Function

Volume 13 Part 1 February 1989

MAGIC SQUARE FOR THIS YEAR

421	404	381	388	395
403	385	411	394	396
384	386	393	400	426
390	416	399	401	383
391	398	405	406	389

The Magic Number is 1989

A SCHOOL MATHEMATICS MAGAZINE

eg. by Aust. Post Publ. No. VBH0171

FUNCTION is a mathematics magazine addressed principally to students in the upper forms of schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

The magazine is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $12.00*; single issues $2.40. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

* $6.00 for *bona fide* secondary or tertiary students.

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Registered for posting as a periodical - "Category B"
ISSN 0313 - 6825

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Published by Monash University Mathematics Department.

FUNCTION

Volume 13, Part 1 February 1989

Welcome to new readers and to continuing readers! We hope you enjoy reading *Function* this year. Please write to the editors telling us what articles you would like to see in *Function*. Send us articles you have written, or ideas for articles.

The articles by Garnet Greenbury on determining the day of the week on which a given date falls, and his interesting observation about Fibonacci sequences, both touch on subjects that have been discussed in *Function* before, as does the article by Michael Deakin, providing interesting historical detail about the famous Königsberg bridge problem. The article by Esther Szekeres provides yet another elegant alternative solution to the three solutions provided by John Burns (*Function*, volume 12, part 4, p. 99) to the problem of constructing an equilateral triangle with a given vertex and the other two vertices lying on a given pair of lines.

We also have articles on the greenhouse effect – is Melbourne getting warmer? –, on 16th century arithmetic, on random numbers and on the effect of round-off errors.

Take your pick!

CONTENTS

FRONT COVER.

The magic square on the front cover was presented by Stephen Murphy. See also his quintuple binary magic square overleaf.

BINARY MAGIC SQUARE

Stephen Murphy, Year 9, Haileybury College

A magic square is a square array of numbers such that the sums of the numbers in each column, in each row, and in each diagonal are all the same. Their common sum is called the magic number for the square. Below is a binary magic square.

0010	1111	0100	1001
0101	1000	0011	1110
1011	0110	1101	0000
1100	0001	1010	0111

It remains magic when:

(1) read as printed above;
(2) reflected in a mirror;
(3) turned upside down;
(4) turned upside down and reflected in a mirror.

The magic number is 11110 (in binary).

Note also that this magic square is effectively made up of 4 component magic squares, namely those made up by the square of 1st digits of the above numbers;

First digits

0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

the square similarly made up by the 2nd digits; that made up by the 3rd digits; and that made up by the 4th digits. The magic number for each of these squares is 2 (or 10 in binary). We have five magic squares in one.

* * * * *

HOW TO FIND THE DAY OF THE WEEK OF ANY DATE IN THE 20TH CENTURY

Garnet J. Greenbury, Brisbane

This method is valid from March 1, 1900, to February 29, 2000.

One need not know if the date is in a leap year or not. The only assumption is that the first day of the year is March 1, rather than January 1.

March through December are respectively the first month through the tenth month; January and February are the eleventh and twelfth months of the previous year.

(1) Denote by D, M and Y the number of the day in the month, the number of the month counted as explained above, and the number given by the last two digits of the year, respectively.

For example, March 30, 1988 gives $D = 30$, $M = 1$, $Y = 88$.

(2) Compute the sum N by substituting in the following formula

$$N = D + M + Y + [0.8(2M+1)] + [Y/4].$$

The square brackets mean the integer part of what is calculated.

For the example just given,

$$\begin{aligned} N &= 30 + 1 + 88 + [0.8(2+1)] + [88/4] \\ &= 119 + 2 + 22 \\ &= 143. \end{aligned}$$

(3) Divide the sum N by 7. The remainder gives the day of the week. Count 0 as Sunday, 1 as Monday, and so on.

Continuing our example we have $N = 7 \times 20 + 3$, so the required day is Wednesday.

Examples to test

July 4, 1976 was a Sunday.
February 29, 1980 was a Friday.

On what day of the year were you born?
What is the formula for the twenty-first century?

* * * * *

EDITOR'S NOTE. I have not seen the expression $[0.8(2M+1)]$ used for this calculation before. It really is very clever. It gives a single formula for all months in the year and so saves you having to remember, or individually work out, a table of values, such as given by Liz Sonenberg in her article in the first issue of *Function* (Vol. 1, Pt 1, p.21), a table that gives the change in the day of the week, compared with January, corresponding to each month. Beginning the year on March 1 also ensures that you need make no special changes in this table for leap years.

Function has had several articles on the calendar and its history. Volume 12, part 4 featured on its cover a Central American Maya Calendar. In Volume 1, part 3, pp.24-5, Mark Michell used Liz Sonenberg's formula to show the surprising fact that Tuesdays, Thursdays and Sundays cannot occur as the first day of a century. In volume 11, part 1, pp.4-5, David Johnson gave an account of John Conway's Domesday method of determining the day of the week that a date falls on. The Domesday method also avoids having to give a special treatment to leap years. Effectively the Domesday method starts the year at March 1 as does Garnet Greenbury's method. We start with declaring the last day of February to be a domesday. Then a domesday is obtained for each other month, all domesdays throughout the year always falling on the same day of the week. Knowing what day of the week is the domesday for a given year makes it easy to calculate what day of the week any date in that year is.

* * * * *

THE GREENHOUSE EFFECT

G.A. Watterson, Monash University

Some scientists believe that the temperature of the earth's atmosphere is gradually increasiung. This could be due to increased pollution from factory chimneys, car exhausts, etc., causing the sun's rays to be trapped in our atmosphere rather than being reflected out again, much like heat is trapped in a greenhouse (or glasshouse) for growing tropical plants. If this Greenhouse Effect is operating in our atmosphere, it could cause the melting of the polar ice caps, and hence the flooding of low-lying cities and even of whole countries. So the Greenhouse Effect is important.

I happened to have access to the Bureau of Meteorology's data for average yearly temperatures in Melbourne. So I wondered whether the Melbourne temperatures showed any change over the years, and in particular, whether they tended to *increase* over the years.

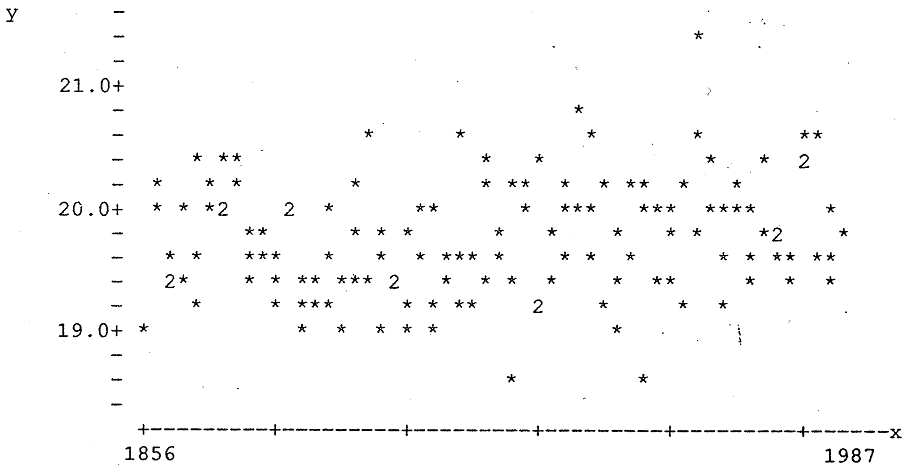

Fig. 1

In Figure 1 is plotted the year's average maximum temperature (i.e. the average of the maximum temperatures over 365 or 366 days) for each of the years 1856 to 1987. [A figure "2" on the plot indicates two overlapping points.] You will see that the averages are usually about 20°C, but they jump about from year to year. To the naked eye, there doesn't appear to be any discernible pattern to the temperatures, and little to hint that they have been increasing.

