between C and Z. Moreover, because X is the mid-point of CZ and AX is parallel to BC, A is the mid-point of ZZ'.

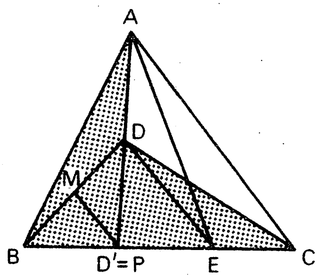
It is known [4], pp 89, 122, that of all triangles with vertex C, sides along CB and CD and base passing through A, the one with the smallest area is obtained when the base is bisected by A. Hence area $CDD' >$ area CZZ'.

We have already shown that $1/2$ area $CYZ >$ area CDD' so we now have $1/2$ area $CYZ >$ area CZZ'. As before, area $CYY' > 1/2$ area CYZ so, by continuity, there is a suitable Q between Y and Z' and a matching point R between Y' and Z so that QAR bisects both the quadrilateral and the triangle as required. This completes the investigation of case (B2c).

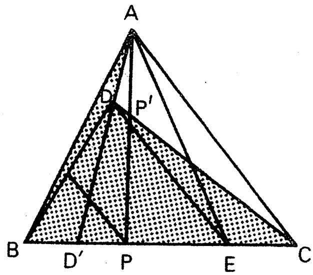
We turn now to case (B3) in which the reflex angle is adjacent to the vertex A. The construction is carried out exactly as before and is illustrated in figure 4. With D' defined as before as the intersection of AD and BC, we this time distinguish three cases (B3a), (B3b), (B3c) according as BP is less than, equal to, or greater than BD'.



(B3a)



(B3b)



(B3c)

Figure 4.

In all three cases,

$$\begin{aligned} \text{area } ABP &= 1/2 \text{ area } ABE \\ &= 1/2 \text{ (area } ABED + \text{area } DEA) \\ &= 1/2 \text{ (area } ABED + \text{area } DEC) \\ &= 1/2 \text{ area } ABCD . \end{aligned}$$

Hence, in cases (B3a), (B3b), AP is the required bisector. In case (B3c) however, the construction is not achieved as the segment AP does not lie wholly within the quadrilateral.

In case (B3c) we see that

$$\text{area } ABD' < \text{area } ABP = 1/2 \text{ area } ABCD .$$

Thus area CDD' > 1/2 area $ABCD$; hence there is a line ARQ (figure 5) which bisects the quadrilateral, cutting CD in R and BC and Q . We proceed to construct such a line.

This is easily done if we first construct as in the front cover figure triangle DFC equal in area to the quadrilateral $ABCD$ by drawing AF parallel to DB and joining DF . Since A is outside CDF , using the construction discussed in [3], we obtain the unique line ARQ through A to bisect triangle DFC and to cut CD in R and BC in Q . The triangle CRQ thus produced has area equal to half that of triangle DFC and hence to half that of the quadrilateral. The line ARQ is accordingly the required bisector of the quadrilateral and the investigation of case (B3c) is complete.

Our conclusion is that the construction described initially for the case of a convex quadrilateral produces in all cases except (B2c) and (B3c) a line AP which bisects the quadrilateral and we have provided alternative constructions for the required line in each of the two exceptional cases.

Finally, we remark that it is not difficult to find examples of quadrilaterals $ABCD$ for which we can draw more than one line through the vertex A to bisect area $ABCD$. There is therefore scope for investigating the number of bisectors through A and the circumstances in which given numbers of bisectors occur.

REFERENCES

[1] H.G.Forder, *A School Geometry*, Cambridge University Press, Cambridge (1939).

[2] Mathematical Association, *A Second Report on the Teaching of Geometry in Schools*, G. Bell and Sons, London (1946).

[3] J.C.Burns, *Construction of a line through a given point to divide a triangle into two parts with areas in a given ratio*, Elemente der Mathematik, 41/3, 58-67 (1986).

[4] Nicholas D. Kazarinoff, *Geometric Inequalities*. New Mathematical Library, Random House, New York (1961).

