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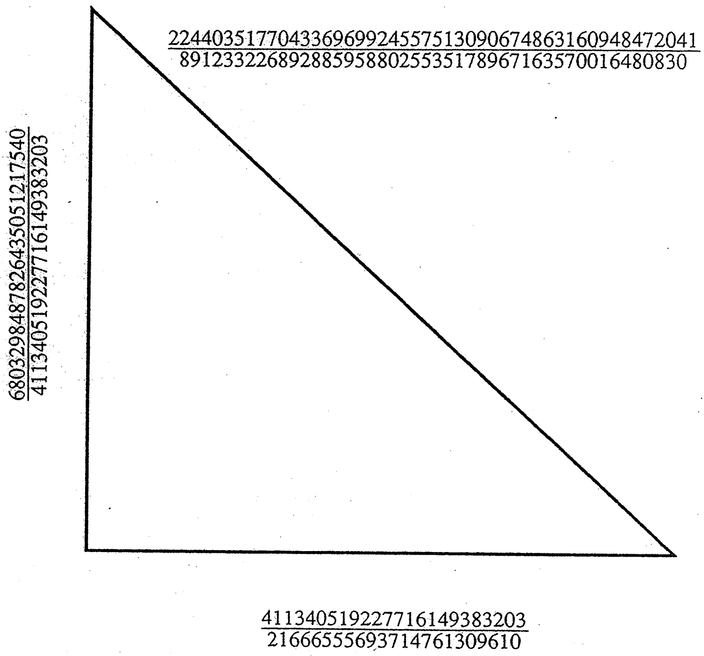

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

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Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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The world is full of mystery, magic, uncertainty and even pessimism, or is it not? In this FUNCTION issue we encounter enigmatic formulae from India, two pieces of mathemagic, as well as strange logic which accepts answers such as "don't know", and we discover that nothing ever goes wright – here we go. A computer program will make your Q-zapping (?!) easy, and at last we receive reassurances that the abacus can be beaten by a quick-witted human calculator. And watch for mathematicians: some of them are very clearly politicians! Finally, Perdix informs us about the Australian Mathematical Olympiad and also tells us who will represent Australia at the International Mathematical Olympiad this year.

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THE FRONT COVER

John Stillwell
Monash University

This picture shows the evil which can lurk in the heart of even a humble right-angled triangle. The monstrous numbers for the lengths of the sides come from asking the question: for what whole numbers n is n the area of a right-angled triangle with rational sides? The question goes back over 1000 years. A tenth-century Arab manuscript listed 30 values of n which can be areas of rational right-angled triangles, the first few being 5, 6, 14, 15, 21, 30. The area 6 is for the well-known (3,4,5) triangle, and 30 is for the (5,12,13) triangle. Fractional sides are needed to get the other areas mentioned. For example, 5 is the area of the triangle with sides 3/2, 20/3, 41/6. (Use Pythagoras' theorem to check that this triangle really is right-angled.)

A computational search for areas n can be made by using the formulas

$$x = a^2 - b^2, \quad y = 2ab, \quad z = a^2 + b^2$$

which generate sides x, y, z of right-angled triangles. The area is $xy/2$, and by removing any square factor f^2 one gets the triangle with sides x/f, y/f, z/f and integer area $xy/2f^2$. The area 5 turns up in this way when $a = 5$ and $b = 4$, giving $x = 9$, $y = 40$, $z = 41$ and area $180 = 6^2 \times 5$. By going to slightly larger values of a, b you can find some areas the Arabs missed, for example area 7 when $x = 175/60$, $y = 288/60$, $z = 337/60$. The next smallest area they missed is 13 ; can you find the sides required? (Hint: confine the search to values of a, b which are squares.)

Obviously something cleverer than this is needed to find the triangle on the cover, which is the simplest one with area 157. The problem is now attacked within the theory of elliptic curves, one of the most active areas of modern number theory. The problem of deciding which n can be areas is not quite settled, even yet, but the theory of elliptic curves gives answers which can be checked without prohibitive amounts of computation.

It should be mentioned that $n = 1, 2, 3, 4$ are *not* the areas of rational right-angled triangles. The case $n = 1$ (and $n = 4$, which follows because $4 = 2^2$) was settled by Fermat around 1640. In doing so, he also showed that there are no positive integers p, q, r such that

$$p^4 + q^4 = r^4.$$

In fact, the latter result is what Fermat is better known for. It was the first step actually taken towards proving the so-called "Fermat's last theorem" that none of the equations

$$p^3 + q^3 = r^3, \quad p^4 + q^4 = r^4, \quad p^5 + q^5 = r^5, \ldots$$

have a solution in positive integers.

SOME FASCINATING FORMULAE OF RAMANUJAN

D. Somasundaram
Madras University[1], India

Editor's Note. Last year the mathematician Srinivasa Ramanujan would have been 100, and the world paid attention. He left many a mystery: formulae that seemed to tell little of their meaning.

Unriddled they sometimes turned out to be deep mathematical theorems. Godfrey Hardy, who wrote the book *Ramanujan*, thought it was a shame this "Hindu clerk" was not born a century earlier, in the "great age of formulae".

About 250 km south of Madras, the city where he studied, is Ramanujan's birthplace. He died when 32.

As late as 1976 one of his notebooks, his legacy, was found in an English library. Others had then already been published (in two volumes by Tata Institute of Fundamental Research, Bombay 1957), and from these the following formulae were taken.

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The object of this short note is to bring to light some formulae of Ramanujan appearing in the "working pages" of his Notebooks. Though many formulae are scattered in these "working pages", we choose the following four which will interest mathematics students at school level.

FORMULA 1. [1, Vol.1. Page 240]

$$(a+1)(b+1)(c+1) + (a-1)(b-1)(c-1) = 2(a+b+c+abc)$$

FORMULA 2. [1, Vol.1, Page 202]

$$\{6m^2 + (3m^3-m)\}^3 + \{6m^2-(3m^3-m)\}^3 = \{6m^2(3m^2+1)\}^2$$

FORMULA 3. [1, Vol.2. Page 384]

(i) $\sqrt[3]{2} = 1.259921\ 049894\ 873164\ 767208\ \ldots$

(ii) $\sqrt[3]{2} = \frac{5}{4}\left(1+\frac{24}{1000}\right)^{1/3} = \frac{63}{50}\left[1+\frac{188}{1000000}\right]^{-1/3}$

---

1 Dr. D. Somasundaram is Professor and Head of the Department of Mathematics.

FORMULA 4. [1, Vol.1. Page 244]

$$\left\{x + \frac{x^2}{2} + \frac{21\ x^3}{64} + \frac{31\ x^4}{128} + \ldots\right\}^{11}$$

$$= x^{11} + \frac{11}{2}x^{12} + \frac{1111}{64}x^{13} + \frac{111111}{2688}x^{14} + \ldots$$

Since the proofs of formulae 1 and 2 are easy, we shall outline the proofs of formula 3 and 4.

PROOF OF FORMULA 3. The decimal representation for $\sqrt[3]{2}$ can be verified with a mechanical or electrical device.

On simplifying the first expression with power $\frac{1}{3}$ for $\sqrt[3]{2}$, we get $2^{10/3}/2^3$ which yields the result.

For the other expression with power $-\frac{1}{3}$, we have after simplification,

$$\frac{63}{50}\left[\frac{10^6}{1000188}\right]^{1/3} = \frac{126}{(1000188)^{1/3}}$$

Since $\frac{(126)^3}{(1000188)} = 2$, we have formula 3.