But perhaps the "naked eye" is deceiving us. Suppose we try to fit a straight line, with equation

$$y = mx + c,$$

as best we can through those data points in Figure 1 where y = year's average maximum temperature and x = year. Then if the slope, m, turns out to be positive, it will indicate the rate at which the temperature is increasing per year. Of course, if m turns out to be zero or negative we can conclude that the temperature is *not* increasing but is either stationary or actually decreasing.

There are many different ways of fitting a straight line through data points. Of course, if all the data points happen to lie *exactly* on a straight line there is no problem, but that is not the case with Figure 1. Whatever the straight line we draw through those points, at least some of the points will not lie exactly on the line. Which is the "best" straight line depends on what you mean by "best". There is a computer program called MINITAB. which fits "best" straight lines by what is called the "least squares" method. Roughly speaking, the "least squares" method ensures that the sum of all the squares of the distances, parallel to the y-axis, of the points from the line, is not exceeded for any other line. Without going into details, I will just quote the results.

The least squares fitted line, for the data in Figure 1, is

$$y = 0{\cdot}002351x + 15{\cdot}18. \tag{1}$$

Thus we might conclude that Melbourne is, indeed, warming up but at the very slow rate of $0{\cdot}00235^\circ$C per year.

There is even some evidence in the data that the rate of increase hasn't been constant over the years, but that temperatures are actually accelerating. The MINITAB program will fit a *curve* to the data, if it is commanded to do so. I asked it to fit a quadratic (parabolic) curve, and the least squares fitted curve turned out to be

$$y = 0{\cdot}0000538199x^2 - 0{\cdot}204479x + 213{\cdot}812. \tag{2}$$

The rate of increase in temperature at year x is then

$$\frac{dy}{dx} = 0{\cdot}00010764x - 0{\cdot}204479.$$

In $x = 1987$, this rate of increase is equal to $0{\cdot}0094^\circ$C per year, a rather faster rate than predicted using the constant-rate model (1).

There are, of course, many possible explanations for a gradual increase in temperature in Melbourne. It may be that temperatures, in cities generally, increase as the population gets larger. Perhaps this warming does not apply elsewhere in unpopulated areas and over the oceans. See if you can find out more about the Greenhouse Effect.

* * * * *

IS IT JUST "SMALL" OR IS IT "NEGLIGIBLE"?

King-wah Eric Chu* , Monash University

Abstract

In this paper, we explain why some classical "black-and-white" concepts like invertibility and rank in linear algebra are meaningless in practical applications. Analogous continuous or "distance" type concepts will be discussed briefly. A simple 2×2 example, which can easily be interpreted geometrically, is used throughout the paper for illustrative purposes.

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*Dr Chu's research interests include numerical analysis, especially numerical linear algebra, and mathematical control theory.

1 *A TALE OF TWO LINEAR EQUATIONS*

Let me bore you with the following simple catering problem:

A half-mad Chinese restaurant owner fixed the prices (P_i) on his menu based on the prices of salt (S) and mono-sodium-glutamate (M), using the formula

$$P_i = a_i S + b_i M,$$

where a_i and b_i measure respectively the quantities of salt and MSG used in cooking the i-th dish, in some funny units.

One day, he went walk-about and his number-one-son[1] took over as boss. Aware of number-one-son's inexperience, the supplier of salt and MSG presented the new boss an inflated bill. Being a Monash Mathematics graduate (2nd Class Honours, Upper Division), number-one-son tried to estimate S and M (and thus the unfair increases in prices) by cooking a couple of dishes, weighing the amount of salt and MSG used (a_i and b_i, $i = 1,2;$), and then looking up the corresponding prices P_1 and P_2 on the menu. The problem was then reduced to solving for S and M the two linear equations

$$a_1 S + b_1 M = P_1, \quad a_2 S + b_2 M = P_2.$$

Substituting in the numerical values from the cooking experiment, he had to solve the linear equations

chu.chi

$$10S + 10M = 2,$$

$$5S + 4M = 1;$$

or in matrix-vector notation,

$$\begin{bmatrix} 10 & 10 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{1}$$

Note that a matrix equation $Ax = b$ like (1) is solvable provided that the matrix A is *invertible*, or, equivalently, the *determinant* of A is nonzero. When a matrix is not invertible, it is said to be *singular*.

Geometrically, the two linear equations in (1) define two straight lines on the S–M plane (analogous to the x–y plane in the usual notation), and solving equation (1) is equivalent to finding the intersection point of the straight lines. The matrix equation (1) is solvable, i.e. the matrix is invertible, if and only if the two straight lines are not parallel. The singular matrix thus corresponds to the parallel situation.

The solution to (1) is easily found to be $S = 0.2$, $M = 0$.

---

[1] Mr Charlie Chan.

Recognising that MSG cannot be free (i.e. M cannot be zero), number-one-son repeated the experiment, obtained a new set of data (i.e. new values for a_i and b_i, $i = 1,2$, of the linear equations) and ended up with the new set of equations

$$\begin{bmatrix} 10 & 10 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{2}$$

the solution of which is $S = 0$ and $M = 0.2$.

Knowing that salt cannot be free either (i.e. S cannot be zero), number-one-son could not accept the result[2]. *What went wrong?!* His old notes on Numerical Methods did not help at all!

What went wrong was, being not a very good cook, he put in the wrong amount of salt and MSG, and the correct amount should yield the equations

$$\begin{bmatrix} 10 & 10 \\ 5.5 & 4.5 \end{bmatrix} \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{3}$$

with the correct solution being $S = 0.1$ and $M = 0.1$.

The story is fictitious, but the mathematics is real. The solutions from the slightly perturbed data vary wildly from one physical impossibility to another, and the solutions S and M of Equation (1) differ totally from those of Equation (2). Using the correct equation in (3) as reference, the solutions of Equations (1) and (2) went wrong by 100%! The shocking thing is that the maximum error in the data or the elements of the matrix in Equations (1) or (2) is only 11.1%!

2 *CONDITIONING, CONDITION NUMBER AND SCALING*

The trouble with number-one-son's experiment is that he chose the wrong dishes to cook, and ended up having to solve a set of badly-behaved or *ill-conditioned* equations.

Notice that the linear equations in (3) can be written as

$$10S + 10M = 2, \quad 5.5S + 4.5M = 1, \tag{4}$$

and the two equations are nearly linearly dependent, with the second equation in (4) nearly the same as $5S + 5M = 1$, which is just the first equation in disguise.

Geometrically, the two linear equations in (4) correspond to two nearly parallel straight lines. Using graph paper to find the intersection point of two such straight lines, it is easy to demonstrate the ill-conditioning or the sensitive nature of the intersection point with respect to the changes in the data – tilting one line slightly (using the edge of a ruler to represent it) while keeping the other one fixed, the intersection point can jump off the graph paper easily! In contrast, if the straight lines are perpendicular to each other, moving the lines will move the intersection points by the same distance and the problem of finding the intersection point is thus well-conditioned.

---

[2] not even with a pinch of salt.

In terms of the cooking experiments, the ill-conditioning of the matrix equation indicates that the two dishes are very similar (for the purpose of determining S and M, if nothing else) and the experimental results do not contain enough information for the solution of S and M. Two quite different dishes should have been cooked to yield two independent pieces of information for the problem to be solved.

The *conditioning* of the problem can be characterised by the *condition number* κ, where

$$\kappa \approx \frac{|\text{relative error in the solution}|}{|\text{relative error in the data}|}. \tag{5}$$

Without getting into details[3], the condition number is always greater than or equal to one, and can be calculated for a given problem. A large condition number indicates ill-conditioning for the problem, and a small error in the data of the problem will be magnified to an unacceptable level in the solution. Note that errors in the data can be the result of human fallibility, inexact representations of data in computers, bad programming techniques, nature of the data collecting process, etc..

