FUNCTION

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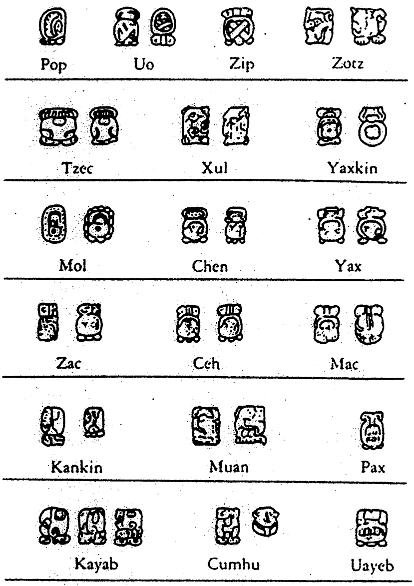

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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In this FUNCTION issue things are done in more than one way. From J.C. Burns we learn that there is more than one procedure of solving a problem and there are "best" and "right" solutions: all we need is a "better" method. Drink problems certainly have more than one formulation, and they can have surprising solutions, as R.B. Potts convinces us. And M. Sved tells us of a geometry which permits the drawing, through each point, of more than one parallel to a given line. It is known that Fibonacci sequences have more than one application, but how explosive some are is intimated by Michael A.B. Deakin.

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THE FRONT COVER

Hans Lausch
Monash University

The 19 sets of symbols originate from a Central American calendar. Before Mexico and Guatemala were conquered by the Spaniards in the sixteenth century, the calendar was part of a civilization known as Maya.

A considerable amount of Maya literature is no longer extant as it was burnt by Spanish monks for political and religious reasons. However, there were scholarly exceptions amongst them, permitting scribes to save at least a few ancient Maya writings by translating them into Spanish, which resulted in the Books of Chilam Balam, now an important source of knowledge that would otherwise have been lost.

Very intriguing are the Maya calendars – there were several in use side by side. The "civil" year consisted of 18 months, each having twenty days plus a fragment, the "Uayeb", considered an extremely unlucky five-day interval, and these 19 periods are symbolized on the front cover.

In cosmic dimensions was time viewed by the Maya. A period of 20 years was called a *katun*, 20 katuns a *baktun*, 20 baktuns a *pictun*, and so their system based on 20 continued until it reached the *hablatun*, an incredible 460,800,000,000 days!

Hand in hand with Maya chronometry went the Maya number notation. One dot stood for the unit, one horizontal bar meant 5, and then there were characters representing 0 and 20. Systems based on 20 are not all that exotic as they might appear at first sight. In Western Europe the ancient Celts lived with one, and its traces are discernible in the French word for eighty, "quatre-vingt" (= four twenties), and in English, when we refer to scores.

With their "civil" calendar the Maya entwined a "religious" calendar in which each day of a "civil" month bore a name and the "religious" cycle numbered 13 days. Since the numbers 20 and 13 are relatively prime, combining a 20-day "civil" cycle with a 13-day "religious" cycle creates a bigger cycle of $13 \times 20 = 260$ days.

This was recognized by the Maya. While our dates have the format DAY NUMBER + MONTH NAME, e.g. 1 April, theirs followed the pattern "RELIGIOUS" DAY NUMBER + "CIVIL" DAY NAME. E.g., the fourteenth day of each "civil" month was named *Ix* and the fifteenth *Men*, and consequently 10 *Ix* was followed by 11 *Men* (not by 11 *Ix* !).

The Maya "religious" calendar, it seems, related to the moon. For 405 new moons to occur they thought that 46 periods, each having 260 days, i.e. 11960 days altogether, had to pass – the modern figure is 11959.888 days.

Relying on the celestial motions of the planet Venus was a third Maya calendar. Venus was referred to by the Maya as *Lahun Chan*, who was a male god with the head of a jaguar, rabbit teeth, and the withered body of a dog.

CONSTRUCTION OF AN EQUILATERAL TRIANGLE

J.C. Burns
Australian Defence Force Academy

It is often possible to solve problems in Euclidean geometry by a variety of methods, and one of the fascinations of the subject is the challenge to find the method which is in some sense the "best" one. This may be, for example, because the solution fits the problem just as, in a less happy context, the punishment should fit the crime or, if one is fortunate, because the solution reveals what can be seen, at least after the event, as the true essence of the problem. Of course we can hardly start looking for the "best" solution until we have actually solved the problem and students generally find this difficult enough. Nevertheless, they should be encouraged to look critically at their first solutions with a view at least to uncovering any errors and improving the presentation and, more fundamentally, to finding other and "better" methods of proof. This search for the "right" solution, the one which ensures that the problem takes its proper place in the broader context within which is it posed, is an essential feature of mathematical research. A problem given by Vern Treilibs in *The Australian Mathematics Teacher* (April 1986) provides an opportunity to illustrate how students can be given experience of this process even in a reasonably simple problem in Euclidean geometry, arriving eventually at a solution which, although not obvious initially, can be seen to relate naturally to the problem.

THE PROBLEM

From an arbitrary point $A(x,y)$ in the Euclidean plane as one vertex, we are asked to draw an equilateral triangle ABC with the vertex B on the x-axis and the vertex C on the y-axis.

SOLUTION 1

We may assume that the triangle has unknown side length ℓ and that $x > 0$, $y > 0$. There are two cases to consider according as the points A, B, C have positive or negative orientation (Figure 1). Let the angle XBA be θ and draw AX′ parallel to 0X. Then angle X′AC is $\theta + \alpha$ where

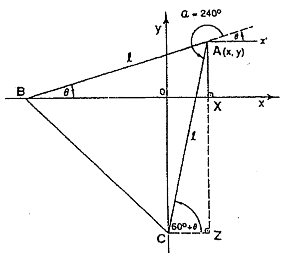

Figure 1a

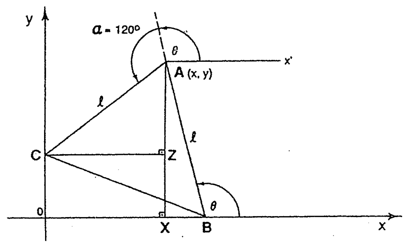

Figure 1b

$\alpha = 240°$ or 120^0 in the two cases.

In the case where $\alpha = 240°$ (Figure 1a),

$$y = \ell \sin\theta \quad (\Delta AXB),$$

$$x = \ell \cos(60° + \theta) \quad (\Delta AZC).$$

Eliminating ℓ between these two equations for ℓ and θ gives

$$y \cos(60° + \theta) = x \sin\theta,$$

i.e. $y(\cos 60° \cos\theta - \sin 60° \sin\theta) = x \sin\theta,$

i.e. $y\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) = x \sin\theta,$

i.e. $y \cos\theta = (2x + \sqrt{3}y)\sin\theta,$

i.e. $\cot\theta = (2x + \sqrt{3}y)/y.$

In the case where $\alpha = 120°$ (Figure 1b),

$$y = \ell \sin(180° - \theta) \quad (\Delta AXB)$$

$$= \ell \sin\theta,$$

$$x = \ell \cos(\theta - 60°) \quad (\Delta AZC).$$

Proceeding in a similar manner as above, we obtain

$$\cot\theta = (2x - \sqrt{3}y)/y.$$

Since x and y are given, it follows that the angles θ which determine the required positively and negatively oriented equilateral triangles can be calculated. (It may be noted that for the positively oriented triangle θ is always acute, while for the negatively oriented triangle θ is acute, a right angle or obtuse according as $2x >, =, < \sqrt{3}y$.) Moreover, if A is given simply as a point in the plane (rather than by a pair of numerical coordinates), the lines BA included at angle θ to the x-axis can be constructed by ruler and compass and so, then, can the required triangles.

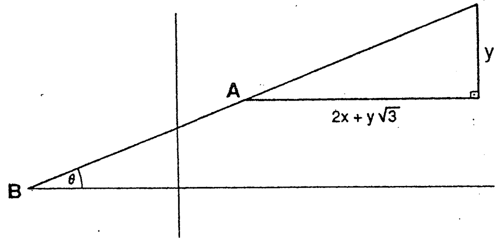

Figure 2

SOLUTION 2

Now that we have a solution we can seek out its implications and perhaps arrive at a new and improved method. A first suggestion is that construction would be simpler if we could obtain the points B and C directly. It is a simple exercise in coordinate geometry to show that the coordinates of B are $(-x \mp \sqrt{3}y,\ 0)$ and of C are $(0, -y \mp \sqrt{3}x)$ in the two cases. The construction of B and C is just as simple as that for the line BA suggested above and, as it leads directly to the triangle ABC, is to be preferred.