PROOF OF FORMULA 4. We can expand the left-hand side of formula 3 as follows.

$$x^{11} + 11\ x^{10}\left[\frac{x^2}{2} + \frac{21\ x^3}{64} + \frac{31\ x^4}{128} + \ldots\right]$$
$$+ \frac{11.10}{1.2}x^9\left[\frac{x^2}{2} + \frac{21\ x^3}{64} + \frac{31\ x^4}{128} + \ldots\right]^2$$
$$+ \frac{11.10.9}{1.2.3}x^8\left[\frac{x^2}{2} + \frac{21\ x^3}{64} + \frac{31\ x^4}{128} + \ldots\right]^3$$
$$+ \ldots$$

Collecting the required coefficients, we can write the above in the increasing powers of x starting from x^{11} as follows:

$$x^{11} + \frac{11}{2}x^{12} + \left[\frac{11.21}{64} + \frac{11.10}{1.2} \cdot \frac{1}{4}\right]x^{13}$$
$$+ \left[\frac{11.31}{128} + \frac{11.5.21}{64} + \frac{11.5.3}{8}\right]x^{14} + \ldots$$

$$= x^{11} + \frac{11}{2}x^{12} + \frac{1111}{64}x^{13} + \frac{5291}{128}x^{14} + \dots$$

Since $5291 \times 21 = 111111$ and $128 \times 21 = 2688$, we get the desired expansion.[2]

REMARK. It is interesting to find formulae similar to (ii) of formula 3 for $\sqrt[3]{5}$, $\sqrt[3]{7}$ and so on.

* * * * *

LETTERS TO THE EDITOR

Correspondent Colin Wilson of Highett contributed the following letter, dated 16.3. :

An elementary solution of Problem 11.3.1 (Vol. 11, Pt. 3, p. 96) is given by Perdix in Vol. 12, Pt. 1, p. 31. Here is another solution utilizing two simple inequalities that are often useful for establishing other inequalities:

Given that a and b are positive numbers with $a + b = 1$, we show that $(a+\frac{1}{a})^2 + (b+\frac{1}{b})^2 \geq \frac{25}{2}$ with equality if and only if $a = b = \frac{1}{2}$.
Noting that

$$x^2 + y^2 = \frac{1}{2}((x+y)^2 + (x-y)^2) \geq \frac{1}{2}(x+y)^2 \qquad (1)$$

and

$$xy = \frac{1}{4}((x+y)^2 - (x-y)^2) \leq \frac{1}{4}(x+y)^2 \qquad (2)$$

for all real x and y, with equality if and only if $x = y$, it follows that

$$(a+\frac{1}{a})^2 + (b+\frac{1}{b})^2 \geq \frac{1}{2}(a+\frac{1}{a}+b+\frac{1}{b})^2 \quad \text{by (1) with } x = a + \frac{1}{a},\ y = b + \frac{1}{b},$$

$$= \frac{1}{2}(a+b+\frac{a+b}{ab})^2$$

$$= \frac{1}{2}(1+\frac{1}{ab})^2$$

$$\geq \frac{1}{2}\left[1+\frac{4}{(a+b)^2}\right]^2 \quad \text{by (2) with } x = a,\ y = b,$$

$$= \frac{25}{2},$$

2 It should be noted that whilst formula 4 works for the term shown, there is no obvious pattern; and it may not be possible to generate the next term on either side of the formula. In his notebooks Ramanujan frequently recorded formulae such as this for which a pattern is clear for a few terms, but does not continue (Ed.).

with equality if and only if $a + \frac{1}{a} = b + \frac{1}{b}$ and $a = b$, i.e. if and only if $a = b = \frac{1}{2}$.

Note that for non-negative x and y we can write inequalities (1) and (2) in the form

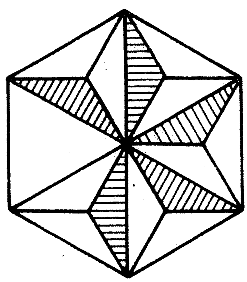
$$\sqrt{xy} < \frac{x+y}{2} < \sqrt{\frac{x^2+y^2}{2}} \qquad (3)$$

where $\sqrt{xy}$, $\frac{x+y}{2}$ and $\sqrt{\frac{x^2+y^2}{2}}$ are called the geometric mean, arithmetic mean and root mean square of x, y respectively. Inequalities (1) and (2), or the equivalent (for non-negative x, y) inequalities (3), might well be considered standard mathematics competition equipment.

* * * * *

FUNCTION editor M.A.B. Deakin reported from India in a letter of 7/2:

I was looking around in Madras yesterday and saw a curious gewgaw. It was a cheap thing of plywood inlay, but maybe Function readers could make one out of cardboard. Here's how it could be made.



Take a regular hexagon and discard one sixth. Use centroidal division of the remaining 5 triangles and colour dark in a cyclic pattern as indicated. The remaining sub-triangles in each triangle are coloured cyclically in 2 light colours. Now close up, so that the colouring lies on the inside of an icosahedral cap.

It makes a salad or fruit bowl that is stood up by putting the point into (say) a suitably-sized serviette ring.

The illusion of a set of pyramids towards or away from the viewer is remarkably strong.

* * * * *

TWO PIECES OF MATHEMAGIC

Bruce Henry
Victoria College, Rusden Campus

This article explores two magic tricks with digits of numbers, each using the test for divisibility by nine - a number is divisible by 9 if the sum of its digits is divisible by 9.

- Write down a four-digit number with all the digits different. (2794)
- Write down that number with the digits reversed. (4972)
- Calculate the difference between these two numbers. (4972-2794 = 2178)
- Multiply this new number by any positive number you choose less than 100 (2178 x 6 = 13068)
- From your answer, keep one non-zero digit to yourself and tell me all the others; I will tell you the missing one. (keep 6, say 1308)
- The missing digit is 6.

You may like to try a few more examples and look for pattern and an explanation before reading on.

The trick works because the difference between the first two numbers is always a multiple of 9. All I have to do is add the digits you give me and subtract the result from the next multiple of 9 (in the examples above, 1 + 3 + 0 + 8 = 12, 18 - 12 = 6). If the sum of the given digits is a multiple of 9 already, then subtract it from the NEXT multiple to get 9 as the missing digit - you are only allowed to keep NON-ZERO digits. Multiplying the number by any other number does not alter the fact that the number will be divisible by 9 - this is just a bit of smoke-screen !

To show why the trick works, and to extend it, we consider a general case - the number abcd where a, b, c and d are different digits and $a > d$. We prefer to write the number as $1000a + 100b + 10c + d$. The reverse number is $1000d + 100c + 10b + a$ and the difference is $999a + 90b - 90c - 999d$ which is clearly divisible by 9, since 9 divides 999 and 90.

In fact a similar situation arises if the digits are simply scrambled, not reversed - can you prove this ? What happens if the digits are not all different? Does the trick work with five- or three-digit numbers? There are a lot of possibilities for extending this one.

What about this one ?

- Write down a three-digit number with all digits different. (397)
- Reverse it and find the difference (793 - 397 = 396)
- Reverse this new number and ADD it on (396 + 693 = 1089)
- The answer is 1089 !

Again you should try some more cases and check that the answer is always 1089 before you read on. This trick can be made very spectacular if the magic 1089 is hidden away somewhere first ("I will write the answer to your sum on this piece of paper and you can put it in your pocket.") or revealed unexpectedly. (Write it on your arm in soap in advance, then burn the paper with the answer on it - unseen by you - and rub the ashes on your arm - 1089 magically appears! Practise first!)