In geometrical terms, κ is related to the quantity $|\cot \frac{\theta}{2}|$, where θ is the angle between the two straight lines defined by the linear equations[4]. Note that $\lim_{\theta\to 0}|\cot \frac{\theta}{2}| = \infty$, indicating ill-conditioning for nearly parallel straight lines.

For the example in Equation (3), the condition number[5] κ is 25.01, although from the discussion from the last paragraph in Section 1, we have

$$\frac{|\text{relative error in the solution}|}{|\text{relative error in the data}|} = \frac{100\%}{11.1\%} = 9,$$

for the perturbed systems in Equations (1) and (2).

The different results of the two calculations result from the fact that the condition number κ is defined to be the maximum of the quantity on the right-hand-side of Equation (5), taken over all possible perturbations in the data.

In the case of Equation (3), we know the extra information that prices of things have to be positive, and thus recognise the unacceptable solutions, despite the comparatively small condition number. In general and without any extra information, it will be difficult to judge whether an answer is acceptable or not.

---

[3] See [1] for details.

[4] The formula $\kappa = |\cot \frac{\theta}{2}|$ holds after we scale the rows of the matrix so that the sum of squares of elements on each row equals to unity.

[5] which can be calculated from the angle θ or by other methods in [1].

Mathematically, Equation (3) is ill-conditioned because the matrix is nearly singular. In general, it can be the result of nearly-dependent equations (like in Equation (3)), or badly scaled data, e.g. as in

$$\begin{bmatrix} 1000000 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$

The matrix in the above equation is approximately $\begin{bmatrix} 1000000 & 0 \\ 0 & 0 \end{bmatrix}$, which is singular.

3 *INVERTIBILITY AND RANK*

The example and the discussions in the previous sections can be interpreted in a different way – the classical discrete or "black-and-white" concepts like invertibility[6] and rank[7] are really inadequate in practical applications. As in the catering example in Section 1, the matrix in Equation (3) is clearly invertible, but modifying it slightly will make it singular. *It will be far more sensible to consider the size of the smallest possible perturbation which will make the matrix singular.* Similar perturbations which change the rank of matrices can also be considered.

If we imagine different matrices are just different objects (or points) floating around in space, we can create the notion of a "distance" between different matrices. Given two matrices, the distance between them can be calculated.

The size of the smallest perturbation which makes a matrix singular can be considered as the distance of a matrix (and the problem it defined) from its nearest ill-conditioned neighbour[8]. (For the example in Equation (3), the distance from the nearest ill-conditioned neighbour is $15.81\kappa^{-1} = 0.6323$, thus this distance behaves like the reciprocal of the condition number.) Similar distances can be defined for problems of the change in rank or other structures for matrices.

---

[6] You can only have matrices which are invertible or singular, but not in between.

[7] Rank of a matrix is the number of independent pieces of information contained in the matrix. For the 2×2 case, it is the number of straight lines defined by the related linear equations – 0 for the matrix with only zero elements, 1 for the matrix which defines two parallel lines, and 2 for the matrix which defines two intersecting lines. Obviously, rank can only be of integer value and cannot distinguish ill-conditioned cases with nearly parallel lines from other rank 2 cases.

[8] It is easy to prove that any singular matrix has invertible neighbours which are arbitrarily close, but an invertible matrix can be quite distant from its nearest singular neighbour. In other words, a randomly selected matrix is likely to be invertible – as a result, the set or *subspace* of invertible matrices is said to be *dense* in the space of matrices.

More generally, one can only determine most quantities in mathematics "theoretically". In practice, we can only do so numerically, and only if equipped with information about the size, i.e. the level, at which a *small* number is *negligible* and can be ignored and treated as zero. Thus, we have to know the reliability of the data, the noise level in our physical model etc., before we can tell what a particular quantity is "most likely" to be. The details of the determination of invertibility and rank of matrices involve the use of Singular Value Decomposition, and can be found in standard advanced texts on numerical linear algebra (e.g. [1]).

4 *AN UNCERTAINTY PRINCIPLE*

The discussion in Section 3 on invertibility and rank can be extended to other discrete or "black-and-white" concepts in linear algebra (and for general matrices of appropriate dimensions), e.g. the signature of a matrix, various indices in the Jordan canonical form, nilpotency, whether two subspaces have empty intersection, etc.[9]. The list is endless.

Similar philosophy can also be applied to other branches of mathematics.

We can now present an *Uncertainty Principle in Linear Algebra* for these concepts or properties:

One cannot analyse a "black-and-white" property for a given problem in linear algebra with accuracy better than the level of error, noise or resolution in the data allows.

When analysing a property or structure (say conditioning of a problem) in practice, it is better (or numerically more stable) to consider the distance between the given problem and its nearest neighbour with a different property or structure (ill-conditioning). Such philosophy is sound, and the corresponding problems of estimating these distances constitutes an exciting area for future research.

5 *CONCLUSIONS*

How small is small?
Is it just small or is it negligible?
These are the seemingly trivial but crucial questions we keep on asking ourselves in some applications in mathematics.

Personally, I find it is the *uncertainties*, like the ones related to the above questions and discussed in this paper, which make the supposedly exact science of mathematics such a fascinating subject, making mathematics on its highest level an art as much as a science. It takes a lot of experience and problem-solving-sense to determine whether a "small" number is just small or in fact negligible, whether a 3% significant level is good enough for a test of a hypothesis, or whether 3 is a good enough safety factor or a good enough redundancy factor in an engineering design. These uncertainties and limitations of mathematical models, as well as the ever-growing area of future applications, make mathematics an evolving, dynamic system and a really challenging and interesting subject.

---

[9] Cf. [1].

So, you see, there is life after Mathematics at school!

Bonne Chance, Bon Travail, et Bon Courage[10] *!*

References

[1] G.H. Golub & C.F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 1983.

* * * * *

ARITHMETIC - 16th CENTURY STYLE

Margaret Jackson

THE
Grounde of Artes
teaching the work and pra-
ctiſe of Arithmetike both in whole numbers
and Fractions, after a more easyer and
exacter ſorte than any like hath hither-
to bin ſet forth :

Made by M. ROBERT RECORD
Doctor in Phyſike, and now of late diligently
overſeene and augmented with newe
and neceſſarie Additions.

The above wording forms part of the title page of a book printed in 1575. The earliest edition of the *The Grounde of Artes* was published in 1542 and is thought to have been the first book of arithmetic printed in English. Except for the occasional use of block letters and modern style lower case type, Old English script is used throughout. Its 508 pages are unnumbered and the spelling is arbitrary. Several poems are included.

The author is the mathematician credited with devising the familiar "equals" symbol. His colourful life included posts as Controller of the Mint at Bristol and physician to both Edward VI and Mary Tudor. He died in 1558 while in prison for debt.

As behoved a wary 16th Century writer, Recorde begins his work with a 16-page fulsome dedication:

"TO THE MOSTE mightie Prince Edwarde the ſixthe, by the grace of God King of Englande, Fraunce, and Irelande", etc.

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10 *Good Luck, Wish You Well in Your Future Work, and, wait for it, Wish you be Cheerful in Your Coming Ordeals!*

* * * * *

This chapter is followed by a lengthy commendation addressed

"TO THE LO-
ving Reader
The Preface of M. Robert Recorde".

reminding him among other things that "Englyſſe men are inferior to no menne in mother witte".

The contents of the book are listed and the reader is warned that "Before the Introduction of Arithmetike, theſe Figures muſt be learned.