Before proceeding, we observe that now that the coordinates of B and C are available, we can calculate the lengths of the three sides of the triangle ABC in each case and so verify that it is indeed equilateral.

How would we construct B and C? When we plot the two positions of B, for example, as in Figure 3, we see that they are the vertices of two

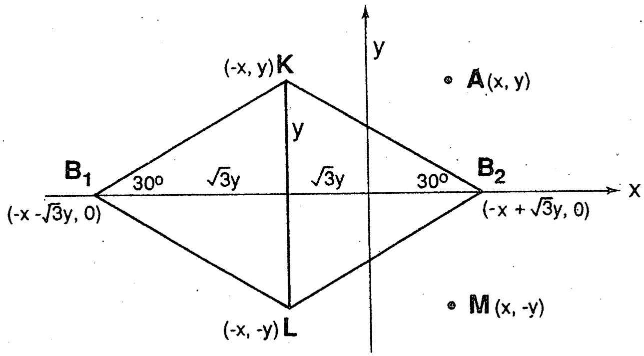

Figure 3

equilateral triangles which have as common base the line segment KL where K, L are the images of A in the y-axis and the origin respectively. Similarly, we find that the two positions of C $(0,\ -y \mp \sqrt{3}x)$ and the vertices of two equilateral triangles on the base LM where M is the image of A in the x-axis.

We can now see that the solution of our problem involves only the construction of four equilateral triangles on given bases. Not only is this very easily carried out by ruler and compass, but it seems especially appropriate that it would be by the construction of these equilateral triangles that we obtain the vertices of our required equilateral triangles.

The procedure is now simply stated. First the images of A in the axes and the origin, the points K, L, M defined above, are constructed. Then equilateral triangles LKB_1, MLC_1 are constructed with positive orientation (and lying wholly outside the rectangle AKLM) to give the vertices B_1, C_1 of the positively oriented equilateral triangle AB_1C_1; and the equilateral triangles LKB_2, MLC_2 are constructed with negative orientation (to overlap the rectangle) to give vertices B_2, C_2 of the negatively oriented equilateral triangle AB_2C_2. (See Figure 4.)

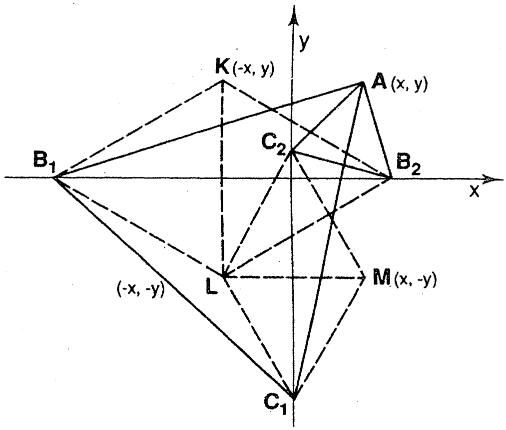

Figure 4

SOLUTION 3

This may well be considered to be a very satisfactory end to the investigation, but to anyone with some familiarity with the use of complex numbers in elementary geometry, the appearance of equilateral triangles both in the statement and in the solution of the problem is likely to bring to mind the simple necessary and sufficient condition that three points represented by the complex numbers a, b, c form an equilateral triangle, namely $a + \omega b + \omega^2 c = 0$ for positive orientation or $a + \omega^2 b + \omega c = 0$ for negative orientation ($\omega = \text{cis } 2\pi/3$ so that $\omega^3 = 1$).

In our problem, we are given a and wish to find b and c so that either

$$(1)\quad a + \omega b + \omega^2 c = 0 \quad\text{or}\quad (2)\quad a + \omega^2 b + \omega c = 0.$$

In addition, for B to lie on the real axis, we must have $b = \bar{b}$ and for C to lie on the imaginary axis we need $c = -\bar{c}$; and we note that $\bar{\omega} = \omega^2$. If we now take the complex conjugates of the conditions that the triangle be equilateral and use these extra conditions, we obtain the equations

$$(3)\quad \bar{a} + \omega^2 b - \omega c = 0 \quad\text{or}\quad (4)\quad \bar{a} + \omega b - \omega^2 c = 0.$$

We solve (1) and (3), and in the other case, (2) and (4) for b and c in terms of a and $\bar{a}$ and, making use of the relation $1 + \omega + \omega^2 = 0$, express the results in the form

$$-a + \omega(-\bar{a}) + \omega^2 b = 0 \quad\text{and}\quad \bar{a} + \omega(-a) + \omega^2 c = 0$$

for the positively oriented triangle; and

$$-a + \omega^2(-\bar{a}) + \omega b = 0 \quad \text{and} \quad \bar{a} + \omega^2(-a) + \omega c = 0$$

for the negatively oriented triangle.

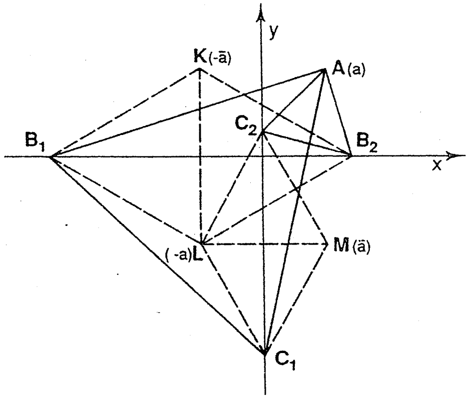

Figure 5

These equations connecting b and c with a and ā show at once that, as before, the triangles LKB_1 and MLC_1 are positively oriented equilateral triangles and that the triangles LKB_2 and MLC_2 are negatively oriented equilateral triangles. (Figure 5.)

While judgement of these matters is of course subjective, it may be that this complex variable solution will appeal to others as it does to me as the "right" one for this problem.

* * * * *

There is no royal road to geometry.

Menaechmus
(to Alexander The Great)

TO THE POINT

Not the lacking width but the lacking length in a point must be blamed for our inability to say that a line is composed of points.

A.L.F. Meister,
professor of Mathematics at Göttingen,
when critically reviewing a mathematics text in 1775

QUENCH YOUR MATHEMATICAL THIRST

R.B. Potts[1]

University of Adelaide

Not everyone likes coke. Not everyone likes coffee. Not everyone likes mathematics.

Those who do like mathematics like solving mathematical problems. Here's an old problem, from the very first first-year mathematics examination paper set by the first Professor of Mathematics in the University of Adelaide's first year, 1876.

> "A man sets apart £28 a year to be spent in drink, and considers that he requires in the year a quantity of alcohol amounting to 24 (reputed) quarts. He prefers claret to ale, but claret costs 40s. a dozen, ale only 12s. a dozen. The percentage of alcohol in the claret being 10, and in the ale 6, how much does he buy of each? If the price of ale rises, will he drink more ale, or less, than before?"

To translate this problem into 1988 language, it needs to be de-sexed, the drug perhaps changed, and delightful units such as reputed quarts replaced by mundane metric litres:

Problem

A woman/man sets aside $12 a year to be spent on coffee and coke and considers she/he requires from these drinks at least 0.28 litres of caffein a year. She/he prefers coffee to coke but coffee costs $1 a litre, coke only $0.3 a litre. Assuming that the percentage of caffein in the coffee is 2%, and in the coke 1%, how much does she/he buy of each? If the price of coke rises, will she/he drink more coke or less than before?

The numbers in the problem are not meant to be realistic but are meant to make the subsequent arithmetic easy.

I hope it increases your thirst for mathematical knowledge when I let you know that the answer to the last question is: she/he drinks *more* coke! Surprise!

Mathematical Formulation

To proceed, we need yet another version of the problem – a mathematical formulation – and I guess that those who do not like mathematics will stop right here.