* * * * *

In his book [*The Psychology of Mathematical Invention* (1945), Jacques] Hadamard [a famous French mathematician] tried to find out how famous mathematicians and scientists actually thought while doing their work. Of those he contacted in an informal survey, he wrote "Practically all of them . . . avoid not only the use of mental words, but also . . . the mental use of algebraic or precise signs . . . they use vague images." (p.84) and ". . .the mental pictures of the mathematicians whose answers I have received are most frequently visual, but they may also be of another kind - for example kinetic." (p.85)

Albert Einstein wrote to Hadamard that "the words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. . . . The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined. . . . The above mentioned elements are, in my case, of visual and some of muscular type.Conventional words or other signs have to be sought for laboriously only in a secondary stage . . . "(p.142) Several recent studies on the way in which nonmathematical adults perform simple arithmetic seem to suggest the same is true for non-mathematicians as well."

Philip J. Davis and Reuben Hersh : *The Mathematical Experience*, Penguin, 1983.

A TALE OF THREE CITIES

Michael A.B. Deakin, Monash University

Take 3 cities called (very imaginatively) A, B & C. They will be supposed to lie in flat terrain and they therefore form the vertices of a triangle in the "plane of the plain".

We wish to site some facility, say a school or a telephone exchange, to serve all three communities, and we want to do so in such a way as to minimise the total cost involved.

It will make sense to locate the facility at some point D inside the triangle ABC and in the plane of the three vertices. We now need to formulate more precisely what mathematics our problem involves.

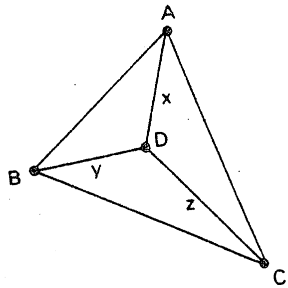


Figure 1.

Suppose, for example, D is to be a telephone exchange and that A has 50,000 inhabitants, B 30,000 and C 20,000. Then we could imagine that the amount of wire involved in connecting D to each of A, B, C would be in the proportion

$$DA:DB:DC = 5:3:2.$$

So if we write $DA = x$, $DB = y$, $DC = z$, we have (in some suitable units):

$$\text{length of wire} = 5x + 3y + 2z. \qquad (1)$$

So we would seek to locate D in such a way as to minimise the quantity $5x + 3y + 2z$.

More generally, the proportions need not be 5:3:2 but could be $\lambda:\mu:\nu$. In this case, we would seek to minimise $\lambda x + \mu y + \nu z$. So the problem becomes :

> Given a triangle ABC, and three positive numbers λ, μ, ν, locate that point D inside the triangle which minimises $\lambda x + \mu y + \nu z$, where $x = DA$, $y = DB$, $z = DC$. (2)

As you can see, the problem is a practical one, but it first arose as a piece of Pure Mathematics. Classical, by which I mean ancient Greek, Geometry knew four different centres for the triangle: the circumcentre (centre of the circle through A, B, C), the incentre (centre of a circle inside ABC and tangent to the three sides), the centroid (or centre of mass) and the orthocentre (where the three altitudes intersect - this last stretches the idea of centre a bit, as it need not lie inside the triangle).

But the point D that we seek turns out not (except in special cases) to be any of these. The first person to consider it was *JAKOB STEINER* (1796–1863), one of a number of mathematicians of his era who were influential in reviving interest in geometry and showing that there were still results to be found, results not known to the ancient Greeks.

Now the problem Steiner considered was not exactly the problem I have labelled (2). The quantity he sought to minimise was

$$x^n + y^n + z^n , \qquad (3)$$

where x, y, z were as described above and n was a real number (not equal to zero). He gave his answer to this in 1835, and later (1837) in an address to the Berlin Academy of Science, he drew particular attention to the case $n = 1$. This is the special case of Problem (2) for which $\lambda = \mu = \nu$.

The answer to that special case is available to us in geometric form on pages 354-361 of a book that should appeal to many of *Function's* readers: COURANT & ROBBINS' *What is Mathematics?* (Oxford University Press, 1941). Courant and Robbins, indeed, consider a different generalisation:

Given n points $A_1, \ldots, A_n$ to find a connected system of straight line segments of shortest total (4) length such that any two of the given points {are connected by the segments] of the system.	(4)

This too is a very practical and important problem for road or rail links, air schedules, etc. But let's get back to our three points A, B, C, and the matter of locating the point D . So our problem is to minimise

$$x + y + z\ . \tag{5}$$

This is the very simplest case of all and all the others depend on it.