	1	2	3	4	5	6	7	8	9
Figures of Number	i	ii	iii	iiii	v	vi	vii	viii	ix

10	x
20	xx
30	xxx
40	xl
50	l
60	lx
70	lxx
80	lxxx
90	xc
100	c

Figures of Money

c a cee the xvi
q a kewe the viii
q̈ a farthing the iiii
ob. an halfe pennie
} part of a penny

∂ a pennie
$ a ſhilling
lb a pounde

Thou (O God) haſte ordered all things in Meaſure, Number, and weyght."

The text proper takes the form of "A DIALOGUE BETWEENE THE Mayſter and the Scholer, teaching the arte and uſe of Arithmetike with the penne". The "scholer" is convinced of the necessity for a knowledge of arithmetic by such arguments as:

Mayſter: How many dayes in a weeke? how many weekes in a yeare? What landes hath your father? How many men doth he keepe? How long is it ſithe you came from him to mee?

Scholer: Mum.

Mayſter: So that if number you want, you anſwere all by Mummes: Howe many myle to London?

Scholer: A poke full of plummes.

Arithmetic with a pen, in contrast to that using a counting board or an abacus requires an understanding of the Hindu-Arabic system of numeration and the use of algorithms impossible to perform using the then familiar Roman numerals. Recorde teaches the Hindu-Arabic system using the labelled columns familiar to many modern primary school children. The general emphasis is on the learning of rules of operation rather than on the whys and wherefores of each algorithm.

Now for a 16th century version of long division – omitting the lengthy instructions given at each stage. See whether you can follow it.

It is required to divide 263 845 by 64.

$$\begin{array}{l} 263845) \\ 64 \end{array}$$

$$\begin{array}{l} 2 \\ \cancel{2}\cancel{6}3845(4 \\ \cancel{6}4 \end{array}$$

$$\begin{array}{l} \cancel{2}7 \\ \cancel{2}\cancel{6}\cancel{3}845(4 \\ \cancel{6}\cancel{4} \end{array}$$

$$\begin{array}{l} \cancel{2}7 \\ \cancel{2}\cancel{6}\cancel{3}845(4 \\ \cancel{6}\cancel{4}4 \\ 6 \end{array}$$

$$\begin{array}{l} 1 \\ \cancel{2}\cancel{7} \\ \cancel{2}\cancel{6}\cancel{3}845(41 \\ \cancel{6}\cancel{4}4 \\ \cancel{6} \end{array}$$

$$\begin{array}{l} 1 \\ \cancel{2}\cancel{7}4 \\ \cancel{2}\cancel{6}\cancel{3}\cancel{8}45(41 \\ \cancel{6}\cancel{4}\cancel{4} \\ \cancel{6} \end{array}$$

$$\begin{array}{l} 1 \\ \cancel{2}\cancel{7}4 \\ \cancel{2}\cancel{6}\cancel{3}\cancel{8}45(41 \\ \cancel{6}\cancel{4}\cancel{4}4 \\ \cancel{6}6 \end{array}$$

$$\begin{array}{l} \cancel{1}2 \\ \cancel{2}\cancel{7}\cancel{4} \\ \cancel{2}\cancel{6}\cancel{3}\cancel{8}45(412 \\ \cancel{6}\cancel{4}\cancel{4}4 \\ \cancel{6}\cancel{6} \end{array}$$

$$\begin{array}{l} 1 \\ \cancel{1}\cancel{2} \\ \cancel{2}\cancel{7}\cancel{4}6 \\ \cancel{2}\cancel{6}\cancel{3}\cancel{8}\cancel{4}5(412 \\ \cancel{6}\cancel{4}\cancel{4}\cancel{4} \\ \cancel{6}\cancel{6} \end{array}$$

$$\begin{array}{l} \cancel{1} \\ \cancel{1}\cancel{2}4 \\ \cancel{2}\cancel{7}\cancel{4}\cancel{6} \\ \cancel{2}\cancel{6}\cancel{3}\cancel{8}\cancel{4}5(4122 \\ \cancel{6}\cancel{4}\cancel{4}\cancel{4}4 \\ \cancel{6}\cancel{6}\cancel{6} \end{array}$$

$$\begin{array}{l} \cancel{1}3 \\ \cancel{1}\cancel{2}\cancel{4} \\ \cancel{2}\cancel{7}\cancel{4}\cancel{6}7 \\ \cancel{2}\cancel{6}\cancel{3}\cancel{8}\cancel{4}\cancel{5}(4122 \\ \cancel{6}\cancel{4}\cancel{4}\cancel{4}\cancel{4} \\ \cancel{6}\cancel{6}\cancel{6} \end{array}$$

Thus the quotient
is 4 122
"and 37 remayneth."

Of course a problem arises if the divisor is a number such as 29, and the way to tackle these cases is prescribed in detail.

As well as the four operations on whole numbers and fractions, *The Grounde of Artes* has chapters on money, length and weight, proportion, progressions and many problems to be solved. How would *you* solve this one?

"A man lying uppon his death bed, bequeatheth his goodes (which were woorth 3600 crownes) in this ſorte. Bicauſe hys wyfe was great with childe, and he yet uncertaine whether the childe were a male or a female, hee made his bequeſt conditionally, that if his wife bare a daughter, then ſhould the wyfe have halfe his goodes, and the daughter $\frac{1}{3}$, but if ſhe were delivered of a ſonne, then that ſonne ſhoulde have $\frac{1}{2}$ of the goodes, and his wife but $\frac{1}{3}$. Now it chaunced hir to bring forthe both a ſonne and a daughter, the queſtion is: How ſhall they parte the goodes agreeable to the teſtatour his will".

The scholer shows some knowledge of the world and replies that:

"If ſomme cunning Lawyers had this matter in ſcanning they woulde determine this Teſtament to be quite voyde ...".

However, after some long-winded reasoning, the mayſter concludes that the son receives $1705\frac{5}{19}$ crowns, the wife $1136\frac{16}{19}$ and the daughter $757\frac{17}{19}$. I leave you to discover how he reaches this conclusion. (Hint: $\frac{1}{2}$, $\frac{1}{3}$ shares are taken to mean a distribution in the ratio of $\frac{1}{2}$ to $\frac{1}{3}$.)

I thought an appropriate ending to this article should be the final stanza of the poem which graces the title page of Recorde's *Grounde of Artes*. Here it is.

Of Numbers vſe, the endleſſe might,
No witte nor language can expreſſe,
Applye and Trie, bothe day and night,
And then this truthe thou wilt confeſſe.

* * * * *

William Pitt (1759-1806) was prime minister of England from 1784 (when he was 25) until 1801. The 9th Edition (1885) of the Encyclopaedia Britannica, in its article devoted to Pitt at one stage comments: "This narrative has now reached a point [1784], beyond which a full history of the life of Pitt would be a history of England, or rather of the whole civilised world; ...".

This article also comments about Pitt in his undergraduate years at Cambridge: "The work in which he took the greatest delight was Newton's *Principia*. His liking for mathematics, indeed, amounted to a passion. ... The acuteness and readiness with which he solved problems was pronounced ... to be unrivalled in the university".

THE MIDDLE SQUARE RANDOM NUMBER GENERATOR

Robert Dunne, Murdoch University

What are random numbers? Clearly there is no sense in which any given number is random, 2 is not a random number, it is always equal to 2. Randomness is a difficult concept to define, but we all think that we can recognise it in practice. It is clearly a property of sequences of numbers rather than of individual numbers and is recognised, in the first instance, by a lack of pattern.