Let x be the number of litres of coffee bought and y the number of litres of coke bought. Then

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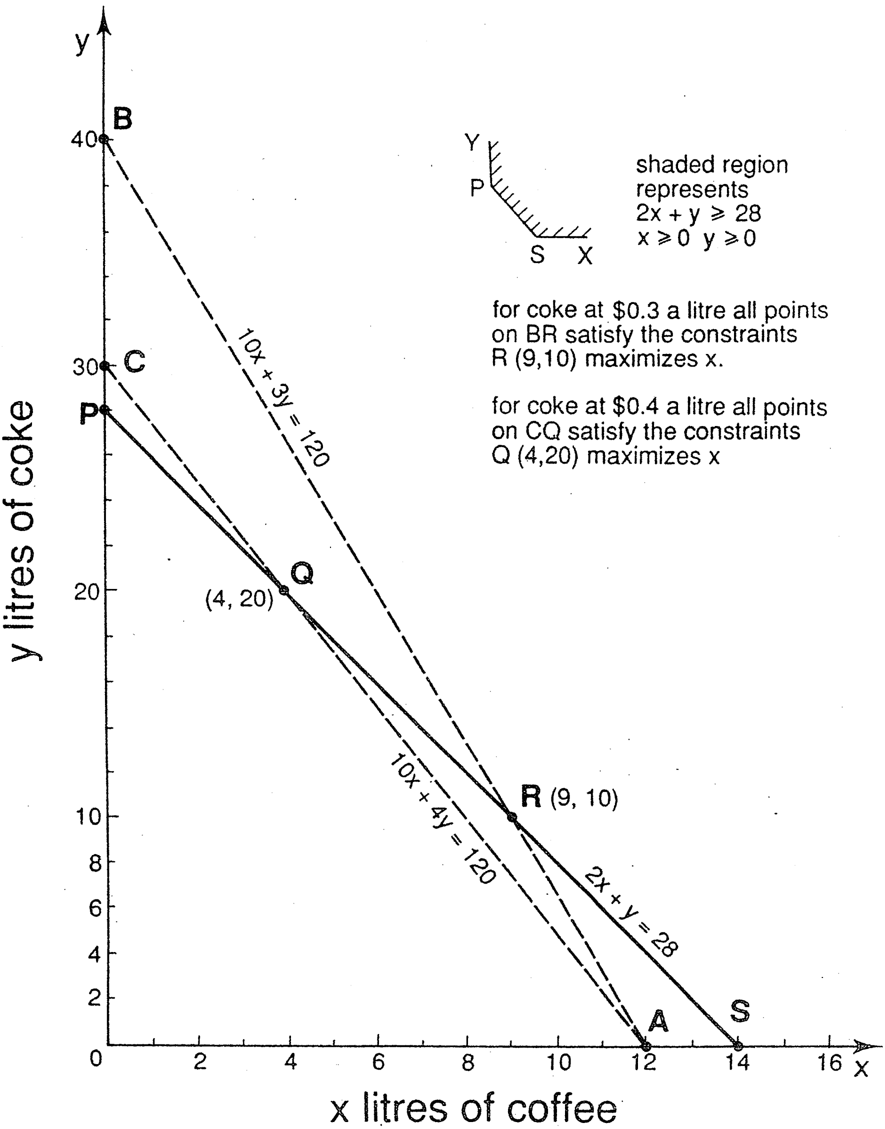
[1] The article appeared in a magazine produced by Westminster School, Adelaide.


y
B
40
Y
P
S X
shaded region
represents
2x + y ≥ 28
x ≥ 0 y ≥ 0
for coke at $0.3 a litre all points
on BR satisfy the constraints
R (9,10) maximizes x.
for coke at $0.4 a litre all points
on CQ satisfy the constraints
Q (4,20) maximizes x
10x + 3y = 120
30
C
P
y litres of coke
Q
(4, 20)
20
R (9, 10)
10x + 4y = 120
10
8
6
4
2
2x + y = 28
A
S
0
2
4
6
8
10
12
14
16
x
x litres of coffee

$$x + 0.3y = 12$$

$$0.02x + 0.01y \geq 0.28$$

and, as she/he prefers coffee, x has to be maximized. We can express this more neatly and, in modern mathematical jargon, as an *optimization problem* as follows:

maximize x subject to the constraints

$$10x + 3y = 120$$

$$2x + y \geq 28$$

$$x \geq 0 \quad y \geq 0.$$

The last inequalities are trivial in a sense (you can't buy -3 litres of coke) but have to be included for completeness.

Solution

A graphical solution should give you the answers:

$x = 9$ = number of litres of coffee bought

$y = 10$ = number of litres of coke bought.

If the price of coke rises (say, to $0.4 a litre), then

$x = 4$ and $y = 20$.

She/he drinks twice as much coke as before! What happens if the price of coke rises to $0.5 a litre?

Let's drink to mathematics!

* * * * *

PROSIT!

At any time must one be able to replace the words "points", "lines", "planes" by the words "tables", "chairs", "beer glasses".

David Hilbert (1862-1943),
mathematician

NON-EUCLIDEAN GEOMETRY

M. Sved
University of Adelaide

If you have the chance to look at texts of geometry used by older generations of students (your grandparents or your great-grandparents) you will find in them a collection of definitions, postulates, propositions and theorems meticulously arranged and numbered. Old English texts followed faithfully the exposition in the "*Elements*" of Euclid. The work of Euclid (about 300 B.C.) is the summary of some three-hundred years of work of Greek scholars (Thales, Pythagoras, Eudoxus and their schools). Geometry, as a practical skill, was pursued by other civilizations preceding the Greeks, but the Greeks were the first who strove to free the art of geometry from trial and error methods and results and turn it into a deductive science: a structure built on a few fundamental "obvious" postulates (or axioms, as we prefer to call them today) and theorems which can be proved, i.e. deduced logically from the axioms. When we talk today about "elementary geometry" we still refer to Euclidean geometry. Engineering, architecture, surveying (confined to areas small enough to neglect the curvature of the earth) still deal with Euclidean planes when talking about the horizontal plane, vertical planes, etc. At the turn of the century and through the earlier years of this century the foundations of this geometry have been somewhat modified by the work of Hilbert (1862-1943) and others. We do not insist any more on defining the basic terms: point, line, incidence, betweenness and congruence. Euclid defines the term point and (straight) line, but these definitions involve terms which are not simpler than those which he tends to define. There were also some small gaps which were filled in: some axioms, not explicitly stated by Euclid, were added, e.g. the idea of "betweenness" is somewhat neglected in the classical, Euclidean geometry. With these minor alterations Euclidean geometry still stands as a theoretical structure in which no contradictions have been found through more than 2000 years.

However, one of the postulates of Euclid did cause some controversy throughout the centuries, namely, the "fifth postulate" or what we call sometimes the "parallel postulate".

Euclid states, in the fifth postulate, that if a transversal line t forms angles α and β with the lines ℓ and m (see Fig. 1), such that the sum of α and β is less than 180°, then the lines ℓ and m intersect on that side of t on which the angles α and β lie. Mathematicians felt somewhat uneasy about classing this statement as a postulate. They felt that it is rather a theorem which should be deduced from the other postulates. In fact, the *converse* of this postulate can be deduced from the others as we are going to show now.

The first postulate of Euclid states that through two distinct points there exists one and only one line. From this it follows at once that two lines cannot intersect in *more* that one point. We are going to show that there are lines which do not intersect in any point. The converse of the fifth postulate can be stated now. In Fig. 2 we have the situation that

$$\alpha + \beta = 180^\circ.$$

We will prove that ℓ and m cannot intersect.

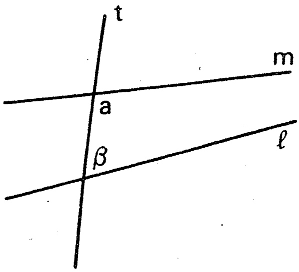


Fig. 1

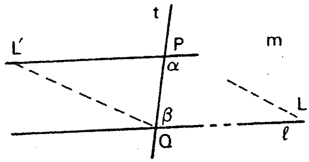


Fig. 2

Proof: Suppose that ℓ and m intersect in L. Let L′ be on line m, as shown such that the segments QL and PL′ are equal. Since $\alpha + \beta = 180°$, the angle measure of L′PQ equals that of LQP, namely β. Hence the triangles LQP and L′PQ are congruent (SAS) and so the angles LPQ and L′QP are of equal size. Since by assumption L lies also on m, LPQ measures α and so does LPQ = α, hence L′ lies on ℓ. This, however, means that ℓ and m intersect in two distinct points, L and L′, which contradicts the first postulate. We conclude that ℓ and m cannot intersect.

Thus we can deduce from the other postulates of Euclid that if P is a point not on line ℓ, we can draw *at least one* line parallel (non-intersecting) to ℓ, (namely by choosing the angle α so that $\alpha + \beta = 180°$). If we accept the fifth postulate of Euclid, it follows that

through a point P *not on the line* ℓ *there is one and only one line parallel to* ℓ.

Fig. 3 brings home to you a theorem with which you are familiar and which can be regarded as the equivalent of the parallel postulate:

In every triangle of the plane the angle sum is $180°$.