Why does this one work? Call the first number abc = 100a + 10b + c. The reverse is 100c + 10b + a. Call the difference xyz = 100x + 10y + z. Since b appears as the number of tens in both numbers and $a > c$ so there will be a carry number in the subtraction, y must be 9. We now have $100(x + z) + 180 + (x + z)$, remembering that $y = 9$. But $x + z = 9$, so this sum is $900 + 180 + 9 = 1089$!

You may like to try to extend this one too - what happens with four-digit numbers? What if not all the digits are different?

There are plenty of these kind of tricks based on pattern work with numbers and it is not too difficult to make up your own. Become a mathemagician!

* * * * *

INFINITY SURROUNDED

"The solution of the difficulties which formerly surrounded the mathematical infinite is probably the greatest achievement of which our age can boast".

Bertrand Russell
(1872-1970)

"Altogether we believe that the new mathematici, who introduced the infinitely small magnitudes into geometry, do not deserve many thanks. Proofs that rely on them are ever so short of the perspicuity, profundity and conviction met in all other geometrical demonstrations. Neither do we need them [the infinitely small magnitudes] in geometry nor in any case, where infinitely small things occur: one can employ the method that Archimedes used in his books *de circuli dimensione* and *de figuro et cylindro*, whereby much stronger conviction is obtained.

Franz Ulrich Theodosius
Aepinus (1724-1802)

* * * * *

YOU'RE A MATHEMATICIAN! OH! I NEVER WAS MUCH GOOD AT MATHS!

Brian A. Davey
La Trobe University

I wish I had a dollar for every time I have heard the title of this article or something very similar to it. The fact that you're reading this indicates that you would almost certainly not say this yourself. Nevertheless the unfortunate fact remains that somewhere in their education most people have a negative experience with mathematics which has a lasting effect upon them. Preventing this from happening is probably the most important problem facing mathematics educators.

I don't intend to tackle this problem here. Rather, I'd like to tell you about the rest of the conversation. Typically it goes something like this (since the natural abbreviation for my name is BAD, let's call the other person GOOD):

GOOD: "What do you do for a crust?"

BAD: "I'm a mathematician."

GOOD: "You're a mathematician! Oh! I never was much good at maths!"

BAD: "Unfortunately lots of people say that. Actually it's lots of fun. I get a real kick out of my research."

GOOD: "Fun, you've got to be kidding! And how can you do research in mathematics? Surely it's all been done. What do you actually do?"

BAD: "If you've got three hours I'll tell you. Actually it isn't hard to explain, but to do it justice would take a bit of time even if you'd already done some maths at uni. If you'd really like me to try, I'd be glad to. You can choose anything from a 10 minute snapshot to the full three-hour crash course."

Some people decide not to go any further and the conversation quickly changes to more important topics like "When will Collingwood win its next flag?" Most people opt for the 10 minute snapshot. On three occasions I've been asked for and delivered the three-hour crash course! What I'm about to give you is something in between. I should mention that I am a pure mathematician and the maths that I do is closer to philosophy than to engineering. Consequently, somewhere along the way GOOD usually asks, "*What is all this good for?*" I then explain that I do what I do because I enjoy it and don't seek or expect any applications. It comes as a surprise to me (and to GOOD) to discover that electronics engineers and computer scientists are actually interested in these ideas - but that's not why I study them.

TWO-VALUED LOGIC

In our everyday life we work with two-valued logic, by which I mean that any statement which makes sense is either *true* or *false*. Statements like "*Two plus two equals four*" or "*There are seven days in a week*" are true. Statements like "*Two plus two equals five*" or "*There are six days in a week*" are false. It might be hard to decide whether a statement like "*President Reagan is a good actor*" is true or false, but this is only because we haven't decided what "good actor" means. Once this is decided the statement will be either true or false.

Let <u>STATE</u> be the (rather large) collection of all English statements which make sense. Just as there are natural operations "+" (plus), "." (times) and "−" (negative) on the number line which produce new numbers from old ones, there are natural operations on <u>STATE</u> which produce new statements from old ones: they are "∨" (or), "∧" (and) and "¬" (not). Consider the following statements –

p: "*I will play basketball tomorrow*",

q: "*It will rain tomorrow*".

Then $p \vee q$ is the sentence "*I will play basketball tomorrow OR it will rain tomorrow*", and $p \wedge q$ is the sentence "*I will play basketball tomorrow AND it will rain tomorrow*".

Note that $p \vee q$ is true provided p is true or q is true (or both); $p \wedge q$ is true provided both p and q are true; $\neg p$ is true exactly when p is false. This is most easily expressed in some form of table. We let 1 stand for "is true", 0 stand for "is false"; then the *truth tables* for ∨ (or), ∧ (and) and ¬ (not) are as given in Figure 1.

p	q	p ∨ q
1	1	1
1	0	1
0	1	1
0	0	0

p	q	p ∧ q
1	1	1
1	0	0
0	1	0
0	0	0

p	¬p
1	0
0	1

Figure 1

If you look at these tables they give us operations on the (rather small) collection consisting of just 1 and 0: the first line of the ∨-table says $1 \vee 1 = 1$, (which is said "1 or 1 equals 1") the third line of the ∧-table says $0 \wedge 1 = 0$, etc. In this way we produce the more compact tables of Figure 2.

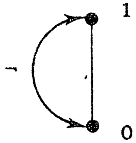
$\vee$	1	0
1	1	1
0	1	0

$\wedge$	1	0
1	1	0
0	0	0

	1	0
$\neg$	0	1

Figure 2

Finally we note that on 1 and 0, the operation $\vee$ is just "maximum" and the operation $\wedge$ is "minimum" while $\neg x$ equals $1 - x$. This gives us a diagramatic way of visualizing the three operations $\vee$, $\wedge$ and $\neg$ on 1 and 0 as shown in Figure 3. (The vertical line from the blob representing 0 to



$$x \wedge y = \max\{x, y\}$$

$$x \wedge y = \min\{x, y\}$$

$$\neg x = 1 - x$$

Figure 3

the blob representing 1 is there to remind you that 0 is less than 1 when you do calculations like $0 \vee 1 = \max\{0, 1\} = 1$ and $1 \wedge 0 = \min\{1, 0\} = 0$.)

The operations $+$, $\circ$ and $-$ on the number line satisfy certain natural laws such as :

$$x + y = y + x, \qquad x \cdot y = y \cdot x,$$

$$x \cdot (y + z) = x \cdot y + x \cdot z, \qquad x + (-x) = 0.$$

Similarly the operations $\vee$, $\wedge$ and $\neg$ on the truth values 1 and 0 satisfy certain laws known as the (rather pompous) *Laws of Thought* or *Laws of the Propositional Calculus* and nowadays better known as the laws of *Boolean algebra*. (Named after George Boole who, in the middle of last century, was the first to attempt to give an algebraic formulation of the "laws of thought".) Below are some of the laws which hold for $\vee$, $\wedge$ and $\neg$ on the truth values 1 and 0: for all x, y and z we have

$x \vee (y \vee z) = (x \vee y) \vee z$ $\quad$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ $\quad$ (*associative*)

$x \vee y = y \vee x$ $\quad$ $x \wedge y = y \wedge x$ $\quad$ (*commutative*)

$x \vee x = x$ $\quad$ $x \wedge x = x$ $\quad$ (*idempotent*)

$x \vee (x \wedge y) = x$ $\quad$ $x \wedge (x \vee y) = x$ $\quad$ (*absorption*)

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $\quad$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ $\quad$ (*distributive*)

$0 \vee x = x, \quad 1 \vee x = 1 \qquad\qquad 0 \wedge x = 0, \; 1 \wedge x = x \qquad$ (*zero-one*)

$x \vee \neg x = 1 \qquad\qquad x \wedge \neg x = 0 \qquad$ (*complementation*)

The laws state that no matter how you put in the truth values 1 and 0 for x, y and z, when you calculate the left-hand-side and the right-hand-side they will be equal. When checking the distributive law $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ there are 8 possible ways of assigning 1 and 0 to the variables x, y and z; as the truth table below shows, in each of these eight cases the left-hand-side and the right-hand-side are equal. (Complete the proof by using Figure 2 or Figure 3 to calculate the missing entries. Then go on and construct the truth tables for each of the

x	y	z	$x \wedge (y \vee z)$	$(x \wedge y) \vee (x \wedge z)$
1	1	1	1	1
1	1	0		1
1	0	1	1	
1	0	0	0	0
0	1	1		
0	1	0	0	
0	0	1		
0	0	0	0	0

other laws.) You should also be able to see what these laws mean back on the collection STATE of statements. For example, the commutative law $x \wedge y = y \wedge x$ says that if p and q are statements, then the compound statement $p \wedge q$ (i.e. "p and q") has exactly the same meaning as the compound statement $q \wedge p$ (i.e. "q and p").