For example, consider that standard of examples in probability, a single die being thrown repeatedly, however in our case we use a die with 10 faces[1] numbered 0 to 9. If you recorded the numbers on the successive faces of the die you would obtain a sequence of random numbers. It might seem the least interesting of all possible sequences of numbers in that it has no order or structure, but this is not the case. If you were to try to write down a sequence of random numbers it would probably not be too hard to distinguish your sequence from the true random sequence. When people try to write down random numbers they tend to get the proportions correct, i.e. about $\frac{1}{10}$ of them will be 1's and about $\frac{1}{10}$ will be 2's, etc., but generally they don't include enough runs of the same number. In a genuine random sequence about $\frac{1}{10}$ of adjacent pairs of random numbers should be the same. In addition a number of patterns, groups of high or low numbers, ascending or descending numbers, etc., are likely to occur with a certain regularity in genuine random sequences. As various people have noted before, global randomness must include a certain degree of local order. It is this aspect of randomness that leads to many of the misconceptions of people who think that they can derive systems for winning at games of chance. For example, contrary to at least one piece of commercially available software for designing such systems, the fact that a number has appeared in a "lotto game does not mean that it is less likely to appear again.

I will use the word "random" to mean a sequence with the same properties as such a die generated sequence, regardless of how it is produced. This leaves a large number of questions unanswered, as we have barely looked at these properties; however, it will do for the time being, and is, in any case, not totally unreasonable. If we went on to explore, and understand the properties of a truly randomly generated sequence, we would still want any sequence we called random to share those properties.

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[1] Such a die could be constructed as a diamond shape with 5 triangular faces meeting in a point on the upper half and the lower half a mirror image of the upper half; alternatively it could be made like a "pencil" with 10 flat facets and a curved surface at each end. Either way, we would have a figure that had 10 facets each with an equal probability of being selected by virtue of the symmetry of the object.

A source of random numbers, believe it or not, has many uses. Random numbers are used in areas as diverse as games and nuclear physics. Anywhere that we want something different and unpredictable (within a range) to happen on successive trials may be a potential place to use random numbers.

The earliest use of random numbers, apart from their widespread use in games of chance, was in the area of statistical sampling. Imagine the situation where you have 1000 objects and you can afford to test only 10 of them for some property. How do you pick out the 10 without introducing some bias? If you just take the ten that you come to first, you might have an unrepresentative sample. One approach is to number the objects 0 to 999 and then consult a list of random numbers between 0 and 999, take the first 10 numbers from the list and sample those objects.

Other uses of random numbers are largely tied up with computing. Computers have given us both the ability to generate sequences of numbers with random properties, and also the need for them. The development of modern high-speed computers has made it possible to simulate many real events on a computer, so that processes that cannot be subjected to direct experiment can now be explored.

It is now almost routine in many areas of science to produce a computer simulation of an event before or instead of trying to produce the actual event. The availability of this tool has been of great benefit to such people as astro-physicists and cosmologists who can now use a mixture of computer simulation and observation in their work. For example, current theories about the early stages in the evolution of the universe can now be used as the basis of computer simulations to see if they lead to the sort of distribution of matter that we observe around us.

Clearly, such a theory cannot specify the position and velocity of every particle. What it does is to give the general range within which they lie, and then the simulation uses a random number generator to give reasonable values to each particle.

How do we go about the task of picking random numbers? There are several possible approaches:

1) one could use a roulette wheel, or a die, and record the results of successive trials. One could then use this record as a source of random numbers.

2) one could abandon the truly random numbers and generate a sequence of numbers with properties "random" enough to suit the particular application.

Both of these approaches have been used in the past. The most famous example of the first approach is contained in a book published by the Rand Corporation in 1955, called "*A million random digits with 100,000 normal deviates*", and which contains just what it says. The problems with this approach can be appreciated just by looking at the book – it takes a lot of space to store such a large list of numbers.

The second approach has a number of attractions. If we could find a way to generate random numbers (or more correctly, "pseudo" random numbers) as we needed them, we would have as many random numbers as we needed without any storage problems.

Note that the desirable qualities of a random number generator depend on the use to which it will be put. The sort of generator used in a video game only has to be random enough to stop the player predicting what will happen next, whereas one used in a cosmologist's simulation would have to be a much more convincing generator.

There is a number of possible approaches to generating pseudo random numbers. One is to take a number like π and calculate it to the appropriate number of places, $\pi \approx 3.1415926536\ldots$; the digits appear to be random and exhibit no discernible order for as far as they have been calculated. The problem with this method is that it takes a lot of computing effort to calculate a number like π to a very large number of places.

An alternative method, and the one generally used, is to use a recursive function on the integers. This is a function that takes integer values such that its $(n+1)^{\text{th}}$ value is defined in terms of its n^{th} value, e.g.,

$$X_{n+1} = f(X_n)$$

so that given X_0 we can calculate $X_1, X_2, \ldots$ etc.. Recursive functions form a rich and important area of mathematics with many applications, particularly in the area of computing. It is clear that as the X_{n+1} term depends on the X_n term, if our random number generator ever picks the same number twice, it will repeat the same cycle of numbers indefinitely. This is not what occurs in a sequence of true random numbers and is a serious limitation of the pseudo-random generators that we are considering. Clearly all of these generators are going to give a recurring cycle eventually. The only defence against this is to make the range of the generator as large as possible. For example, consider a random generator giving the integers 0,1,2,3,4,5,6,7,8,9. This can give at most 10 digits before starting to repeat a cycle, and it is unlikely that a list of ten or fewer random digits would have wide applications. We call the length of the run the period of the generator.

The Middle Square

One function that can be used for this is the middle square function, first suggested by the noted mathematician John von Neumann in 1946. This is readily illustrated with an example: suppose we were looking for random numbers between 0 and 100. We would start with a "seed" (an X_0 value), for example 57, we would then square it to get 3249 and our next random number would be the middle digits 24. We would then simply repeat the process.

If you try to continue this example, you will discover a major problem with this sort of generator is that it may have a very short period, in this case only two numbers, 24 and 57. Often this method will give a sequence of apparently random numbers before falling into a very short repeated cycle, in some cases the same number over and over again. I wrote a short program to calculate the middle square repeatedly and found that the method has a number of problems. The best "run" I found started with a "seed" of 69 and proceeded like this:

69,76,77,92,46,11,12,14,19,36,29,84,5,2,0.

When a 0, 10, 50 or a 60 appears the sequence repeats that number endlessly, and all the "seed" values between 1 and 99 degenerate to the 24, 57 cycle or to one of these numbers very quickly, most commonly to 0. If the middle square is 09 or less then the sequence will go to 0 (try an example to see why), and we would expect that about $\frac{1}{10}$ of the numbers generated will be in this range if the sequence is mimicking the properties of random numbers adequately. Hence it is not surprising that many of the sequences end up at 0. What is interesting is that for two digit numbers there is only one 2-cycle, namely, 24, 57 (this occurs just three times with a seed between 1 and 99) and no cycles of longer duration.

Here is a program in the language TURBO-PASCAL that you can use on your PC. It prints out the seed, on one line, and then on the next line the successive numbers generated by this seed. You enter the seed of your choice when asked to do so.

```
program MIDSQUARE (INPUT, OUTPUT);

var
    N,K :INTEGER;

begin
    WRITELN ('Enter an integer between  1  and  99:');
    READLN (N);
    WRITELN (LST,N);
    K := 0;
    for  K := 0  to  15  do
         begin
              N := SQR(N);
              N := N div 10;
              N := N mod 100;
              WRITE (LST,N, ' ');
         end;
              WRITELN (LST)
end.
```

Numerous attempts have been made to modify the original "middle square" method so that it gives better results, including such things as adding a constant before doing the squaring, and using larger numbers to reduce the probability of the sequence ending up at 0. Some of these changes have made improvements but in general this method has fallen out of favour as other methods have been found to give better results.

You might care to write your own program to do this and see if you can improve on my results.

* * * * *

A WALK ACROSS THE BRIDGES

Michael A.B. Deakin, Monash University

North of the eastern part of Poland and south and west of Lithuania is a piece of land now part of the Russian Soviet Socialist Republic, although it is not connected with the rest of that state. Before World War II, however, this area was part of Germany and it was called East Prussia. In those days, its principal town, now called Kaliningrad, was known as Königsberg (or Koenigsberg) and (though other names like Queenisbrig and Ginsburg have also been current), this is the name by which mathematicians commemorate it.