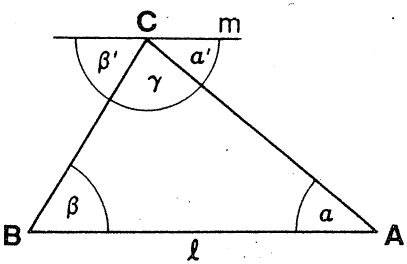


Fig. 3

Indeed, this means that if one day someone ascertained by indisputable accurate measurement that the angle sum in some triangle is different from 180°, then the parallel postulate could not hold. Suppose that the triangle ABC of Fig. 3 is our offensive triangle. We know (by the converse of the parallel postulate which was proved above) that if we construct the line m through the vertex C so that the angles α' and α are equal, then m is parallel to ℓ. However, since $\alpha + \beta + \gamma \neq 180°$ by assumption, the angles β and β' are not equal. Thus we can construct through C at least two lines parallel to ℓ ; m and the line which makes an angle β with BC.

Obviously we are not in the position of measuring accurately the angle sum of every triangle. This can make you understand why mathematicians looked for a proof of the parallel postulate. Many attempts were made, many "proofs" appeared (see [4]), all of them false, but until the early years of the 19th century, the problem remained unresolved. The "prince of mathematicians", Carl Friedrich Gauss (1777-1855), was deeply interested in the problem from the days of his youth, and his correspondence with various mathematician friends indicates clearly that he arrived at the same findings as the founders of the new geometry, yet he did not publish his results, finding them too revolutionary, and hard to be accepted by his contemporaries.

The discovery of hyperbolic geometry was left to the Russian Nicolai Ivanovich Lobachevsky (1792-1856) and the Hungarian János Bólyai (1802-1860). Their results (independent from each other) were published in 1829 and 1831 respectively. The important departure taken by Lobachevsky, Bólyai (and Gauss) was that they did not try any more to deduce the parallel postulate from the other postulates. Nor did they follow the path of some previous worthwhile attempts (notably Girolamo Saccheri (1667-1733)) in trying to prove that it is impossible to draw more than one parallel line through a point to a given line. Instead, they assumed the validity of all the postulates of Euclid, save the fifth, which they substituted with the *hyperbolic postulate:*

> *If ℓ is a line and P a point not on the line, then there are at least two lines through P which are parallel to ℓ.*

What resulted was a new geometry to which Bólyai referred jubilantly: "out of nothing I have created a strange new universe".

Before we discuss some features of this "strange new universe" (not so new, 150 years later, after the work of other mathematicians and physicists of the 19th and 20th centuries, the extension and generalization of the ideas, and the proliferation of geometrical systems), we should say a few words to make you more ready to accept such an alternative geometry. Even admitting that the geometry of the hyperbolic plane is just as perfect logically as that of the Euclidean plane, we all have a natural resistance to it; we tend to deny its "physical truth". So at this point we should analyse our feelings. We favour Euclidean geometry, "because it is simpler, because it is ingrained in our way of thinking, because it agrees with our experience". We can answer all these arguments. It is "simpler" to think of the sun rising in the East and setting in the West than to think of the earth revolving in an elliptical orbit around the sun, yet we have come to prefer the latter description. Throughout the years the individual life and throughout the collective life of hundreds of generations the concepts of Euclidean geometry, such as the simple properties of rectangles and circles, or the angle sum of triangles have been with us, so it is no wonder that they are "naturally ingrained in our way of thinking". As for agreement with our experience, we can answer that Euclidean geometry has certainly sprung out of experience. We can measure with our protractors the angles of a triangle and find that the sum is 180° or "something very near". We have to make special arrangements to ensure the precision of the measurements. However, when we deal with very large triangles, we have to reappraise the object of our measurement. The word geometry means "measuring the earth". A physical horizontal plane on the earth exists as a good approximation to the surface of a lake, if the weather is still and the lake is not too large. We know that the surface of the sea, however calm, is not a horizontal plane. In fact we know that the thing which we think of as a straight line does not exist at all on the surface of the earth. A North-South line is a meridian – a great circle going through the poles. It can be shown that the shortest distance joining two points on the surface of the earth, along that surface, is the arc between the two points of the great circle joining the two points, i.e. the arc of the circle m in which a plane containing the points and the centre of the earth cuts the surface. Such a line is called the *geodesic line* joining the two points.

The sides of a triangle on the surface of the earth are segments of great circles. We call such a triangle a spherical triangle. Its angles can be measured locally and can be taken to coincide with the angles between the tangents drawn to the circles at the vertices. Fig. 4 sketches the spherical triangle formed by the Equator, the meridian through Greenwich (called the 0° meridian) and the 90°E meridian, the vertices being G, N and P. (G is not Greenwich!) It can be seen easily that each of its angles is 90°. So we have a spherical triangle with the angle sum of 270° and which has in fact three right angles (even two are denied to a Euclidean plane triangle). It can be shown that the angle sum in a spherical triangle is always more than 180°.

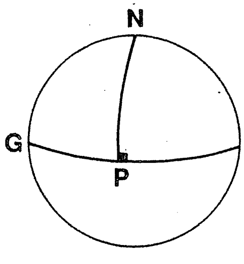


Fig. 4

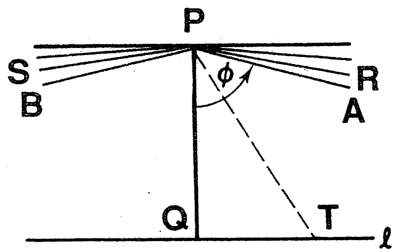


Fig. 5

We must point out hurriedly that hyperbolic geometry is different from spherical geometry. One important difference is that in spherical geometry two "lines", i.e. geodesics, meet in two points instead of one (e.g. two meridians meet in the N and S poles). In hyperbolic geometry, however, the first postulate of Euclid is valid and so two lines intersect in at most one point.

We are now ready to list a few interesting consequences of the hyperbolic postulate. From it follows immediately that, in a hyperbolic plane, through a point P not on a line ℓ there are an *infinite* number of lines parallel to ℓ. The situation is illustrated in Fig. 5. The rays PA and PB shown play a distinguished part. They are called limiting parallel or bounding parallel rays, which are symmetrical about the perpendicular PQ from P to ℓ and are inclined at an angle ϕ to it, which is named the angle of parallelism. Each ray enclosing an angle less than ϕ with PQ intersects ℓ (e.g. the line PT) while rays enclosing non-obtuse angles greater than ϕ on either side of PQ (e.g. PR or PS) are parallel to ℓ. (Some texts call the limiting parallel rays PA and PB parallel, while all the other parallel lines are classified as "hyperparallels".)

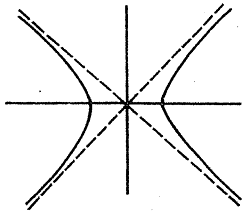


Fig. 6

It was found that ϕ, the angle of parallelism, depends on the distance PQ and it decreases from 90° to 0° as the distance PQ increases from 0 to infinity. We may see from this that when PQ is small, the rays PA, PB are very nearly perpendicular to PQ, hence very nearly coincide with the Euclidean parallel through P. This means that in "small" regions hyperbolic geometry is not distinguishable from Euclidean geometry. You are justified to ask the question: what is "small"? Remember here that on the earth we can take a spherical triangle to be a Euclidean plane triangle if the region is small in comparison to the total surface of the earth. For any sphere, the surface area is $4\pi R^2$, where R is its radius, so "small" means small compared with $4\pi R^2$. There is also a natural way of defining "small" in hyperbolic geometry. As there are many different spherical geometries – spheres with different radii – so there are also many different hyperbolic geometries or hyperbolic planes. For given a line ℓ on a hyperbolic plane and a point Q on the line, we can erect a perpendicular at Q. There will be some point P on this perpendicular for which the angle of parallelism is 45°. Different hyperbolic planes through the line ℓ will have different values for the length of PQ. If we fix our attention on a particular plane, a "small" region near Q, say, will mean a region whose size or diameter is small compared with PQ, described above. You may raise another objection: "PA and PB approach the line ℓ, so they must surely intersect it". However, we assume that you are already familiar with the asymptotes of a hyperbola which approach the curve, but do not intersect it (Fig. 6).

We may proceed now to list a few more properties of this strange geometry. You have to part with the notion of squares and rectangles. They do not exist in hyperbolic geometry. The distance between two parallel lines is not constant. The case of hyperparallelity is illustrated in Fig. 7. There are at most two points on m which are equidistant from ℓ, and there is exactly one line which is perpendicular to both ℓ and m and determines the shortest distance between them. In the case of limiting parallelness, m asymptotically approaches ℓ (Fig. 8).