This is where I come in! I am an algebraist. I study algebraic systems like the number line with its operations +, · and − (*high-school algebra*) and equally well the collection STATE with its operations ∨, ∧ and ¬ or the much smaller algebra of truth values 1 and 0 with the operations ∨, ∧ and ¬ (*Boolean algebra*). Most of the questions that I would consider are unfortunately beyond the scope of this short article, but here are a couple for starters.

QUESTION 1. *We have listed 18 laws of Boolean algebra. Is this list minimal?* (The list will be minimal provided none of the given laws follows from the others.) The answer is certainly "no". For example, the idempotent laws follow from the absorption laws:

$$
\begin{aligned}
x &= x \wedge (x \vee x) && \text{(absorption with } y = x) \\
\therefore\; x \vee x &= x \vee (x \wedge (x \vee x)) \\
&= x \vee (x \wedge y) && \text{(where } y = x \vee x) \\
&= x && \text{(absorption)}
\end{aligned}
$$

As a somewhat trickier exercise you might like to show that, given the associative, commutative, idempotent and absorption laws, each of the distributive laws follows from the other. Thus both idempotent laws and at least one distributive law can be deleted. Reducing the list of 18 laws to a minimal list is a non-trivial exercise.

QUESTION 2. *We have listed 18 laws of Boolean algebra. Is this list complete?* That is, given any other law of Boolean algebra (which has been shown to be true by working out the corresponding truth table), does it follow (can it be proved) from the 18 laws already listed? (For example, below on the left is the truth table calculation (with some gaps for you to fill in) which shows that $(x \wedge y) \vee (x \wedge \neg y) = x$ is a law of Boolean algebra, and on the right is a proof that it follows from the original list of laws.) The answer to this question is "yes". Again this is not at all

x	y	$\neg y$	$(x \wedge y) \vee (x \wedge \neg y)$	x
1	1	0		1
1	0	1	1	1
0	1	0	0	0
0	0	1		0

$$y \vee (\neg y) = 1$$
$$\therefore x \wedge (y \vee \neg y) = x \wedge 1$$
$$\therefore (x \wedge y) \vee (x \wedge \neg y) = x$$

obvious and requires, for example, a careful definition of what we mean by a "proof" like the one on the right-hand-side of the example above.

At this stage GOOD usually throws in the first "*What's all this good for?*" To which I'm able to reply that a computer is "just" a massive collection of two-valued switches. Each switch is either closed, and so lets an electrical current flow through it, (this corresponds to the truth value 1), or open, and so does not let the current flow through it (this corresponds to the truth value 0). Connecting switches in parallel corresponds to the operation $\vee$ while connecting switches in series corresponds to the operation $\wedge$. Thus Boolean algebra is used to determine the flow of the current through the computer. Boolean algebra is fundamental to both electronics and computer science. The mention of applications to computing always seems to placate GOOD and we are able to continue.

THREE-VALUED LOGIC

I hope by now we have taken two steps forward and you have some feeling for the sort of mathematics which I, as an algebraist, do. But we must also take one step backwards, for I have to admit that everything we've discussed so far about 2-valued logic and Boolean algebra was worked out late last century and early this century and (therefore) dates from before my birth. (It is important to note that it also predates the invention of the computer which to the modern pragmatist, like our friend GOOD, is Boolean algebra's raison d'être.) So what sort of algebra do I actually do? Read on.

There are various philosophical objections to two-valued logic. The most contentious of our 18 "laws of thought" is

$$x \vee \neg x = 1$$

which says that if p is any meaningful statement, then either p or its negation ¬p is true. Put more simply, this says that any meaningful statement is either true or false – this is known as "*the law of the excluded middle*" since it allows no middle ground between the two extremes of falsity and verity (= truth). Consider again the statement

q: "*It will rain tomorrow.*"

It could well be argued that, while we will know by the end of tomorrow whether the statement q is true or false, at the moment it should be assigned some other truth value which stands for "don't know". We shall return later to this three-valued logic in which the law of the excluded middle fails. First let's look at a more subtle argument against the law of the excluded middle.

Consider the following proof that 1/0 is not a number. (In other words, you can't divide by 0.) In this proof 0 and 1 are representing the numbers "zero" and "one", not the truth values "false" and "true". We need the following facts about numbers.

(a) If y is any number, then $y \cdot 0 = 0$.

(b) If x and 1/x are both numbers, then $(1/x) \cdot x = 1$.

Let p be the statement "1/0 *is not a number*." We wish to prove that the statement p is true. Suppose that p is false; then 1/0 is a number. Hence

$$0 = 1/0 \cdot 0 \quad \text{by (a) with } y = 1/0,$$

$$= 1 \quad \text{by (b) with } x = 0.$$

Thus the assumption that p is false leads to the contradiction $0 = 1$. Consequently p is not false; that is, p is true.

This form of argument is very common within mathematics but quite rare in everyday arguments and so may seem a little strange to you. It is called *reductio ad absurdum* or simply *proof by contradiction.* In order to show that a statement p is true, we show that the assumption that p is false leads to a contradiction (like $0 = 1$) and hence p must be true. This definitely uses the law of the excluded middle, since we need to know that if p is not false then it is true.

There is a school of philosophical thought known as *intuitionism.* Intuitionists believe that if I wish to prove the existence of some thing or other, then I must actually produce it; in a sense they insist that I be able to walk into the room and show it to them. At first sight, that probably seems like a perfectly reasonable viewpoint. But if you accept it then you must, in general, reject proofs by contradiction (and consequently you must also reject the law of the excluded middle), as the following example illustrates.

In the game of Hex, which is marketed by Parker Brothers Inc., two players, black and white, take turns to place pieces of their colour onto a diamond-shaped board with the aim of forming an unbroken chain of their pieces from one side to the opposite side. It is quite easy to show that there can never be a draw. Our intuition would tell us that the first player should have an advantage and indeed it can be proved that there is a winning strategy for the first player; that is, there is a set of instructions which, if followed by the first player, will always lead to a win for that player. Unfortunately, the only known proof of the existence of this set of instruction is a proof by contradiction – if we suppose that no such set of instructions exists, then we can derive a contradiction. An intuitionist will not accept such a proof, since the set of instructions has not actually been produced. (A detailed discussion of the game of Hex along with a sketch of this proof may be found in Martin Gardner's "*Mathematical Puzzles and Diversions*", published by Pelican.)

The two-valued logic based on the laws of Boolean algebra is known as classical logic; all others are known as *non-classical logics.* The non-classical logic which replaces Boolean algebra in the intuitionist's view of the world is an important one, but is too complex to describe here. Let's now return to the non-classical, three-valued logic with truth values "true", "false" and "don't know" to which we alluded earlier.