Königsberg stood at the confluence of two branches of the river Pregel (now called the Pregolya). The New Pregel and the Old Pregel joined together to form the Pregel, but the geometry is complicated by the fact that there were two channels uniting the branches and cutting off an island as shown in Figure 1.

Back in 1735, the different parts of the city (the north, south and east banks and the island) were connected by seven bridges, also shown in Figure 1.

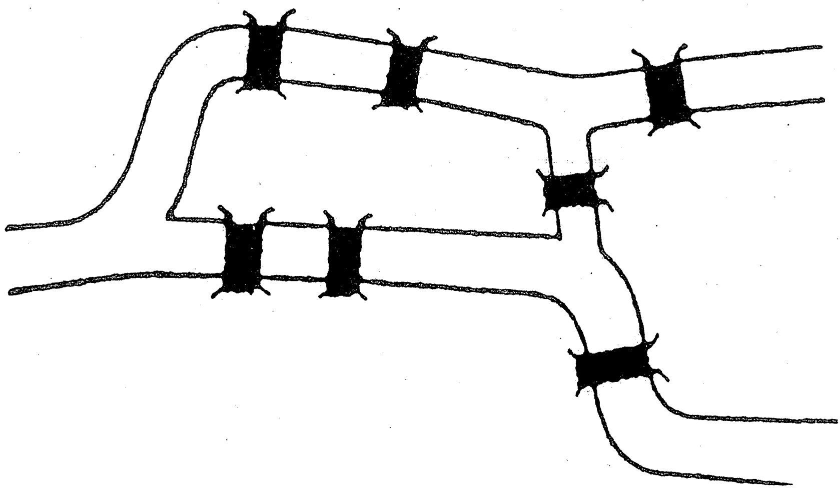


Fig. 1

Someone, we don't know who it was but it may have been Carl Ehler, the mayor of Danzig (now Gdansk), became interested in a problem this geographic situation posed for the inhabitants of Königsberg. Interested enough to write to Leonhard Euler (the name, incidentally, is pronounced *Oiler*), the greatest mathematician of the day. Euler at this time was in St. Petersburg (now Leningrad), then the capital of Russia.

The problem which so intrigued the inhabitants of Königsberg was this. Was it possible to start from some point in the city and return to that point after a journey in which each bridge was crossed once and only once?

Euler was able to answer the question (the journey can't be done) and he addressed the St Petersburg Academy of Sciences to present his analysis. This was on the 26th of August 1735. Later, in 1741 (though the publisher's date was 1736 – these discrepancies were common then and are not unknown today), the Academy published a Latin text of his address and this may now be read – the more easily as an English translation was published in 1976.

In the paper, a slightly altered version of Figure 1 is presented. This is reproduced here as Figure 2. Note that the apparent differences between Figures 1, 2 make *no difference whatsoever* to the problem as posed. What is important is that there are 4 distinct land areas – already Euler calls them A, B, C, D – and 7 bridges (a, b, c, d, e, f, g) and that these stand in a certain relation to one another. For example, it is important to the problem that Bridge c connects land areas A, C, etc. But every point on the island A is equivalent to every other point, for these points may all be reached, the one from the other, without crossing any bridges. The same is true for the banks B, C, D.

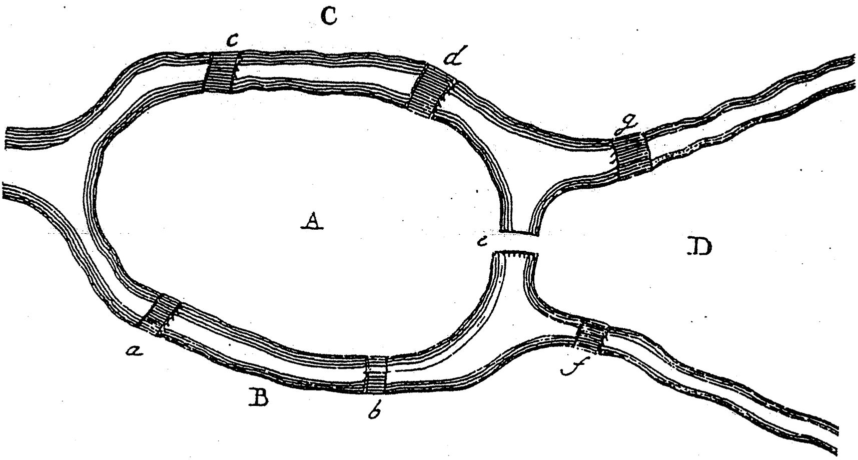


Fig. 2

We are not interested in the usual questions that classical geometry posed but in a different sort of question that Euler characterised as belonging to "positional" geometry, a geometry of "where we are". Euler next generalised the problem. Clearly the *specific* problem of the Königsberg bridges can be solved by the tedious, but quite practical, method of trying all available routes. But what interested Euler was the problem of finding whether *any* such problem can be solved. Even here we could list all possible routes and test each one, but Euler rejected this way of attacking the problem.

Rather, he considered journeys as sequences of capital and lower case letters. So, for example, the sequence

$$A\ c\ C\ d\ A\ e\ D\ f\ B\ b\ A$$

represents a journey from the island back to the island, visiting all banks and crossing five of the seven bridges.

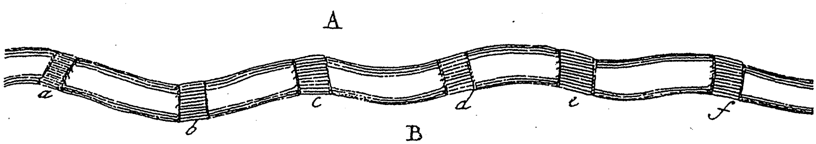


Fig. 3

We can use this same principle in other problems. Figure 3, for example, shows a number of bridges spanning one river with two banks A, B. Now if the number of bridges is even, we can set out from A, cross to B, return to A, etc. by a route

$$A\ a\ B\ b\ A\ c\ B\ d\ A\ e\ B\ f\ A$$

for example, crossing each bridge exactly once and finally reaching A. If, on the other hand, the number of bridges is odd, we end up at B and cannot return to A without violating the conditions of the problem.

But this now gives us a handle on the Königsberg bridge problem. For consider one of the areas, say A, and lump together all the others as "not-A". Then if an odd number of bridges leads from A to not-A, a journey beginning in A and crossing all these bridges lands us in not-A with no way of getting back. There must, therefore, if the problem is to be soluble, be an even number of bridges connected to A, and similarly for all the other areas, B, C, etc.

Thus in the case of the Königsberg bridges, where A is met by five and each of B, C, D by three, clearly no solution is possible.

But if we go back to Figure 3 (but with an odd number of bridges), we *can* make a journey crossing each bridge once and once only if we are content to drop the condition about our journey's being a round trip – making it a journey "from here to there". Would *this* be possible in Königsberg?

Such a journey is in fact possible in the case of Figure 4. (Can you find it?) It starts in D and ends in E (or vice versa). Here Euler showed, and it isn't difficult to reconstruct his reasoning (a task I leave to the reader), that such journeys are impossible unless we have precisely two regions met by an odd number of bridges (the count for all others being even). Thus even this is not possible in Königsberg.