I will now explain how to find the point D in this case. Two possibilities arise. Either :

(a) One of the angles A, B, C in the triangle ABC exceeds 120° (if this happens, there will only be one such angle and we can suppose it to be A),

or (b) The angles A, B, C are all less than 120 .

What Courant and Robbins show is that in Case (a), the point A itself is the point we seek while in the more usual Case (b), the point D is such that the angles ADB, BDC, CDA are all equal to 120°. [Two related problems for the reader.

1. Show that such a point exists and that it is unique.

2. Give a method for finding it.]

Figure 2 shows the result in this case. For a 'pure mathematical' proof, see, as I said, Courant and Robbins' book, which also answers the questions set above, in case you need help.

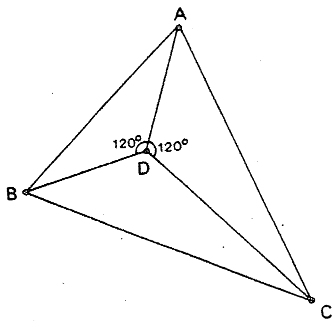


Figure 2.

But I want to take a different course here and refer to a treatmentby a Polish mathematician HUGO STEINHAUS (1886-1972)†. In a remarkable piece of lateral thinking, Steinhaus saw that the problem could be replaced by another. He proposed making a model.

† I derive these dates from an obituary that (strangely) neglects to give Steinhaus' date of birth, but tells us he died on February 25, 1972 at the age of 85. What is the probability that he was born in 1887?

Draw *A*, *B*, *C* on top of a table (Figure 3) and drill smooth holes at each of these points. Through each hole, thread a string and tie the top ends together in a knot. To each of the other ends attach a weight of 1 kg. Now let the device go and see where *D* turns up.

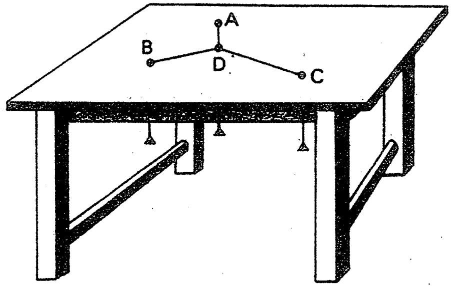


Figure 3.

Figure 4 shows what happens in Case (b). The knot at *D* is subject to three equal "pulls". By symmetry, these align themselves at equal angles, i.e. each of the three angles involved is 120°.

In Case (a), the knot "tries" to reach a 120° - 120° - 120° configuration, but gets tangled up in the hole at point *A* before this is possible.

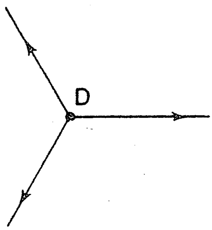


Figure 4.

I take this analysis from Steinhaus' book *Mathematical Snapshots*, another introduction to mathematical thought that I would thoroughly recommend - but draw your attention to one point I have glossed over.

Are we solving the same problem with our doctored table as is posed in the minimisation of Expression (5)?

Steinhaus addresses this matter, but you will find it better dealt with by the Russian mathematician USPENSKII (*Some Applications of Mechanics to Mathematics*, Pergamon Press, 1961).

Those readers studying Physics will understand Uspenskii's argument very easily - minimising $x + y + z$ is the same as minimising the potential energy of the system depicted in Figure 3. The mathematical questions involved are the same.

[As an applied mathematician, I take great pleasure in the paradoxical note sounded by Uspenskii's title. The section (pp.20-22) I refer to here gives only one of his many delightful examples. But just one word of warning. Uspenskii relies on Steinhaus. Uspenskii wrote in Russian, and translated Steinhaus' title from either Polish or English into Russian. In due course, Uspenskii's book was translated into English. "Mathematical Snapshots" appears (after this long journey) as "Mathematical Kaleidoscope". This is just one example of the traps that can beset the historian of mathematics.]