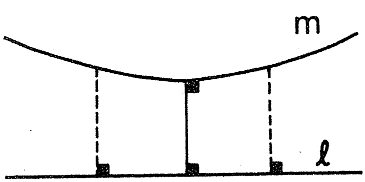


Fig. 7

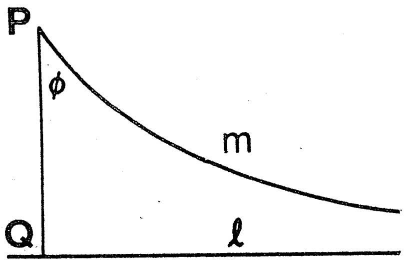
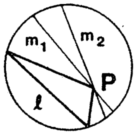


Fig. 8

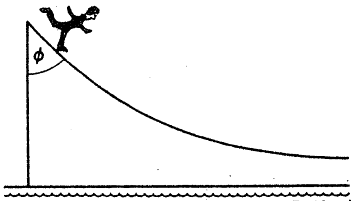
In hyperbolic geometry you lose the concept of "similarity". In fact, in this geometry if two triangles are equiangular, then the triangles are congruent, i.e. their sides are equal in length. Here we must remember *that the angle sum of a hyperbolic triangle is always less than 180°*. The difference between 180° and the actual sum is called the "defect". An interesting connection exists between this angular defect and the area of a triangle. It can be shown that the area is proportional to the defect. It follows from this that since the defect is always finite, however large the sides become, the area remains bounded (!).

These few startling results should make you appreciate the boldness of the discovery of this geometry. Gauss was reluctant to publish his findings and neither Lobachevsky nor Bólyai received any recognition during their lifetimes. Mathematicians with minds sufficiently open not to dismiss the theory, asked the relevant question: admitting that no logical contradictions are apparent in hyperbolic geometry, is not there a possibility that some inconsistencies will appear at some later stage? The same question can be asked about Euclidean geometry, but there no inconsistencies appeared in more than 2000 years. The idea occurred to mathematicians at the later years of the 19th century and the early years of the 20th century (Beltrami, Felix Klein, Henri Poincaré) to use Euclidean geometry to prove that hyperbolic geometry is consistent. Since the basic concepts of each geometry, line, point, congruence, incidence, betweenness are not defined, one may start within the framework of Euclidean geometry and build a "model" for hyperbolic geometry. For example, in the model of F. Klein, the points in the interior of a circle and the chords of the circle (excluding the points on the circumference) are made to play the roles of the points and lines of the hyperbolic geometry. With congruence suitably defined in this model, situations similar to hyperbolic geometry arise (e.g. through each point P more than one chord can be drawn, not intersecting a given chord ℓ). The results of hyperbolic geometry can then be identified with theorems in Euclidean geometry. So if Euclidean geometry is consistent, so is hyperbolic geometry.



$m_1 \parallel \ell, m_2 \parallel \ell$

An article like this can merely give you a glimpse of this fascinating chapter of mathematics. Many good books are available on the subject.



Heading for the great river Kwai
I turned off at an angle of ϕ.
When a voice cried from the jungle:
– Friend, you made a bungle –
So I stopped and shouted back: – why?

The answer came with a sigh:
– You'll never reach the great river Kwai.
It may cause anguish and pain
That you are in a hyperbolic plane
And the angle of parallelism is ϕ.

References

[1] H.S.M. Coxeter, "Introduction to Geometry" (Chapter 16), J. Wiley & Sons Ltd.

[2] H. Meschkowski, "Non-Euclidean Geometry", Academic Paperbacks.

[3] M.J. Greenberg, Euclidean and Non-Euclidean Geometry", W.H. Freeman.

[4] T. Heath (Ed.), "Euclid's Elements", Dover, Vol. 1, pp. 202-220.

[5] R. Bonola, "Non-Euclidean Geometry", Dover. [Contains translations of J. Bólyai, "The Science of Absolute Space" and N. Lobachevski, "Geometrical Researches on the Theory of Parallels".]

FIBONACCI SEQUENCES AND CHAIN REACTIONS

Michael A.B. Deakin, Monash University

The Fibonacci sequence is well known and much written about, but continues to be fascinating in all sorts of ways. It was the subject of an article in the very first issue of FUNCTION and there have been many other popular and indeed technical works written on it. There is no shortage of reading matter, if you want to learn more of the subject.

The Fibonacci sequence is the list of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ..., \tag{1}$$

where each number from the third entry (2) onwards is the sum of the two preceding numbers. That is to say that if we indicate the nth numbers by f_n (so that $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3$, etc.), we have

$$f_{n+2} = f_{n+1} + f_n. \tag{2}$$

This relation, together with the values of the first two numbers,

$$f_1 = f_2 = 1, \tag{3}$$

serves to determine the sequence.

The sequence has an unlikely origin, in that it was first introduced as a mathematical model of the breeding of rabbits. On one account (not quite that first adduced), it is supposed that a pair of rabbits mates each month and that after a two-month gestation period they produce a new pair of rabbits who are mature enough themselves to mate immediately, and so on.

It should perhaps be said at this stage that as an attempt to provide a realistic mathematical description of rabbit population dynamics, this model has several severe deficiencies – but perhaps we can just say that these are mathematical rabbits and we can attribute to them what properties we like, including immortality, for these rabbits don't die.

Anyhow, start with a single pair in, let us say, January. They mate but have nothing to show for this in February, when they mate again. However, by March the first of the younger generation, the pair conceived in January, arrive, so now we have two pairs of rabbits. Both pairs mate, but this will have no effect till May. However, in April a second litter is born to the original pair, so we now have three pairs. By May the two litters conceived in March arrive and so now there are five pairs. And so it goes on.

In August, say, we have all those pairs that were around in July, together with the new ones conceived by the pairs that were there in June.

In mathematical language we can express this as

$$f_8 = f_7 + f_6, \tag{4}$$

or more generally by Equation (2). The model clearly also involves Equation (3) and so the numbers each month are given by the consecutive terms of the Fibonacci sequence.

Given this background, we might expect that as each pair of rabbits ultimately produces a second pair of rabbits, the growth would follow a geometric sequence. To check this idea, try substituting $f_n = ar^n$ into Equation (2). This gives $ar^{n+2} = ar^{n+1} + ar^n$, which simplifies to

$$r^2 - r - 1 = 0, \tag{5}$$

as obviously $a \neq 0$. This has solutions

$$r = \tfrac{1}{2}(1 \pm \sqrt{5}). \tag{6}$$

We now need also to satisfy Equation (3), and it can easily be checked that this is not possible if we try to set $f_n = ar^n$. For either of these values of r, as given by Equation (6), no single value of a will do. However, if we set

$$f_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n, \tag{7}$$

we will find that Equations (2), (3) are both satisfied. You could do this as an exercise, and also check, both on a calculator and by exact multiplication of the quantities, that the first few numbers of the sequence (1) are in fact produced by this formula.

As n becomes large, the first term of Equation (7) becomes large (very rapidly), while the second becomes small (also very rapidly). E.g., when $n = 10$, the second term is only 0.0036, the extent by which the first term exceeds the true value of 55. For larger n, the discrepancy is even smaller and so we can put

$$f_n \simeq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n, \tag{8}$$

as long as n is not too small. In other words, although the Fibonacci sequence is not a geometric sequence, it soon settles down into something that very nearly is.

Moreover, if we go back to those rabbits, they are not breeding quite as fast as another species that simply doubled its numbers every month, producing the sequence whose general term is 2^n. On the other hand, they are multiplying more rapidly than a species that doubled its numbers every two months, producing (if n is even) the sequence whose general term is $2^{n/2}$ or $(\sqrt{2})^n$. The growth rate is intermediate between these and so we would expect

$$\sqrt{2} < \tfrac{1}{2}(1 + \sqrt{5}) < 2, \qquad (9)$$

which you can easily check to be the case.

I recently attended a meeting of the South-East Asian Mathematical Society in Chiang Mai, Thailand, and heard a Thai mathematician, Sompop Krairojananan, discuss another model that also gives rise to the Fibonacci sequence. He took a relatively simple account of the reaction between oxygen (O_2) and hydrogen (H_2) to form water (H_2O). You may have seen it done in your school chemistry lab. The two gases are mixed and a brief spark initiates their combination – boom! According to one model, this spark dissociates some of the oxygen molecules into highly reactive oxygen atoms (O).

Each oxygen atom then attacks a hydrogen molecule to produce two highly reactive species, a hydrogen atom (H) and a hydroxyl radical (OH):

$$O + H_2 \longrightarrow H + OH. \qquad (10)$$

These then attack other hydrogen and oxygen molecules according to the reactions:

$$OH + H_2 \longrightarrow H_2O + H \qquad (11)$$

$$H + O_2 \longrightarrow OH + O. \qquad (12)$$

Now see what's happened. After one round of reactions one oxygen atom has been replaced by another. So we still have one oxygen atom. But also in this round, we produced two hydrogen atoms and used only one of them up. This hydrogen atom will, on the *next* round, give rise to a new oxygen atom. Then there will be two oxygen atoms, because the actual atom of oxygen will also be replaced, and so on. The number of atoms of oxygen will follow the Fibonacci sequence.