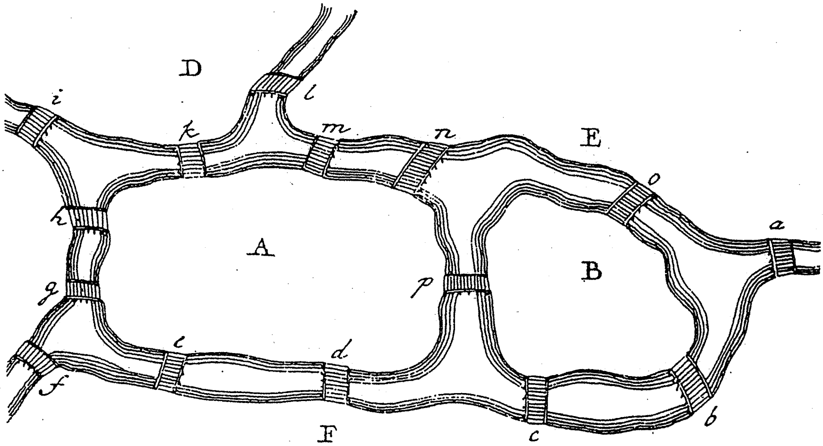


Fig. 4

There are other things in Euler's paper, but I will not pursue these except for the final remark it contains, on which I'll comment later.

It is clear from the tone of parts of Euler's paper (e.g. "I do not ... think it worthwhile giving any further details ...") that he regarded the question as a relatively trivial one, and this is the attitude adopted, by and large, in his correspondence on the topic, some of which has recently come to light. There is almost certainly more and it is quite possible that further items may one day be discovered.

There are three letters in what has been found. First is a letter from Ehler to Euler, dated 9/3/1736, and so later than Euler's solution. Ehler had heard of that solution and asked for a copy. It is clear that Ehler and Euler had exchanged earlier letters, but these are not currently available. Ehler wrote not only on his own behalf but also on behalf of his friend Heinrich Kühn, a professor of mathematics.

We do have Euler's reply to this letter. He wrote on 3/4/1736, and he rather dismisses the problem: "... this type of solution bears little relationship to mathematics ... the solution is based on reason alone and ... does not depend on any mathematical principle ...".

Similar thoughts occur in the third of the three letters, addressed to the Italian mathematician Giovanni Marinoni and dated 13/3/1736, "This question [the Königsberg bridge problem] is so banal, but seemed to me worthy of attention in that neither geometry, nor algebra, nor yet arithmetic was sufficient to solve it."

In this letter, however, he goes on to say that the other interesting thing about it is its possible relation to "positional" geometry, a topic Leibniz had envisaged, but in a rather vague way. This "positional" geometry has developed into topology in today's mathematics and Euler's contribution is now seen as the very first result in *graph theory*, a branch of that discipline.

[Oddly enough, Euler (in 1750) made another contribution to graph theory with his formula connecting the number of vertices (V), the number of faces (F) and the number of edges (E) of a simple polyhedron. The formula reads $V + F = E + 2$, and it is now basic to graph theory (see *Function, Vol.* 1, *Part* 1, p. 12), but Euler did not relate it to his "positional" geometry.]

Nowadays, we see the Euler analysis of the Königsberg bridge problem as being very much a part of graph theory – the origin of that theory, but more than that, the source of its first clear theorem. To put Euler's result into modern-day terms, we take "to its logical conclusion" the argument that identified Figure 2 with Figure 1. If the size or shape of the regions A, B, C, D is immaterial, there is nothing to stop us shrinking each down to a single point.

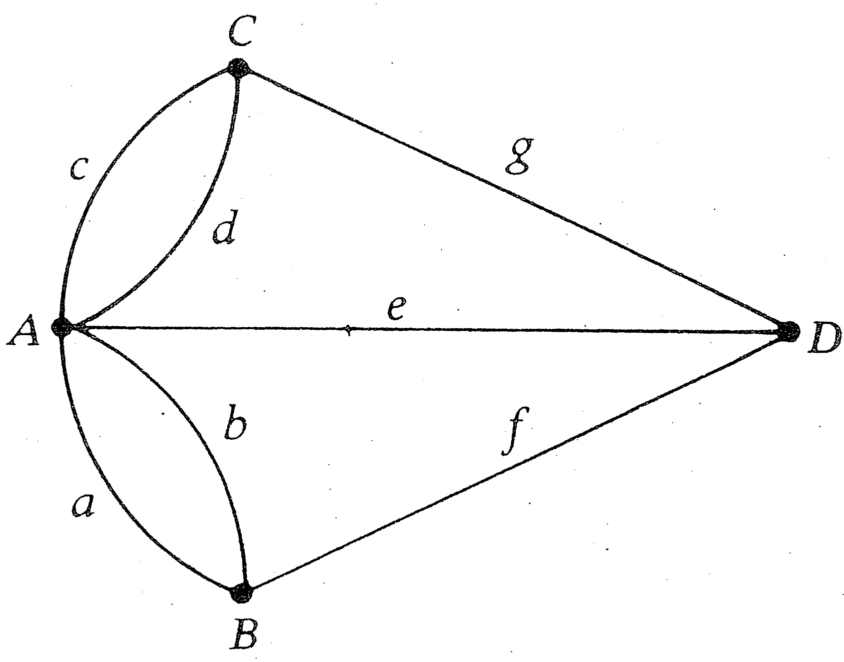


Fig. 5

Figure 5 shows the bare "positional" geometry of the situation. (This figure is *not* due to Euler as is often wrongly asserted. It first appeared in W.W. Rouse Ball's book *Mathematical Recreations and Problems*, the first (1892) edition.) It reduces the map of Königsberg to what we would now recognise as a *graph* in the sense in which that word is used in graph theory: i.e. a set of points some of which are connected to others by lines.

A journey traversing each of these lines once and once only and starting and ending at the same point is nowadays known (in honour of Euler and the Königsberg bridges problem) as an *Euler path*. The existence or otherwise of an Euler line for any graph at all is easily settled.

We need two definitions:

1. A graph is *connected* if every point is joined (by one or more lines) to every other point.

2. The *degree* of a point is the number of lines that connect to it.

We then have a *theorem*.

If a graph has an Euler path, it must be connected and all its points must have even degree, and vice versa.

As we have seen Euler proved half this theorem. His proof is not very different from the ones found in standard texts today (see, e.g., Oystein Ore's *Graphs and their Uses*). He did not, however, prove the "vice versa" part, which is rather harder.

If we check that a graph is connected and that all its points have even degree, this means that we can't rule out the possibility of an Euler path (in the way that we can for Figure 5); we are not guaranteed such a path unless we prove, as Euler did not, the second half of the theorem.

Euler has some passages that may be read as implying an oversight of this point - or it could be that subconsciously he intuited the proof and assumed that others would do so as well.

The missing part of the proof was not supplied explicitly till much later by the German mathematician Carl Hierholzer, who proved the result in 1871. In point of fact, Hierholzer discovered the result independently for he was unaware of Euler's paper. Hierholzer died before he could publish his work and the paper that finally appeared (in 1873) and supposedly was authored by him was in fact written jointly by two colleagues (though the ideas presented are of course Hierholzer's).

That proof was in fact a method by which an Euler path may be constructed. A more accessible account appears in Ore's book, referred to above. It runs like this.

Suppose we begin a path at some point A and continue as far as we can, always leaving a point by means of a line not travelled before. As the number of lines is finite, sooner or later we must run out of lines. But at each point there is an even number of lines and so there will always be an exit line corresponding to an entry line. Except at A, where one of the even number of lines was used up at the outset, leaving an odd number. Thus the path ends at A.