But now we have established the validity of Steinhaus' lateral thoughts, we can go back to Problem (2). What if there are λ people in A, μ in B, ν in C? All we need do, says Steinhaus, is to use Figure 3 again, but to hang a weight of λ kg on the string through A, μ kg on the string through B and ν kg on the string through C.

Once again, and for the same reason, the position taken up by D (the knot) determines the solution. Once again we can analyse that solution in terms of the forces pulling on D. See Figure 5.

When the point D comes to rest, we have, by a result known as LAMY's Theorem:

$$\frac{\lambda}{\sin\theta} = \frac{\mu}{\sin\phi} = \frac{\nu}{\sin\psi} .$$

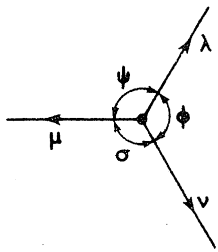


Figure 5.

Steinhaus produces this result but with a bit more song and dance, as he seems to have been unaware of Lamy's Theorem as such.

[The theorem is said to be found in B.Lamy's book *Traitez de Mécanique* (1679), to which I do not have access. However, many more recent books give the result and the name "Lamy's Theorem" is standard. The proof is not difficult. I leave it as an exercise to the reader.]††

Equation (6) applies if a point D satisfying these conditions can be found - otherwise one of the vertices, and let's keep calling it A, swallows it up.

Again the question arises as to whether the point D exists and is unique and, in effect, the Steinhaus approach guarantees that it does.

Questions such as those raised here have become very important because of their economic implicatons, but they derive from Steiner's interest in a purely geometric problem.

* * * * *

"In World War II, one finds mathematical and scientific talent in widespread use in the Army, Navy, and Air Force, in government research laboratories, in war industries, in governmental, social and business agencies. A brief list of the variety of things that mathematicians did would include aerodynamics, hydrodynamics, ballistics, development of radar and sonar, development of the atomic bomb, cryptography and intelligence, aerial photography, meteorology, operations research, development of computing machines, econometrics, rocketry, development of theories of feedback and control. Many professors of mathematics were directly involved in these things, as were many of their students."

Philip J. Davis and Reuben Hersh: *The Mathematical Experience*, Penguin, 1983.

* * * * *

---

††B.Lamy (1640-1715) was a writer of mathematical texts. His biographer states that the theorem should really be credited to the slightly later author P.Varignon (1654-1722).

SOLVING POLYNOMIAL EQUATIONS II

Neil S. Barnett
Department of Mathematics
and Operations Research
Footscray Institute of Technology

In part I of this article, in *Function* Vol.11, part 4, we began a discussion of the solution of polynomial equations. In this final part we begin by giving an interesting method, devised (in 1829) by Jacques Charles François Sturm, modifying an earlier idea due to Fourier, which enables one to discover how many real roots and hence how many complex roots a polynomial has. The method applies only to polynomials without repeated roots. Other considerations can be used to remove any repeated roots before beginning but we do not consider this refinement here: in a practical situation repeated roots would occur only rarely.

THE NUMBER OF REAL SOLUTIONS

If the polynomial equation is represented as $f(x) = 0$ and its derivative as $f'(x) = 0$ then the following functions may be calculated. First write $f_0(x)$ for $f(x)$ and $f_1(x)$ for $f'(x)$. Now divide $f_0(x)$ by $f_1(x)$, getting a remainder that we call $-f_2(x)$, of degree less than $f_1(x)$:

$$f_0(x) = f_1(x)q_1(x) - f_2(x)$$

Now divide $f_1(x)$ by $f_2(x)$ getting remainder $-f_3(x)$, of degree less than $f_2(x)$:

$$f_1(x) = f_2(x)q_2(x) - f_3(x)$$

Continue the process, terminating with

$$f_{m-2}(x) = f_{m-1}(x)\ q_{m-1}(x) - f_m(x)$$

for which $f_m(x)$ is a non-zero constant. The calculations end with a non-zero constant because the polynomial $f(x)$ is assumed to have no repeated roots.