As before, let 1 stand for "is true" and 0 stand for "is false", and now let 1/2 stand for "don't–know". We still have the operations $\vee$, $\wedge$, $\neg$ standing for "or", "and", "not" and our first task is to work out their three-valued truth tables, as we did for the two-valued case back in Figure 1. On 1 and 0 the operations $\vee$, $\wedge$ and $\neg$ should act exactly as they did for two-valued logic. What about $p \vee q$ if p is true and q is don't–know, for example? Since $p \vee q$ will be true provided at least one of p and q is true, in this case $p \vee q$ will be true since p is true. Similarly, if p is true and q is don't–know, then $p \wedge q$ must be don't–know, since $p \wedge q$ is true only if both p and q are true. If p is don't–know, then $\neg p$ must also be don't–know. Arguing in this way we obtain the truth tables given in Figure 4. As in the two-valued case, we can

p	q	p ∨ q
1	1	1
1	1/2	1
1	0	1
1/2	1	1
1/2	1/2	1/2
1/2	0	1/2
0	1	1
0	1/2	1/2
0	0	0

p	q	p ∧ q
1	1	1
1	1/2	1/2
1	0	0
1/2	1	1/2
1/2	1/2	1/2
1/2	0	0
0	1	0
0	1/2	0
0	0	0

p	¬p
1	0
1/2	1/2
0	1

Figure 4.

view $\vee$, $\wedge$ and $\neg$ as operations on 1, 1/2, 0 as shown in Figure 5.

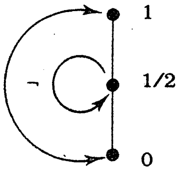
∨	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

∧	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

	1	1/2	0
¬	0	1/2	1

Figure 5.

Note that, as in the two-valued case, on 1, 1/2 and 0 the operation $\vee$ is "maximum" and the operation $\wedge$ is "minimum", while $\neg x$ equals $1 - x$. This leads to the diagrammatic visualization of $\vee$, $\wedge$ and $\neg$ on 1, 1/2 and 0 shown in Figure 6.



$$x \vee y = \max\{x, y\}$$

$$x \wedge y = \min\{x, y\}$$

$$\neg x = 1 - x.$$

Figure 6.

The law of the excluded middle certainly fails here, since

$$1/2 \vee \neg 1/2 = 1/2 \vee 1/2 = 1/2 \neq 1.$$

Nevertheless, the associative, commutative, idempotent, absorption, distributive and zero-one laws still hold. (You could check these laws and the laws given below by writing down all the three-valued truth tables, but it's slicker to observe that each of these laws expresses a natural property of max and min.) In order to obtain a complete set of laws for our three-valued logic (in the sense of Question 2 from the previous section), we must replace the complementation laws by

$\neg(x \vee y) = \neg x \wedge \neg y$ $\quad$ $\neg(x \wedge y) = \neg x \vee \neg y$ $\quad$ (*de Morgan*)

$\neg(\neg x) = x$ $\quad$ (*double negation*)

$(x \wedge \neg x) \wedge (y \vee \neg y) = x \wedge \neg x$ $\quad$ (*Kleene*)

The de Morgan laws are named after the 19th century mathematician Augustus de Morgan and may be familiar to you from set theory, while the Kleene law is named after the logician Stephen Kleene, who introduced this three-valued logic in 1938. As a slightly tricky exercise you might like to show that from these 20 laws it follows that $\neg 0 = 1$ and $\neg 1 = 0$.

In my research I have developed a general theory which applies, in particular, to this three-valued logic and yields interesting and useful information about it.

About now I either discover that GOOD is a closet philosopher and is fascinated by the concept of many-valued logics (Why stop at 3?), or I hear yet again, "*What's all this good for?*". Well, as it happens, back in 1980 when I was doing this research, I was amazed to discover that there was a Japanese electronics engineer working on the same topic. Using completely different methods, we had obtained overlapping results. For me it was a piece of pure research motivated by a simple quest for knowledge, while for him it was a practical piece of research related to the building of computers based on three-valued switches rather than the usual two-valued ones.

BAD: "So now you've got some idea of the sort of things I work on."

GOOD: "Are there many algebraists who work on non-classical logics?"

BAD: "There aren't many who work exclusively on them. In fact, I tend to use non-classical logics as testing grounds for general algebraic results. I actually spend most of my time doing research in universal algebra."

GOOD: "What on earth is `universal algebra`?"

BAD: "If you've got three hours I'll tell you . . . ."

* * * * *

DON'T BE ANXIOUS - BE SUBTLE!

"Histories make men wise; poets, witty; the mathematics, subtle; natural philosophy, deep; moral, grave; logic and rhetoric, able to contend".

Francis Bacon

"Anxious inquiry into ... mathematical problems leads away from the things of life, and estranges men from a perception of what conduces to the common good".

Juan Luis Vives

* * * * *

"Q-ZAPPING"

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Most readers will be familiar with the process of "zapping" - that is summing the squares of the digits of an integer and repeating the process until 1 is reached or until the endless cycle with 4 is reached.

If the digits are cubed, instead of squared, some interesting results occur. With zapping one can't predict which numbers will be "happy" - that is those which zap to 1, or those which will be "unhappy" - that is those which find themselves in the 4-cycle. With "Q-zapping", the name I give the cubing of the digits, results can be predicted.

If we divide the positive integers into three groups with the general forms of -

3n   3n + 1   3n + 2

it can be proved that when we Q-zap any of these it stays in its own form.

3n	→	3a	→	3b		etc.
3n + 1	→	3a + 1	→	3b + 1		etc.
3n + 2	→	3a + 2	→	3b + 2		etc.

From my investigation the following results appear to hold for all positive integers.

(1) If N = 3n, numbers of this type Q-zap to 153
($1^3 + 5^3 + 3^3 = 153$)

(2) If N = 3n + 2, numbers of this type Q-zap to 371 or 407
(N.B. $3^3 + 7^3 + 1^3 = 371$, $4^3 + 0^3 + 7^3 = 407$)

(3) If N = 3n + 1, numbers of this type Q-zap to 370 or 1
(N.B. $3^3 + 7^3 + 0^3 = 370$, $1^3 = 1$)
Or find themselves in the following endless cycles

(a)	136	→	244	→	136	etc.		
(b)	55	→	250	→	133	→	55	etc.
(c)	217	→	352	→	160	→	217	etc.
(d)	1459	→	919	→	1459	etc.		

Editor's Note: The results (1) to (3) above can be shown to be true using a computer, combined with some reasoning. First, a simple argument shows that Q-zapping enough times will always produce a number of at most 4 digits. Then a computer can be used to determine the behaviour of the numbers 1 to 9999.

If a number has k digits, then each digit is at most 9 and so Q-zapping produces at most $9^3 + 9^3 + ... + 9^3$ (k times) = $k \cdot 9^3$. Since $10^{k-1} > 9^{k-1}$ we see that Q-zapping produces a smaller number whenever $9^{k-1} \geq k \cdot 9^3$, i.e. $k \leq 9^{k-4}$. This is true whenever $k \geq 5$. Since any number of 5 or more digits always decreases on Q-zapping, it will eventually decrease to 4 or less digits.

A simple basic program to examine all 4 digit numbers is given on page 51.

Exercises.

(i) Show that a number of at most 4 digits must Q-zap to a smaller number, if it exceeds 2916 (= 4×729).

(ii) Show that a number of at most 4 digits must Q-zap to a smaller number if it exceeds 2195.