Indeed, the oxygen atom is strictly analogous to the pair of rabbits in the first example, while the hydrogen atom is the analogue of the "potential rabbit pair" that has been conceived, but not yet born.

We may think of the chemical model (which somewhat simplifies the real-life thing, though not nearly as drastically as the rabbit model simplifies *its* corresponding reality) as a chain reaction. This term is especially used in nuclear situations, and it is from these that I draw the final illustration, again due to Professor Krairojananan. The model, once again, is a simplification, but not as drastic a one in this case.

Let a high-energy sub-atomic particle enter a mass of fissile nuclei. Eventually it collides with one of them, which then emits two more high-energy particles and one low-energy particle. Low-energy particles can also react with nuclei, but in this case they only emit two high-energy particles (as well as the original low-energy one). So

$$\left.\begin{aligned} h &\rightarrow 3h + \ell \\ \ell &\rightarrow 2h + \ell \end{aligned}\right\} \qquad (13)$$

So, if at the start of the nth round of reactions, there are h_n high-energy particles and ℓ_n low-energy particles, we will have, after this round,

$$\left.\begin{aligned} h_{n+1} &= 3h_n + 2\ell_n \\ \ell_{n+1} &= h_n + \ell_n \end{aligned}\right\} \qquad (14)$$

for the new high-energy particles come from both the old high-energy particles (3 for each of these) and the old low-energy particles (2 for each) according to the reactions (13). Similarly for the new low-energy particles.

Equations (14) are two simultaneous "recurrence relations", and one way to proceed is to eliminate ℓ_n. When this is done, we get a single relation for h_n, that reads

$$h_{n+2} = 4h_{n+1} - h_n. \qquad (15)$$

(Do this as an exercise.)

Equation (15) may be solved and thus also the system (14); we also need some analogue of Equation (3) and a simple one to use is $h_1 = 1$, $\ell_1 = 0$. Try this, too, as an exercise. The details differ from those used for Equations (2), (3) but the overall behaviour is very similar. The sequences that emerge:

1, 3, 11, 41, 153, 571, 2131, etc. (for h_n)

and

0, 1, 4, 15, 56, 209, 780, etc. (for ℓ_n)

are very similar to the Fibonacci sequence (1) and indeed are often referred to as Fibonacci sequences because of this similarity. It is not difficult to construct other examples, and you might like to do this yourself, and send your findings to FUNCTION.

* * * * *

> Examples ... which might be multiplied *ad libitum*, show how difficult it often is for an experimenter to interpret his results without the aid of mathematics.
>
> Lord Rayleigh

THE TOWERING CHALLENGE

Tim Hartnell
Chelsea, Victoria

Hour by hour, day by day, monks in the great temple of Benares are playing a game. When they finish it, the world will end. Your computer can give them a hand.

In 1883, a toy invented by the French mathematician Edouard Lucas went on sale. It was an immediate hit. The toy, marketed under the name *Tower of Hanoi*, consisted of eight disks of different sizes, and a base containing three pegs. The aim of the game was to move the disks, one by one, from the initial peg to one of the other pegs in the shortest number of moves. The only rule was that you could never place a disk on top of one which was smaller than itself.

It sounds a simple game, but it is one which demands a great many more moves than you might think. Those monks, in the "great temple of Benares beneath the dome which marks the centre of the world" (as the instructions with the original game explained it), were working with 64 solid gold plates. They have a brass plate, on which rest three diamond needles "each a cubit high and as thick as the body of a bee".

Apparently the creator put the 64 disks on one of the needles at the moment of creation, and told the hapless monks to get on with it, and move them. When they finish the game, they are told, the universe will pack its bags and go home. (The game instructions actually put it a bit more poetically: "When the sixty-four disks shall have been thus transferred from the needle on which, at the creation, God placed them, to one of the other needles – tower, temple, and Brahmans alike will crumble into dust, and with a thunderclap, the world will vanish.")

How long will it take them until Thunderclap Day? If there are n disks, it takes a minimum (assuming you don't make any dumb moves) of 2^n-1 moves. The numbers increase quite rapidly. With three disks, it will take you seven moves (in theory; it always seems to take me 12). Fifteen moves are needed with four disks, 31 moves with five disks, 63 moves with six, 127 with seven, 255 with eight, 511 with nine disks and 1,023 moves when you have ten disks.

That means our monkish persons, with 64 gold plates, will take 18,446,744,073,709,551,615 moves. If a plate was transferred every second, and the monks didn't drink their Horlicks and get any sleep, it would take them thousands of millions of years to finish the task (580,454,204,615 and a bit years, according to my calculations). Whew! I thought for a moment there that the Big Un-Bang was going to happen before the Christmas holiday.

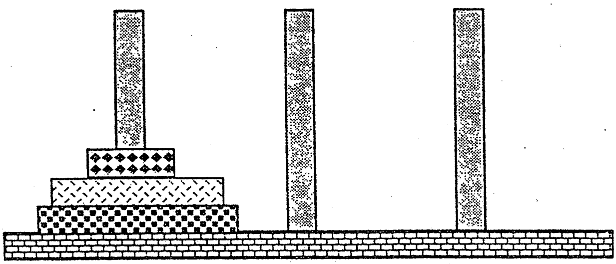
The eight disks provided with the original toy would take, as pointed out before, a minimum of 255 moves to transfer the disks. The program HANOI.BAS allows you to choose how many disks you want to transfer (from two to nine), then draws up the scene for you, and lets you get on with it. Needless to say, the program does not allow you to cheat.

```
10   REM HANOI.BAS
20   REM (C) TIM HARTNELL, 1987
30   REM INTERFACE PUBLICATIONS
40   CLS:PRINT"TOWER OF HANOI - TIM HARTNELL":PRINT:PRINT
50   DIM A$(9),B(20),C(20),D(20):MV=0
60   Q=9:REM CHANGE TO Q=5 ON AN APPLE OR 40-COL DISPLAY
70   PRINT "HOW MANY DISKS DO YOU WANT (2 TO";Q;")";:INPUT DK
80   IF DK=0 THEN END
90   IF DK<2 OR DK>Q THEN 70
100  Q=DK
110  PRINT:PRINT "PLEASE STAND BY ..."
120  REM ** SET UP **
130  FOR J=0 TO Q:A$(J)=""
140  B(J)=J
150  FOR Z=1 TO 10-J:A$(J)=A$(J)+" ":NEXT Z
160  FOR Z=1 TO 2*J+1:A$(J)=A$(J)+"*":NEXT Z
170  FOR Z=1 TO 10-J:A$(J)=A$(J)+" ":NEXT Z
180  NEXT J
190  Q=DK
200  REM ** PRINT OUT **
210  CLS:SC=-1
220  B(0)=0:C(0)=0:D(0)=0
230  PRINT "AFTER MOVE";MV:PRINT
240  REM FOR NUMBERS ONLY, CHANGE NEXT  LINE  TO:PRINT
B(0);"                              ";C(0);"  ";D(0)
250  PRINT A$(B(0));"  ";A$(C(0));"  ";A$(D(0))
260  FOR J=0 TO DK
270  REM FOR NUMBERS ONLY, CHANGE NEXT LINE TO:PRINT B(0);" ";C(0);"
     ";D(0)
280  PRINT A$(B(J));" ";A$(C(J));" ";A$(D(J))
290  B(J+10)=B(J):C(J+10)=C(J):D(J+10)=D(J)
300  IF D(J)=J THEN SC=SC+1
310  NEXT J
320  PRINT "-----------------------------------------------------------------
------------------------":REM FULL WIDTH OF SCREEN
330  IF SC<>0 THEN PRINT "SCORE IS"SC
340  IF SC=DK THEN PRINT "YOU'VE DONE IT IN";MV;"MOVES":END
350  MV=MV+1
360  INPUT "MOVE DISK FROM WHICH COLUMN";A
370  IF A<1 THEN END
380  IF A>3 THEN 360
390  INPUT "                 TO WHICH COLUMN";B
400  IF B=A OR B<1 OR B>3 THEN 390
410  REM ** FIND TOP DISK IN COLUMN **
420  E=0
430  FOR J=DK TO 1 STEP -1
440  IF A=1 AND B(J)<>0 THEN E=J
450  IF A=2 AND C(J)<>0 THEN E=J
460  IF A=3 AND D(J)<>0 THEN E=J
470  NEXT J
480  IF E=0 THEN 360:REM NO DISK IN THAT COLUMN
490  REM ** FIND TOP SLOT TO PLACE DISK **
500  F=0
510  FOR J=1 TO DK
520  IF B=1 AND B(J)=0 THEN F=J
530  IF B=2 AND C(J)=0 THEN F=J
540  IF B=3 AND D(J)=0 THEN F=J
```

```
550  NEXT J
560  IF F=0 THEN 390:REM NO ROOM IN THAT COLUMN
570  REM ** NOW MAKE MOVE **
580  REM ** FIRST REMOVE DISK **
590  IF A=1 THEN TEMP=B(E):B(E)=0
600  IF A=2 THEN TEMP=C(E):C(E)=0
610  IF A=3 THEN TEMP=D(E):D(E)=0
620  REM ** NOW PLACE IN NEW POSITION **
630  IF B=1 THEN B(F)=TEMP
640  IF B=2 THEN C(F)=TEMP
650  IF B=3 THEN D(F)=TEMP
660  REM ** NOW CHECK IF DISK BELOW IS LARGER **
670  OK=1:FOR J=1 TO DK-1
680  IF B(J)>B(J+1) THEN OK=0
690  IF C(J)>C(J+1) THEN OK=0
700  IF D(J)>D(J+1) THEN OK=0
710  NEXT J
720  IF OK=1 THEN 200
730  PRINT TAB(4)"⇒ THAT MOVE IS ILLEGAL ⇐"
740  FOR Z=1 TO 500:NEXT Z:REM ADJUST THIS DELAY FOR YOUR SYSTEM
750  FOR J=1 TO Q
760  B(J)=B(J+10):C(J)=C(J+10):D(J)=D(J+10)
770  NEXT J
780  MV=MV-1
790  GOTO 200
```