Sturm's theorem states that if a and b are not roots of $f(x)$ then the number of real solutions to $f(x) = 0$ in the interval of $a < x < b$ is $N(a) - N(b)$ where $N(a)$ is the number of sign changes in the sequence $f_0(a), f_1(a), \ldots, f_m(a)$ and $N(b)$ the number of sign changes in the sequence $f_0(b), f_1(b), \ldots, f_m(b)$.

Use of this will be illustrated shortly. Although it is not appropriate to prove this result here it shouldn't be surprising that changes of sign are linked with existence of real solutions. Since polynomials are continuous it is clear that if a polynomial $y = f(x)$ is such that at $x = a$, $f(a) > 0$ and at $x = b(>a)$, $f(b) < 0$ then $y = f(x)$ must cross the x axis at least once (so there is at least one real solution) between $x = a$ and $x = b$. The same is true if $f(a) < 0$ and $f(b) > 0$. Thus a change of sign of $f(x)$ in its values for $x = a$ and $x = b$ indicates the existence of at least one real solution between $x = a$ and $x = b$.

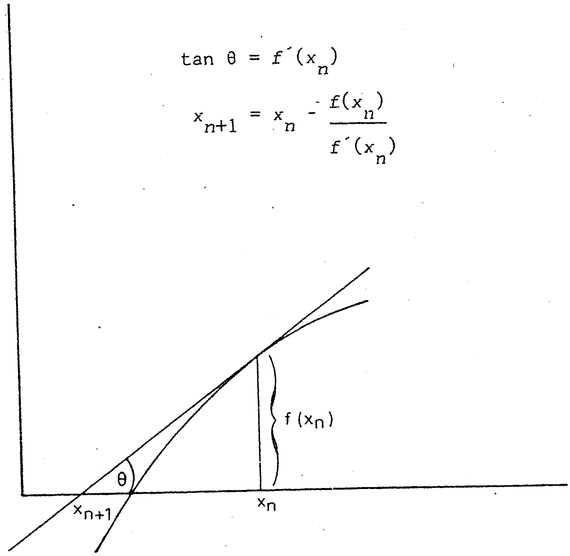
APPROXIMATION AND ITERATION

Once the number of real solutions is known, intelligent application of this sign change principle can often be used to obtain a first approximation to these solutions. Once this first approximation is available there is potential for being able to improve on it. Methods that facilitate this are called iterative techniques; the following is one that is widely used. Such methods proceed with a crude approximation to a particular solution and refine it repetitively until its value is given to a desired degree of accuracy. If the coefficients of the equation have been themselves approximated then this will of course affect the accuracy of the solutions. Because of being systematic and repetitive, iterative techniques are ideally suited to be performed by computer. The functions $f_1(x)$, $f_2(x)$, $f_3(x)$.. $f_m(x)$ can also be obtained via computer.

The method to be demonstrated is called the Newton-Raphson method and uses the given polynomial $f(x)$ and its derivative $f'(x)$. The iteration formula is quite simple (see diagram) and can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots$$

where x_0 is the first rough approximation and x_1 the first 'improvement', x_2 the second, and so on.



To start, therefore, requires finding x_1 from

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \quad \text{and so on .}$$

When the digits of successive approximations do not change at the accuracy required then the iteration process can stop. There are other iterative methods designed to achieve the same end, they vary mainly in how quickly they reach the degree of accuracy required. (In the days when computations were done by hand this was particularly important!). Clearly, the Newton-Raphson method will strike problems when $f'(x_n) = 0$, however, this will usually not be a problem with other than contrived examples.

A SUMMARY EXAMPLE

To conclude, the techniques discussed in this article are used to obtain the largest real root (if one exists) to two places of decimals of the equation $1.2x^4 + 4.8x^3 - 2.4x^2 + 8.4x + 4.8 = 0$.

From what has been learned regarding complex solutions there are either 0, 2 or 4 real solutions. To determine how many, use is made of Sturm's theorem.