(iii) What is the largest number that actually increases when Q-zapped?

(iv) Investigate Q-zapping when numbers are written to bases other than 10 (for example 8, 16).

```
10 REM q-zapping program
20 REM
30 REM ensure all variables are integers
40 DEFINT A-Z
50 REM
60 REM constant - upper limit of search
70 MAX = 9999
80 REM
90 REM array to hold q-zap values
100 DIM Q(MAX)
110 REM
120 REM main part of program
130 FOR I=1 TO MAX STEP 1
140  GOSUB 10000          'calculate q-zap of i in j
150  Q(I)=J
160 NEXT I
170 FOR I=1 TO MAX STEP 1
180 PRINT I;
190  GOSUB 20000          'print trail of q-zaps from i till we recognise it
200 NEXT I
210 STOP
220 REM end of main part of program
9990 REM subroutine to calculate q-zap of i in j (don't change i)
10000 J=0
10010 TEMP = I
10025 REM in some basics which don't have mod, the next line may need to
10026 REM be replaced by k=temp/10:k=temp-10*k
10030 K = TEMP MOD 10
10040 J = J + K*K*K
10045 REM the '\' indicates integer division in gwbasic
10050 TEMP = TEMP\10          'having used last digit, get rid of it
10060 IF TEMP > 0 THEN GOTO 10030     'still more digits to process
10070 RETURN
19990 REM subroutine to follow trail of q-zaps till we recognise it
20000 TEMP = I
20010 LOOP = 0
20020 PRINT " >> ";Q(TEMP);
20030 TEMP = Q(TEMP)
20040 IF TEMP = 1 THEN LOOP = 1
20050 IF TEMP = 55 THEN LOOP = 2
20060 IF TEMP = 153 THEN LOOP = 3
20070 IF TEMP = 160 THEN LOOP = 4
20080 IF TEMP = 244 THEN LOOP = 5
20090 IF TEMP = 370 THEN LOOP = 6
20100 IF TEMP = 371 THEN LOOP = 7
20110 IF TEMP = 407 THEN LOOP = 8
20120 IF TEMP = 919 THEN LOOP = 9
20130 IF LOOP = 0 THEN GOTO 20020
20140 PRINT " loop number ";LOOP
20150 RETURN
```

* * * * *

THEOREMS AMONG MURPHY'S LAWS

Ian W. Wright
Murdoch University

In many areas of human activity people jokingly refer to Murphy's Law, describing the situation where if something can go wrong with an enterprise, it will, and at the most inopportune time! The purpose of this article is to demonstrate that Murphy's Law can be regarded as a Theorem in Probability, but that Mrs. Murphy's Law is false.

While there appears to be some consensus as to Murphy's First Law - "If something can go wrong, it will (eventually)", there is no uniformity of usage regarding what we will call Mrs. Murphy's Law - "Murphy was an optimist" or perhaps - "Nothing ever goes right". [Mrs. Murphy had 23 children all with birthdays between 25 September and 1 October.]

We will first show that Murphy's Law is true in simple circumstances. Suppose that an experiment is repeated, over and over again. It could be a very simple experiment like tossing a coin to see if it turns up heads, or a more complicated experiment like manufacturing a motor car, which can either be a good car or a "lemon" which never goes properly. We assume that our experiment has just two possible outcomes which we will call success, "S", or failure, "F". On any one trial of the experiment, we assume that the probability of success is p (a number such that $0 < p < 1$), and the probability of failure is $q = 1 - p$ (also a number between 0 and 1). We further assume that the trials are "independent", so that the success probability of one trial is not influenced by what happened in earlier trials. Such trials are called "Bernoulli" trials.

Although you might hope that you would get a success every time you perform the experiment, Murphy's Law would say that eventually you would get a failure.

It is easy to see that Murphy's Law is true. The probability of getting n successes in a row is

$$\begin{aligned} Pr(S, S, S, \ldots, S) &= p \times p \times \ldots \times p \\ &= p^n, \end{aligned}$$

and so long as we assume $p < 1$, p^n approaches 0 as n gets larger. The complementary event that we would get at least one failure in the n trials has probability

$$1 - p^n \to 1 \text{ as } n \to \infty.$$

Thus eventually we would get at least one failure, with probability 1 (which is about as close to being a certainty as most people would want!)

But suppose now that we have arrived at our first failure. The trials from now on are still going to produce another failure, according to Murphy's Law, eventually. And when we get to that one, there will be another one, at least, eventually, and so on. We can state the strong form of Murphy's Law as a probability theorem:

THEOREM 1. In Bernoulli trials with success probability $p < 1$, an *infinite* number of *failures* will eventually occur, with probability 1.

Theorem 1 is actually a strengthening of Murphy's Law - not only does it show that the experiment is certain to go wrong, it is certain to keep going wrong (which is intuitively reasonable since we have assumed that no "learning" is taking place and that p isn't increasing).

Our next result relates to Mrs. Murphy's Law.

THEOREM 2. In Bernoulli trials with success probabilty p, with $p > 0$, an *infinity* of *successes* will eventually occur, with probability 1.

The proof of this theorem is the same as for Theorem 1, except we interchange "Success" and "Failure" and thus interchange p and $1 - p$.

Theorem 2 shows that Mrs. Murphy's Law is false - bad luck must eventually take a (however brief) holiday!

We now turn to a weaker form of Mrs. Murphy's Law which *will* be true, but it relies on Mrs. Murphy having a rather selective memory! Suppose that she can accurately recall the outcomes of the k most recent trials. But because of the painful consequences which have ensued, she also recalls in vivid detail each of the previous trials that have gone wrong, and has forgotten altogether the other occasions when the trial went smoothly. Suppose that out of the last k trials, ℓ were failures and $k - \ell$ were successes. Suppose also that before the last k trials there were N failures (all of which Mrs. Murphy remembers). Then the proportion of trials that Mrs. Murphy *remembers* as going wrong is

$$q^* = \frac{\text{Number of trials recalled as going wrong}}{\text{Total number of trials recalled}},$$

$$= \frac{N + \ell}{N + k}.$$

Clearly, because $0 \le \ell \le k$,

$$\frac{N}{N + k} \le q^* \le \frac{N + k}{N + k} = 1.$$

But as time goes on, N keeps increasing (as it must from Theorem 1) and so $\frac{N}{N + k} \to 1$. But this means that $q^* \to 1$ too, which justifies

Mrs. Murphy's confidence in her Law! For the trials *she remembers*, the failures seem to be squeezing out all the successes.

Finally we should remark on the application of Murphy's Law to Murphy's Law. With the large number of propositions annually paraded as Murphy's Law, (our version of) Murphy's Law shows that eventually one will be found which is valid. Could ours be it?

* * * * *

FROM OUR NEWS DESK

Shakuntala Devi, a 43-year old woman from Bangladore, India, has an amazing skill with numbers. Devi is a sari-clad diva of numbers, a math prodigy who can calculate as fast and accurately as any hand-held contraption. She is one of those rare people who somehow – even she does not know how – possesses a skill with figures that amazes computer wizards and intrigues academics.

She is in the Guinness Book of World Records for a 1980 feat at a London university multiplying two 13-digit numbers:

7,686,369,774,870 by 2,465,099,745,779.

In 28 seconds. The answer:

18,947,668,177,995,426,462,773,730.

In 1977, she came up with the 23rd root of a 201-digit number in 50 seconds – faster than a powerful Univac computer, although since then, some people have scoffed that the computer would have won if only it had been properly programmed.