If you want a mathematical project, try working out a proof that you can always move n disks in 2^n-1 moves. I'd be interested in seeing your proof. Another project would be to write a program which would tell you the best way to move the disks. You can do this with the assistance of binary numbers.

Let's say you had three disks. You know, from 2^3-1, that it will take seven moves. You write down the numbers one to seven, in binary, one under the other. In order to work out which disk to move, you count the digits from the right until you reach the first 1. The number of digits you have counted will tell you which disk to move. You can see that, counting from the right with move number one, you'll hit a 1 right away, so you move the first (that is, the smallest, disk).



A three-disk Tower of Hanoi requires seven transfers to move the disks from the left hand needle to the right hand one.

To find out where to put the disk, you start counting again and keep moving across the binary number. If you don't come to any other 1's, then place the disk on the first needle you come to. In this case, you'd place it on needle number 2. If you were on needle number 3, as you will be when using more than three disks, you move it to 1. If there are other 1's to the left of the first 1, you count across from the right until you hit the next 1. This identifies the disk you moved on the previous move. Now, if there are no zeros between the first 1 and the second 1, or there is an even number of zeros, place the disk on top of the one you moved in the previous move. If there is an uneven number of zeros, then you skip that move.

Here's how it works in practice:

1 – 001	Move disk one
2 – 010	Move disk 2
3 – 011	Move disk 1 on disk 2
4 – 100	Move disk 3
5 – 101	Skip this move (odd number of zeros between the 1's)
6 – 110	Put disk 2 on disk 3
7 – 111	Put disk 1 on disk 2

You might also be interested in writing a program which not only works out the above, but actually moves the disk for you. Then you could get it on the 64 gold disk problem, and start packing your bags for Thunderclap Day.

□ □ □ □ □

* * * * *

HALT!

Mathematics permits no free movement.

Immanuel Kant (1724-1802),
philosopher

* * * * *

But there is another reason for the high repute of mathematics: it is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.

Albert Einstein

LETTER TO THE EDITOR

Mark Kisin, who is a year-eleven student at Melbourne Church of England Grammar School, wrote:

In the 1987 IMO, Question 1 asked one to prove that $\sum_{k=0}^{n} kp_n(k) = n!$, where $p_n(k)$ is the number of permutations of the set $\{1,2,3,...,n\}$ with exactly k fixed points. It is clearly true that $\sum_{k=0}^{n} p_n(k) = n!$, as there are $n!$ permutations, each counted once by the preceding sum. It also turns out that $\sum_{k=0}^{n} (k^2-1)p_n(k) = n!$. This seems to suggest that in general there might be a polynomial of degree j (say $F_j(x)$) such that $\sum_{k=0}^{n} F_j(k)p_n(k) = n!$. However, as $\sum_{k=0}^{n} (k^2-1)p_n(k) = n!$ does not hold for $n = 1$, and $\sum_{k=0}^{n} kp_n(k) = n!$ does not hold for $n = 0$, we should impose the additional requirement that $n \geq j$. In fact, this constraint assures the existence of such $F_j(x)$, and in what follows I shall prove this, as well as give a method for constructing the polynomials in question.

Readers who are interested in Mark Kisin's proof should write to the Editors of FUNCTION for further details. (Ed.)

* * * * *

STOP PRESS

1988

INTERNATIONAL MATHEMATICAL OLYMPIAD

This year the IMO took place in Canberra. The examinations were held on July 15 and 16. There were 49 teams competing. Australia was 17th overall. The USSR was first, the People's Republic of China and Romania came equal second.

Australia won one gold medal and one bronze medal (in 1987 our team won three silver medals).

The medallists were:

Gold: Terence Tao, Blackwood High School, S.A.
Bronze: Geoffrey Bailey, St. Aloysius College, N.S.W.

Congratulations from FUNCTION to them and all the team.

PROBLEM SECTION

FUNCTION has received another solution to Problem 11.3.1 (Vol. 11, pt. 3, p. 96; compare PERDIX in Vol. 12, pt. 1, p. 31f. and Colin Wilson in Vol. 12, pt. 2, p. 37f.). The problem is:

If a and b are positive and $a + b = 1$, show that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2} .$$

Here is the latest solution. It is by C.J. Eliezer of La Trobe University and again proves the usefulness of the formula

$$xy \leq \frac{1}{4}(x + y)^2$$

as was indicated by Colin Wilson:

The inequality is equivalent to $a^2 + 2 + \frac{1}{a^2} + b^2 + 2 + \frac{1}{b^2} \geq \frac{25}{2}$

i.e. $a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2} \geq \frac{17}{2}$

i.e. $(a + b)^2 - 2ab + \frac{(a+b)^2 - 2ab}{a^2b^2} \geq \frac{17}{2}$

i.e. $1 - 2ab + \frac{1}{a^2b^2} - \frac{2}{ab} \geq \frac{17}{2}$, because $a + b = 1$,

i.e. $4a^3b^3 + 15a^2b^2 + 4ab - 2 \leq 0$

i.e. $(4ab - 1)(a^2b^2 + 4ab + 2) \leq 0$

i.e. $ab \leq \frac{1}{4}$.

Hence the given inequality is equivalent to $ab \leq \frac{1}{4}$.

But $ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right] \leq \frac{1}{4}$, using $a + b = 1$.

* * * * *

Also, Etta Melman and John Smith of Moriah College, Bellevue Hill, NSW, submitted at last to the challenging fangs ot PERDIX. Their pretty solution to the problem is:

To prove that if $a + b = 1$ then

$$(a+\frac{1}{a})^2 + (b+\frac{1}{b})^2 \geq \frac{25}{2}.$$

In general, $(a+b)^2 = 2ab + a^2 + b^2$

and $(a-b)^2 = -2ab + a^2 + b^2$.

Adding: $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$

i.e. $1 + (a-b)^2 = 2(a^2+b^2)$

$$\therefore \quad a^2 + b^2 \geq \frac{1}{2}.$$

Subtracting $(a+b)^2 - (a-b)^2 = 4ab$

i.e. $1 - (a-b)^2 = 4ab$

$$\therefore \quad 4ab \leq 1.$$

Hence,
$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2+b^2}{a^2b^2}$$

$$\geq \frac{\frac{1}{2}}{\frac{1}{16}},$$

$$\geq 8.$$

$$\therefore \quad a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2} + 4 \geq \frac{1}{2} + 8 + 4$$

i.e. $(a+\frac{1}{a})^2 + (b+\frac{1}{2})^2 \geq \frac{25}{2}$.

Q.E.D.

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John Barton of North Carlton presented two solutions to Problem 12.1.2.