Recall that

$$f_0(x) = 1.2x^4 + 4.8x^3 - 2.4x^2 + 8.4x + 4.8$$

and

$$f_1(x) = f'(x) = 4.8x^3 + 14.4x^2 - 4.8x + 8.4 .$$

Dividing $f_0(x)$ by $f_1(x)$ gives a remainder term of $-4.8x^2 + 7.5x + 2.7$, giving

$$f_2(x) = 4.8x^2 - 7.5x - 2.7 .$$

Dividing $f_1(x)$ by $f_2(x)$ gives a remainder term of $32.11875x + 20.71875$, giving

$$f_3(x) = 32.11875x - 20.71875 .$$

Similarly, $f_4(x)$ is (to two decimal places) -4.14.

Now the functions, $f_0(x)$, ... , $f_4(x)$ are examined at large positive or negative values of x and 0 to see if they are positive or negative. When x is large in each case the highest power of x is the dominant term and so the signs of $f_0(x)$, ..., $f_4(x)$ are determined by the term of highest power. Thus

	Sign at	$-\infty$	0	∞
$f_0(x)$		+	+	+
$f_1(x)$		−	+	+
$f_2(x)$		+	−	+
$f_3(x)$		+	−	−
$f_4(x)$		−	−	−
	Total number of sign changes	3	1	1

Using Sturm's theorem there are 3−1 = 2 real roots (solutions) less than 0 and 1 − 1 = 0 real solutions greater than 0 .Hence there are two real solutions (both negative) and two complex solutions.

The task now remains to find roughly where the largest of the two negative roots lies. The method of obtaining bounds is of little value here because it has been established that there are complex solutions.

By judiciously choosing integer values and substituting them into

$$f(x) = 1.2x^4 + 4.8x^3 - 2.4x^2 + 8.4x + 4.8$$

it should be possible to determine the 'where-a-bouts' of these solutions. We find

$$f(0) = 4.8 ,$$

$$f(-1) = -4.8 .$$

Hence, this change of sign indicates a real root between -1 and 0. It is still necessary to locate the remaining solution at least approximately, so that it can be established whether or not there are one or two solutions between -1 and 0 . Iteration can strike difficulties if there are two close solutions. We have

$$f(-2) = -40.8$$

$$f(-5) = 52.8 .$$

Hence there is a solution between -5 and -2 and since our task is to find the largest, effort can now be concentrated on the single solution between -1 and 0 . Since

$$f(-.5) = -.525 ,$$

thus the solution lies between -.5 and 0; as a first approximation let this be -.3. This can now be used as x_0 in Newton-Raphson's iteration formula ,

$$x_1 = -.3 - \frac{f(-.3)}{f'(-.3)} = -.3 - \frac{1.94412}{11.0064} = -0.4776635412 \quad ,$$

$$x_2 = -0.4776635412 - \frac{-0.220621377}{13.45519694} = -0.4612668$$

$$\text{and} \quad x_3 = -0.461112205 .$$

Hence to two decimal places the largest real solution is -0.46.

As an exercise you may want to find, to the same degree of accuracy, the other remaining real solution. You will save yourself work if you first 'narrow it down' more before using iteration. As an additional exercise you might wish to attempt programming Newton-Raphson's result, the obtaining of $f_0(x)$, ..., $f_m(x)$ of Sturm's result and even the locating of real solutions by the change of sign method. Such programs could then be readily used to help solve many polynomial equations of much higher degree when hand calculation would be prohibitive.

* * * * *

A TRICK WITH DOSEDIT

Jandep

If you use a personal computer with PC or MS DOS then you are probably familiar with EDLIN the line editor provided with DOS. Most books say that nobody uses EDLIN these days and advise a screen editor or word processor to help prepare batch files. It is true that a line editor is not the easiest way to prepare such files and the COPY CON command can also be quite frustrating if you are not a touch typist. A real professional working with large installations of PCs might tell you, however, that he uses EDLIN every day and the reason for that is that it is available on every machine using DOS. For those who work with the Vax/VMS system EDLIN will also have a certain familiarity. So there is some advantage in knowing EDLIN.

DOSEDIT is a public domain program and is therefore free to all PC users. It is easy to get a copy - most PC users will have it. DOSEDIT is a program which saves a stack of DOS commands (up to 256 characters in toto). Before committing yourself to a sequence of DOS commands in a batch file DOSEDIT allows you to go through the sequence and if a command is incorrect to recall it from the stack with the arrow keys to the command line where you can change it. You can scroll backwards and forwards through the stack as you please. An example will make this clear.