From the speedy mental calculation file of math whiz Shakuntala Devi:

28 Seconds

```
      7,686,369,774,870
    × 2,465,099,745,779
---------------------------
18,947,668,177,426,462,773,730
```

10 Seconds

```
Figure the Cube Root of
     2,373,927,704
-----------------------
         1,334
```

40 Seconds

```
              Figure the 7th Root of
455,762,531,836,562,695,930,666,032,734,375
-------------------------------------------
                  46,295
```

* * * * *

MATHEMATICAL POLITICIANS

Michael A.B. Deakin
Monash University

Most readers of FUNCTION would probably find politicians, at least those of today, an innumerate lot - although some make claims to expertise in the mathematically related discipline of Economics.

The usual exception quoted is Eamon De Valera, Ireland's first president. Certainly he was a mathematics teacher and he also was active in the administration of higher learning, being, for instance, made chancellor of the National University of Ireland in 1921 and founding the Dublin Institute of Advanced Studies. To call De Valera a mathematician, however, seems to be stretching matters a little. As far as I can discover, he made no contribution to Mathematics itself, although his teaching and his patronage very probably assisted the development of the subject in Ireland.

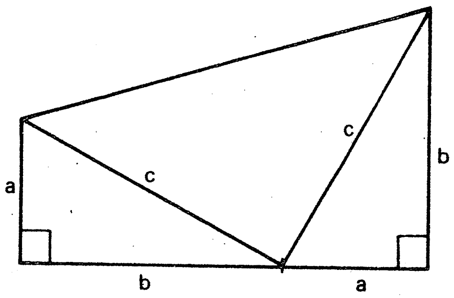
The same can be said of other such figures, of whom the best-known is Daniel Arap Moi of Kenya. But neither he nor De Valera would be remembered for any contribution made to Mathematics. The same is true of Mao Zedong, who is from time to time mentioned in the same connection.

By contrast, Albert Einstein can very definitely be called a mathematician, and he could well have been a politician for he was asked to become Israel's first president. He declined this offer, however, and so cannot really be called a politician, although his influence was sometimes used to political ends - most notably in persuading President Roosevelt to build the A-bomb.

One figure who was very clearly a politician and also made a contribution to Mathematics, albeit an extremely minor one, was James Abram Garfield, the 20th president of the USA. Born in 1831, he grew up in extreme poverty - quite literally in a log cabin. He was a member of a fundamentalist protestant sect, the Disciples of Christ, and taught at their Western Reserve Eclectic School (now Hiram College) while he himself was an undergraduate at Williams College, in the years 1851 - 1854. He later studied law and became active in Politics, was briefly principal of the Eclectic School (1858), and began his life as a representative politician in 1859. This continued, through his election to the presidency in 1880, until his assassination by a psychopath a few months later in 1881.

His mathematical contribution is a proof - quite a nice one - of Pythagoras' Theorem. I read the story in Martin Gardner's column of *Scientific American* (Oct.1964). The proof was published in the *New England Journal of Education* of 1st April, 1876. (One should not read any significance into this date!) The editor stated that Garfield, then a Republican congressman, had given him the proof saying that he'd hit on it while he and some fellow congressmen, of both parties, were engaged (as was their habit! - how times have changed) in mathematical recreations.

The diagram shown is that used by Garfield to provide his proof. Can you give the details?



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TWO VIEWS FROM THE TOP

The reasoning of mathematicians is founded on certain and infallible principles. Every word they use conveys a determinate idea, and by accurate definitions they excite the same ideas in the mind of the reader that were in the mind of the writer. When they have defined the terms they intend to make use of, they premise a few axioms, or self-evident principles, that every one must assent to as soon as proposed. They then take for granted certain postulates, that no one can deny them, such as, that a right line may be drawn from any given point to another, and from these plain, simple principles they have raised most astonishing speculations, and proved the extent of the human mind to be more spacious and capacious than any other science.

John Adams,
second President of the U.S.A.

* * * * *

"It is well ascertained fact ... that mathematicians ... do of all men show the least judgement for the practical purposes of life, and are the most helpless and awkward in common life".

Prince Albert

* * * * *

AS THE ABACUS, SO THE ELECTRONIC CALCULATOR?

Colin Pask
Australian Defence Force Academy

The benefits and disadvantages of using computers and hand-held electronic calculators are widely debated, and I was reminded of some of the arguments when reading the following passage from that wonderful book *Surely You're Joking, Mr Feynman!* (subtitled *Adventures of a curious character*, by Richard P. Feynman, as told to Ralph Leighton, Unwin Paperbacks, 1986). Nobel prize winning physicist Feynman was in a restaurant in Brazil when a Japanese abacus salesman was peddling his wares. They competed (and the Japanese won) on addition and multiplication and then:

We both did a long division problem. It was a tie.

This bothered the hell out of the Japanese man, because he was apparently very well trained on the abacus, and here he was almost beaten by this customer in a restaurant.

"Raios cubicos!" he says, with a vengeance. Cube roots! He wants to do cube roots by arithmetic! It must have been his topnotch exercise in abacus-land.

He writes a number on some paper - any odd number - and I still remember it: 1729.03. He starts working on it, mumbling and grumbling: *"Nmmmmmagmmmmbrrr"* - he's working like a demon! He's poring away, doing this cube root.

Meanwhile I'm just *sitting* there.

One of the waiters says, "What are you doing?"

I point to my head. "Thinking!" I say. I write down 12 on the paper. After a little while I've got 12.002.

The man with the abacus wipes the sweat off his forehead: "Twelve!" he says.

"Oh, no!" I say. "More digits! More digits!" I know that in taking a cube root by arithmetic, each new digit is even more work than the one before. It's a hard job.

He buries himself again, grunting *"Rrrrgrrrrmmmmmm..."* while I add on two more digits. He finally lifts his head to say, "12.0!"

The waiters are all excited and happy. They tell the man, "Look! He does it only by thinking, and you need an abacus! He's got more digits!"

He was completely washed out, and left, humiliated. The waiters congratulated each other.

How did the customer beat the abacus? The number was 1729.03. I happened to know that a cubic foot contains 1728 cubic inches, so the answer is a tiny bit more than 12. The excess, 1.03, is only one part in nearly 2000, and I had learned in calculus that for small fractions, the cube root's excess is one-third of the number's excess. So all I had to do is find the fraction 1/1728, and multiply by 4 (divide by 3 and multiply by 12). So I was able to pull out a whole lot of digits that way.

A few weeks later the man came into the cocktail lounge of the hotel I was staying at. He recognised me and came over. "Tell me," he said, "how were you able to do that cube-root problem so fast?"

I started to explain that it was an approximate method, and had to do with the percentage of error. "Suppose you had given me 28. Now, the cube root of 27 is 3 . . ."

He picks up his abacus: zzzzzzzzzzzzzzz - "Oh yes," he says.

I realized something: he doesn't know numbers. With the abacus, you don't have to memorize a lot of arithmetic combinations; all you have to do is learn how to push the little beads up and down. You don't have to memorize 9 + 7 = 16; you just know that when you add 9 you push a ten's bead up and pull a one's bead down. So we're slower at basic arithmetic, but we know numbers.

Furthermore, the whole idea of an approximate method was beyond him, even though a cube root often cannot be computed exactly by any method. So I never could teach him how I did cube roots or explain how lucky I was that he happened to choose 1729.03.

The whole chapter is called *Lucky Numbers*. It makes one think about number skills - and thinking!

Acknowledgement : FUNCTION very happily thanks Dr Graeme Cohen, Editor of the Australian Mathematical Society Gazette, for his friendly permission to reproduce Professor Pask's article.