First Method

Through E draw EF parallel to BC to meet AB in F. Join CF. Let CF intersect BE in G. Since $\angle BCD = 50°$ and $\angle BCF = 60°$, $\angle BCD < \angle BCF$, so B and D are on the same side of the line CF. $\triangle BCG$ is equilateral, so that

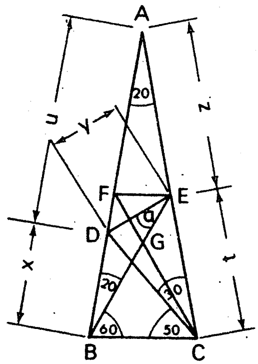
$$BG = BC. \qquad (1)$$

Since $\angle BDC = 50^\circ = \angle BCD$,

$$BD = BC.$$

Hence $BG = BD$, so that

$$\angle BDG = \angle BGD = 80^\circ.$$



Now, by (1), $\angle FGD = 180^\circ - \angle BGD - \angle BGC$

$$= 180^\circ - 80^\circ - 60^\circ$$

$$= 40^\circ.$$

But $\angle DFG \cong \angle BFC = 40^\circ$, so that

$$\angle FGD = \angle DFG.$$

Hence $DG = DF$.

Since ΔEFG is equilateral, $EF = EG$.

Hence $\Delta DFE \cong \Delta DGE$, (SSS).

Hence $\angle DEF = \angle DEG$.

Since $\angle FEG = 60^\circ$, $\angle DEG = 30^\circ$.

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Second Method (Trigonometry)

Let the base BC be 2 units long.

Δ ABC : $u + x = z + t = \text{cosec } 10^\circ$. (1)

Δ ADE : $\dfrac{\sin 20^\circ}{y} = \dfrac{\sin(20+\alpha)^\circ}{z} = \dfrac{\sin(40+\alpha)^\circ}{u}$ (2)

Δ BDE : $\dfrac{\sin \alpha}{x} = \dfrac{\sin 20^\circ}{y}$ (3)

Δ DEC : $\dfrac{\sin 30^\circ}{y} = \dfrac{\sin(70+\alpha)^\circ}{t}$ (4)

(1), (2), (3), (4) are six equations in the six unknowns x, y, z, u, t, α. Eliminate x, y, z, u, t and find, after some straightforward elimination not given here

$$\tan \alpha = \frac{\sin 20^\circ}{2 \cos 40^\circ - \cos 20^\circ} = \frac{\sin 20^\circ}{\cos 40^\circ - \sin 10^\circ}$$

$$= \ldots = \frac{1}{2} \sec 30^\circ = \frac{1}{\sqrt{3}} .$$

Hence $\alpha = 30°$.

[There are other choices of triangles which may give a little shorter elimination.]

Also Leigh Thompson of Bairnsdale has sent us a trigonometric solution that is based on the "sine rule".

* * * * *

Stephen Bigelow, who is in year 12 at Ivanhoe Grammar School, Victoria, offered a solution to Problem 12.2.3:

$\frac{\text{TALK}}{9999}$ simplies to $\frac{\text{EVE}}{\text{DID}}$. So, clearly it is a factor of 9999, which factorizes to $3^2 \times 11 \times 101$. The only factors which fit the structure DID are 101, 303, 909. If DID were 101, then it would be impossible to meet the condition that EVE < DID. If DID = 909, then

$$\frac{\text{EVE}}{909} = \frac{\text{TALK}}{9999},$$

i.e. $\text{EVE} \times 11 = \text{TALK}$,
hence $\text{E} = \text{K}$,

which is against the assumptions. Therefore by elimination we have

$$\text{DID} = 303, \text{ thus}$$

$$\frac{\text{EVE}}{303} = \frac{\text{TALK}}{9999},$$

i.e. $33 \times \text{EVE} = \text{TALK}$.
so, $3 \times \text{E} = \text{K}$

and since EVE < DID = 303, E must be less than 3. If E was 1 then

$$\begin{aligned} \text{K} &= 3 \times \text{E} \\ &= 3 \\ &= \text{D}, \end{aligned}$$

which is against the assumptions. We conclude that E must be 2 and K = 6. It follows that 33 × 2V2 = TAL6. To avoid different letters standing for the same digit, V = 0, 2, 3 or 6 must be eliminated; V = 5 and V = 8 would produce 252 and 282 respectively, which are multiples of three and thus allow cancellation of $\frac{\text{EVE}}{909}$. This leaves the possibilities V = 1, V = 4, V = 7, and by trial and error we find that 4 is the only value which gives no repetition of digits in TALK. So the solution of the problem is:

$$\frac{242}{303} = 0.798679867986 \ldots$$

Leigh Thompson let his computer do all the work, save writing the following BASIC program:

```
1Ø FOR E = 1 TO 8 : IF E/2=INT(E/2) THEN N=2 ELSE N=1
2Ø FOR V = Ø TO 9 : IF V=E THEN 9Ø
```

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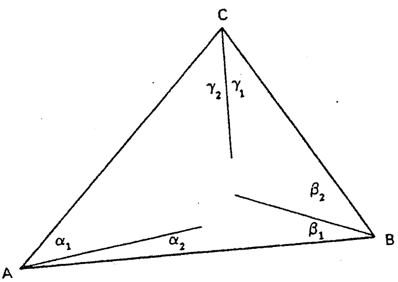
3Ø FOR D = E+1 TO 9 STEP N : IF D=V THEN 8Ø
4Ø FOR I = Ø TO 9 : IF I=E OR I=V OR I=D THEN 7Ø
5Ø TA = 9999*(1Ø1*E+1Ø*V)/(1Ø1*D+1Ø*I)
6Ø IF TA = INT(TA) THEN T=INT(TA/1ØØØ):A=INT(TA/1ØØ)-1Ø*T:L=INT(TA/1Ø)-1ØØ*T
    -1Ø*A:K=TA-1ØØØ*T-1ØØ*A-1Ø*L:IF T=A OR T=L OR T=K OR T=E OR T=V OR T=D
    OR T=I OR A=L OR A=K OR A=E OR A=V OR A=D OR A=I OR L=K OR L=E OR L=V
    OR L=D OR L=I OR K=E OR K=V OR K=D OR K=I THEN 7Ø ELSE PRINT
    E;V;D;I;T;A;L;K
7Ø NEXT I
8Ø NEXT D
9Ø NEXT V
1ØØ NEXT E
```

N.B. : Since $.\dot{T}\dot{A}\dot{L}\dot{K} < 1$, we have $E < D$, and also if E even, then D must be odd; hence lines 1Ø and 3Ø.

* * * * *

Problem 12.4.1 (proposed by A.W. Sudbury of Monash University). From each of the three vertices A, B, C of a triangle, a ray is drawn in the direction of the interior of ABC. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ be the angles as indicated in the diagram below.



Find a trigonometric equation, involving only the angles $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$, which holds if and only if the three rays intersect in one point. Demonstrate directly from their definitions that the orthocentre, in-centre, circumcentre and centroid satisfy your equation.

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Problem 12.4.2 (communicated by M.A.B. Deakin, Chiang Mai, Thailand). The game NORTH-EAST is played on the rectangular array of points in the plane with integral coordinates (n,m), where $0 \le n \le N$, $0 \le m \le M$. Player A selects a point (p,q) and removes all those points for which $n \ge p$, $m \ge q$. Player B then selects a point (r,s) and removes all those points still left for which $n \ge r$, $m \ge s$, etc. The loser is the player who takes (0,0). The problem is to show that A has a winning strategy.

Problem 12.4.3 (proposed by a puckish Lewis Carroll in *The Monthly Packet* beginning in April, 1880). Place twenty-four pigs in four sties so that, as you go round and round, you may always find the number in each sty nearer to ten than the number in the last.

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Problem 12.4.4. Let k be a positive integer and let $A = \{2^i \mid i = 0,1,2,...\}$. Find all positive integers n such that

$$\text{numbers } a_1,...,a_k \ (a_i \neq a_j \text{ for } i \neq j) \text{ from } A,$$

for which the sum $|n-a_1| + ... + |n-a_k|$ is minimal, can be chosen in more than one way.

* * * * *

Problem 12.4.5. Let $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x+y) = f(x)\,f(a-y) + f(y)f(a-x) \text{ for all } x,y \in \mathbb{R}$$

$$f(0) = 1/2.$$

Prove that f is constant.

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Problem 12.4.6. Is it true that 832^n and $832^n + 2^n$ have an equal number of digits?

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Problem 12.4.7. Prove that for any $a,b \in \mathbb{R}$, $a < b$, there exists $n \in \mathbb{N}$ and $c_i \in \{-1,1\}$, $i = 1,...,n$, such that

$$a < c_1 + c_2/2 + ... + c_n/n < b.$$

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> Approach your problems from the right end and begin with the answers. Then, one day, perhaps you will find the final question.
>
> "The Hermit Clad in Crane Feathers", in R. van Gulik's *The Chinese Maze Murders*