Assuming you have DOSEDIT (and have read the document) you let it take over your PC by typing in the command

```
DOSEDIT
```

followed by Enter.

Then key in the following commands:

```
ECHO ONE

ECHO TWO

ECHO THREE
```

If you made any typing errors you can recall and edit the incorrect entry with the cursor control keys (the arrow keys etc.). The short document which comes with DOSEDIT describes this example and exactly how the keys are to be used for editing commands. Of course you can substitute any DOS commands for the ones in this example.

It is when EDLIN is combined with DOSEDIT that you see the real advantage. Suppose you have the stack of commands shown above. Now key in the command

EDLIN TRY.BAT

Enter input mode with command

I

Then call down the stack as before finishing with

CTRL Z

which is the 'end-of-file marker', followed by

EXIT

to return from EDLIN to DOS and your batch file will be saved to disk. To check that your batch file runs properly key in

TRY

followed by Enter to run it.

This method allows you to build a batch file by rehearsing it first with DOSEDIT and then recording it from the stack into an EDLIN file and saving it to disk.

When you run the batch program suggested you may dislike the way commands are echoed to the screen before they are executed so try another experiment. Clear the stack with Ctrl-Pg Up then enter:

ECHO BLAH

ECHO ONE

ECHO TWO

ECHO THREE

Errors in typing can be corrected by calling down the stack as before.

Now enter EDLIN with

EDLIN TRY2.BAT

and enter input mode with

I

Call down your stack with the cursor keys but change ECHO BLAH to ECHO OFF. Continue on as before until Ctrl Z, the end-of file marker, and EXIT from EDLIN as before. Now when you run TRY2 the separate commands will not appear on the screen which will show only as follows:

ECHO OFF

ONE

TWO

THREE

Doing it this way returns your normal prompt when the program finishes. An alternative is to use ECHO OFF before running TRY2 but then you have to remember to key in ECHO ON again as soon as TRY2 has finished or you will get lost in DOS.

DOSEDIT and EDLIN in combination give you a miniature word processor which is amusing to try out. For example you can write some Christmas greetings to your friends this way.

First clear the stack with Ctrl-Pg Up then key in the following

Dear Bill

Can you come

to my party

on Christmas Eve?

5 o'clock in the Caf

Love,

Mary

DOS will complain as each 'command' is entered but each line will go into the stack nonetheless. Then as before key in

EDLIN TRY3.BAT

and

I

to enter input mode. Call down the stack, and finish with

Ctrl Z

and

EXIT

You can TYPE your batch file TRY3.BAT to the screen, or PRINT it to the printer. By the way, REName the file LETTER.TXT if you don't want it to be mixed up with your executable batch files later.

Now repeat the letter as often as you like changing the name each time and your miniature word processor will produce the 'customized' invitations you need.

* * * * *

"At a talk which I gave at a celebration of the twenty-fifth anniversary of the construction of von Neumann's computer in Princeton a few years ago, I suddenly started estimating silently in my mind how many theorems are published yearly in mathematical journals. I made a quick mental calculation and came to a number like one hundred thousand theorems per year. I mentioned this and my audience gasped. The next day two of the younger mathematicians in the audience came to tell me that, impressed by this enormous figure, they undertook a more systematic and detailed search in the Institute library. By multiplying the number of journals by the number of yearly issues, by the number of papers per issue and the average number of theorems per paper, their estimate came to nearly two hundred thousand theorems a year."

Stanislaw Ulam : *Adventures of a Mathematician*, New York, Schribners 1976.

CAREERS IN ECONOMETRICS

Maxwell L. King
Department of Econometrics & Operations Research
Monash University

WHAT IS ECONOMETRICS?