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The mathematician Johann Heinrich Lambert (1728–1777) had an interview with King Frederic II of Prussia. King: "Good evening, Monsieur! Give me the pleasure of telling me which sciences in particular you have learned." Lambert: "All of them." King: "So you are also a skilled mathematician?" Lambert: "Yes." King: "And who was the professor who taught you mathematics?" Lambert: "I myself." King: "In which case you are a second Pascal?" Lambert: "Yes, Your Majesty!"

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PROBLEM SECTION

12.2.1 THE PROOF IS IN THE PUDDING

The host at a party turned to a guest and said, "I have three daughters and I will tell you how old they are. The product of their ages is 72. The sum of their ages is my house number. How old is each?"

The guest rushed to the door, looked at the house number, and informed the host that he needed more information.

The host then added, "The oldest one likes strawberry pudding".

The guest then announced the ages of the three girls.

What are the ages of the three daughters? (All ages are integers)

12.2.2 GROWING OLD TOGETHER

A ship is twice as old as its boiler was when the ship was as old as the boiler is now.

The sum of their ages is forty-nine years.

How old is the ship and how old is the boiler?

12.2.3 TALKATIVE EVE

$$\frac{EVE}{DID} = .TALKTALKTALK\ldots$$

Each letter stands for a different digit, zero included, and the fraction EVE/DID has been reduced to its lowest terms. The solution is unique. What is it?

(Hint: .TALKTALK... = TALK/9999).

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PERDIX

On the 2nd and 3rd of March the 1988 Australian Mathematical Olympiad (AMO) was conducted in all six Australian states and the Australian Capital Territory. Sixty-four selected competitors engaged in the Olympiad.

Most had been selected because of their achievements in the Australian Mathematics Competition. Moreover, some had demonstrated their skill in other contests, and a few had been accepted with excellent references from their mathematics teachers.

The 1987 Inter-State Finals, where Seniors (mostly students in year 12) and Juniors (students in year 11, 10, 9, ...) competed separately, constituted step 1 of the 1988 AMO. Leading towards the ultimate national competition, step 2 was the AMO Correspondence Programme, a course which, for Seniors, comprised 24 problems, two per week, that had to be tackled; each solution was checked, commented upon in detail and returned, together with "official" solutions, by professional mathematicians.

Participation and success in the Australian Olympiad were documented:

6 gold certificates,
12 silver certificates
12 bronze certificates and
34 participation certificates

were given out.

Do you wish to find out how you would have performed in this Olympiad? Look at the end of my article and you will find the original AMO papers reproduced: send me any of your solutions, and let me know any queries you have about the questions.

Held in Canberra this time, the International Mathematical Olympiad will again see an Australian team that is assembled from the best six performers at the Australian Mathematical Olympiad. The team in alphabetical order is:

Geoffrey Bailey	–	St. Aloysius College, New South Wales,
Martin Bush	–	Brisbane State High School, Queensland,
David Jackson	–	Sydney Church of England Grammar School, New South Wales,
Jeremy Liew	–	Duncraig Senior High School, Western Australia,
Martin O'Hely	–	Salesian College, Victoria,
Terence Tao	–	Blackwood High School, South Australia.

Nominated as reserve member of the team was:

Blair Trewin	–	Canberra Grammar School, Australian Capital Territory.

Intensive training will continue for the team members. As in past years, the IBM training school at Sydney, lasting a whole week, will be a vital preparation for the International Mathematical Olympiad on the one hand, on the other it looks further ahead and offers talented Juniors the opportunity to handle the most demanding problems of their kind, supervised by experienced staff: these younger students could be amongst next year's team members.

Amongst those selected to join the team at the school in May are:

James Allnutt	–	Canberra Grammar School, Australian Capital Territory,
Michael Camarri	–	All Saints' College, Western Australia,
Danny Calegari	–	Melbourne Church of England Grammar School, Victoria,
Kevin Davey	–	St. Kevin's College, Victoria,
Andrew Donald	–	James Ruse Agricultural High School, New South Wales,
Christopher Eckett	–	Brisbane Grammar School, Queensland,
Matthew Emerton	–	Glen Waverley High School, Victoria,
Mark Kisin	–	Melbourne Church of England Grammar School, Victoria,
Clement Tien-Hui Loy	–	Sydney Church of England Grammar School, New South Wales,
Brian J. Weatherson	–	Mazenod College, Victoria.

Congratulations to all of the above!

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THE 1988
AUSTRALIAN MATHEMATICAL OLYMPIAD

PAPER 1

Tuesday, 1st March, 1988

Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

Question 1

A function f satisfies the following conditions:

(i) For each rational number x, $f(x)$ is a real number;
(ii) $f(1988) \neq f(1987)$; and
(iii) $f(x+y) = f(x)f(y) - f(xy) + 1$ holds for all rational numbers x and y.

Show that $f(-1987/1988) = 1/1988$.

Question 2

The triangles ABG and AEF are in the same plane. Between them the following conditions hold:

(i) the midpoint of AB is E;
(ii) the points A, G and F are on the same line;
(iii) there is a point C at which BG and EF intersect; and
(iv) $CE = 1$ and $AC = AE = FG$.

Show that if $AG = x$ then $AB = x^3$.

Question 3

Let $e_1, e_2, \ldots, e_r$ be non-negative integers such that $e_1 > e_2 > \cdots > e_r$.

Put $n = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_r}$.

Show that $n!/2^{n-r}$ is an odd integer, where $n! = n(n-1)\ldots 2.1$.

THE 1988
AUSTRALIAN MATHEMATICAL OLYMPIAD

PAPER II

Wednesday, 2nd March, 1988

Time allowed: 4 hours
No calculators are to be used
Each question is to be worth seven points

Question 4

A city has a system of bus routes laid out such that

(i) there are exactly 11 bus stops on each route;
(ii) it is possible to travel between any two bus stops without changing routes; and
(iii) any two bus routes have exactly one bus stop in common.

What is the number of bus routes in the city?

Question 5

In an ancient court of law, 23 seats were arranged for up to 23 judges in a single row.

During a long court session, some judges might leave and others might come in. Judges entering the court room came either alone or in pairs and if one judge of a pair had to leave the room, his partner left with him, whereupon that pairing ceased.

The Court Servant was ordered by the Father of the Court to direct the judges when they entered to their seats in the row, and to do this so that whenever a pair had come in together, they sat next to each other.

Show that the Court Servant could carry out this task, provided that there were never more than 16 judges present at any time of the proceedings.

Question 6

Let $x_1, \ldots, x_n$ be n integers and let p be a positive integer with $p < n$.

Put

$$\begin{aligned} S_1 &= x_1 + x_2 + \cdots + x_p, \\ T_1 &= x_{p+1} + x_{p+2} + \cdots + x_n, \\ S_2 &= x_2 + x_3 + \cdots + x_p + x_{p+1}, \\ &\vdots \\ S_n &= x_n + x_1 + \cdots + x_{p-1}, \\ T_n &= x_p + x_{p+1} + \cdots + x_{n-1}. \end{aligned}$$

For $a = 0, 1, 2$ and 3 and $b = 0, 1, 2$ and 3 let $m(a, b)$ be the number of numbers i, $1 \le i \le n$, such that S_i leaves remainder a on division by 4 and T_i leaves remainder b on division by 4.

Show that $m(1,3)$ and $m(3,1)$ leave the same remainder when divided by 4 if and only if $m(2,2)$ is even.

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Geometry is a true natural science : – only more simple, and therefore more perfect than any other. We must not suppose that, because it admits the application of mathematical analysis, it is therefore a purely logical science, independent of observation. Every body studied by geometers presents some primitive phenomena which, not being discoverable by reasoning, must be due to observation alone.

A. Comte, in
Positive Philosophy.

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