There is no simple answer to this question because econometricians can wear many different hats. Sometimes they are economists who use economic theory to improve the statistical analyses of economic problems of interest. Sometimes they are mathematicians concerned with translating economic theory into mathematical terms so it can be tested. At times they might be considered to be accountants who seek, collect and monitor economic data and who relate theoretical economic variables to observed data values. They may also be business consultants who provide predictions or forecasts of a variety of variables necessary for planning and budgeting. Sometimes they are applied statisticians who spend hours at the computer estimating economic relationships and/or forecasting future values of economic variables. They can also be theoretical statisticians who use their skill to develop statistical techniques to overcome special problems that occur when statistical methods are applied to economic problems.

The computer has played a vital role in the development of econometrics as a discipline. The 60's and early 70's saw the construction of a number of large econometric models of different economies. These models are systems of mathematical equations which attempt to reflect the complex inter-relationships between economic variables. With the aid of a computer they allow economic policymakers to forecast how the economy will behave in the future, particularly if the government makes policy changes. Over the last four decades, econometric techniques have also been used to explore and test economic theories, and relationships. In fact, econometrics is that branch of economics which attempts to quantify economic relationships. In recent years, the widespread penetration of the computer into the office environment, particularly with the advent of the personal computer, has meant that econometric methods have become more widely available and more widely used. The deregulation of financial and foreign exchange markets, plus a heightened awareness of the need for modern businesses to be competitive, has led to an increased demand for economists and financial analysts with quantitative skills. These skills are used to monitor the national and international economies plus markets of particular interest as well as to make forecasts. Undoubtedly, the demand for economists and financial analysts who are trained in econometric techniques will continue to grow.

WHO EMPLOYS ECONOMETRICIANS?

A survey of advertisements for economists placed in the Melbourne newspaper *The Age* over six weekends during March/April this year revealed that:

(i) 71% of the advertisements explicitly required the successful applicant to work with statistical data,

(ii) a knowledge of econometric or quantitative methods was acknowledged as desirable for 47% of the positions,

(iii) 29% of the positions required an ability to collect data,

(iv) 29% of the positions specified an ability to monitor economic series,

(v) 24% of the positions required the successful applicant to forecast economic variables,

(vi) 21% of the positions required a knowledge of sample surveying methods.

Organizations that wished to recruit economists with a knowledge of econometric or quantitative techniques included:

> The Labour Resources Centre, EPAC, Brotherhood of St Laurence, Health Department of Victoria, Department of Conservation Forests and Lands, Australian Dairy Corporation, Victorian Council of Social Service, BASF, Australian Bureau of Statistics, Federal Bureau of Transport Economics, RACV.

A similar survey of senior management positions advertised in the *Financial Review* identified a number of firms seeking Managerial/Financial Accounts or Economic/Business Analysts with quantitative skills. The specified duties of successful applicants included short and medium term forecasting and the monitoring of key economic and financial variables.

Both surveys show that some training in econometric or quantitative methods helps the employment prospects of graduates in economics, finance and accounting.

Some students may wish to train to be econometricians. Graduates who have specialized in econometrics are employed in a variety of key positions in both the private and public sectors. There is ample scope for using and developing further the skills acquired during university training, and the career opportunities open to econometricians frequently lead to senior administrative positions.

In Australia, graduates trained in econometrics, and quantitative economics generally, are employed by such organizaitons as:

> Oil and Mineral Companies, Insurance Companies, Manufacturing Industries, Trading Banks, The Reserve Bank of Australia, The Industries Assistance Commission, the Australian Bureau of Statistics, The Bureau of Transport Economics, Telecom Australia, The Bureau of Agricultural Economics, The Australian Treasury, EPAC, State Government Departments, Economic and Business Consultants, and Economic Research Groups.

TRAINING IN ECONOMETRICS

The training of an econometrician involves courses of study in the main areas of Economics, with special emphasis on the statistical analysis of economic data. The particular techniques which comprise econometrics itself are largely extensions of those found in certain areas of mathematical statistics, these extensions being necessary because of the particular nature of economic systems and economic data.

Econometrics is both a theoretical and an applied subject and its teaching, as well as offering specialised training in econometric principles and theory, places substantial emphasis on the development of sound practical techniques. Courses in applied econometrics put the theory into practice. They offer the opportunity for students to gain practical experience in the handling of economic data, the construction of econometric models, and the use of a range of the latest computer packages.

* * * * *

INDEX, VOLUME 11

* * * * *