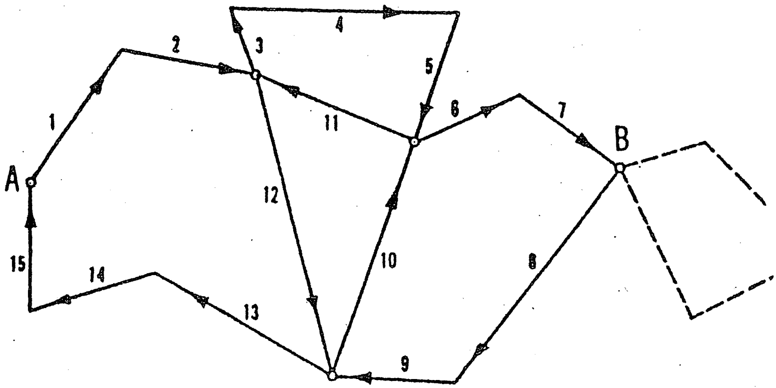


Fig. 6

Now either:

(a) All the lines of the graph have been traversed,

or (b) There are some lines that have not.

In Case (a) our task is complete. In Case (b) there must be some point B lying on the path we have covered but met by lines not so far covered. (This is because the graph is connected.) In Figure 6 (based on a diagram in Ore's book), for example, the point B lies between the 7th and 8th legs of the journey. The rule is to modify the journey by interrupting the old route at B and including a new round trip from B to B (along previously untraversed lines) before resuming.

It may be necessary to add in several such additional journeys, but ultimately (because the number of points is finite) we cover the whole graph.

Euler does not mention this, although it is just possible he took it for granted, and thus Hierholzer is today credited with the full proof of the theorem.

Three points need to be made.

First, if the graph is connected and all points save two, B and C let us call them, have even degree while B, C have odd degree, then although no round-trip Euler path can exist, there is a one-way path from B to C (or vice versa) traversing each line exactly once and these are the only circumstances in which such a journey is possible. (This was in fact the version of the theorem proved by Hierholzer.)

Second, if an Euler path exists, it is usually not unique – often there will be many Euler paths, given that an Euler path can exist. It has become a matter of some interest to see how many. However, I will not pursue this here.

Third, the proof of the "vice versa" part is actually a technique whereby an Euler path can be found. Euler concluded his paper by saying "When it has been determined that such a journey can be made, one still has to find how it should be arranged". He gives some quite useful guidelines, but not the above details – this is the reason for our modern view in awarding the credit for the more difficult part of the theorem to Hierholzer rather than to Euler.

There have been other attempts to analyse the actual situation at Königsberg. In 1984, Roger Cooke, an American mathematician, pointed out that in 1800 or thereabouts a ferry had been in service between the southwest point of the island and the north bank (traversing the New Pregel) connecting therefore (see Figure 2) A and C. By using this ferry, therefore, it would have been possible to travel from B to D crossing each of the bridges (and travelling on the ferry) exactly once.

Another "cheating" approach to the problem was reported in 1876 when it was found that a new railway bridge had been built to connect B to C – i.e. rather downstream from Königsberg proper. This too (supposing that one could manage to walk across the railway bridge) enabled the one-way journey.

It might be of some interest to know about the situation today, but alas, that is not possible. Königsberg was bombed out of existence on 30/8/1944. It has been rebuilt (as Kaliningrad) and from time to time the claim is made that it was restored to its original state. This is almost certainly not so – but it is a claim that cannot easily be checked, as Kaliningrad is now a naval base and subject to very tight security. Even were this not so, accurate maps would be very hard to come by. Recently, in an exercise in *glasnost*, Mr. Gorbachev admitted that the published maps of all Soviet cities were falsified. The only correct map of Moscow is in fact issued by the American CIA. The CIA would also have an accurate map of Kaliningrad (from spy satellites), but because Kaliningrad *is* a Soviet naval base, it is unlikely that we mere mortals could get hold of it. So we don't know if modern Kaliningrad has an Euler path or not.

Further Reading

This article is based on a number of sources, all quite accessible in that they are not especially difficult and can be found in university libraries. A general article is Robin Wilson's "An Eulerian Trail through Königsberg", published in the *Journal of Graph Theory*, *Vol.* 10, *No.* 3 (1986), pp. 265–275. Euler's correspondence is described in the same journal, *Vol.* 12, *No.* 1 (1988), pp. 133–139. An English translation of Euler's paper is given in the book *Graph Theory, 1736-1936*, by N.L. Biggs, E.K. Lloyd and R.J. Wilson (Oxford University Press, 1976). A good introductory text is O. Ore's *Graphs and their Uses* (New Mathematical Library, *Vol.* 10, 1963).

Background data on Königsberg were supplied from German sources by my colleague (and fellow *Function* editor) Hans Lausch. To whom my thanks.

* * * * *

YET ANOTHER CONSTRUCTION OF AN EQUILATERAL TRIANGLE

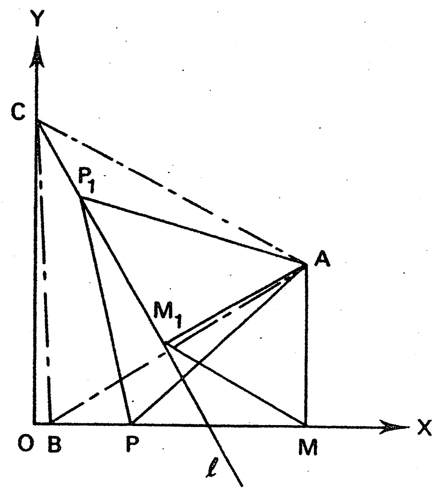
Esther Szekeres

J.C. Burns (*Function* Volume 12, pt. 4, p. 99) in his interesting article in the August 1988 issue of *Function* discusses three different approaches to solve the following problem:

Given the two axes of a rectangular coordinate system OX and OY, and a point A in the system, construct an equilateral triangle ABC, where A is the given point, B is a point on OX, C a point on OY.

I would like to describe a fourth approach which solves the problem without any calculations and can be easily modified to solve a more general problem.

Consider all equilateral triangles with one vertex at A, a second vertex, P, on OX, and ask what is the locus of the third vertex, P_1, as P describes the line OX. For simplicity, take all triangles with negative orientation (see diagram). Drop a perpendicular from A to OX, meeting it in point M. Then ΔAMM_1 is the smallest of these triangles. If APP_1 is an arbitrary equilateral triangle in the above position, the triangle AMP will be congruent to triangle AM_1P_1 as both the sides AM and AP have been subjected to a clockwise rotation of 60° to reach the positions AM_1 and AP_1 respectively. That means that $\angle AM_1P_1 = 90^\circ$ for all positions of P_1, so P_1 describes a straight line ℓ, which is simply the line we get if we rotate OX clockwise by 60° around the given point A. Now ℓ will intersect OY in C and we find B on OX by making $AB = AC$. Symmetrically, we may rotate OX anticlockwise around A by 60°, getting an anticlockwise triangle ABC.



Clearly, this method can be applied without any change if the given lines are any two arbitrary lines, k and m. If the rotation of k produces a line ℓ, which is parallel to m, then there is no solution; if it is coinciding with m then there are infinitely many solutions.

It is not difficult to see that a small modification of this construction will enable us to solve the following more general problem:

Given any triangle XYZ, construct a triangle ABC such that vertex A should be at a given point, B and C should be on two given lines ℓ and m respectively, and ΔABC should be similar to ΔXYZ.

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SUMMATION OF FIBONACCI TERMS AS DECIMALS

Garnet J. Greenbury, Brisbane

Consider the Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 397, 610, 987, 1599, 2584, ...

Summation of every term

Arrange successive terms under one another so that each one is moved one place to the right; add them up, and interpret the sum, as shown, as a decimal.

```
 0
  1
   1
    2
     3
      5
       8
        13
         21
          34
           55
            89
             144
              .
              .
              .
-----------------
 ·011235955056 ...
```

This is the decimal equivalent of 1/89.

Summation of every second term

Now repeat the procedure, starting at 0, but now using only every other term. Show that this time the terms add up to 1/71.

Can you discern a general pattern?

Letter to the Editor

In Volume 10, Part 5 (October 1986) of *Function*, I wrote about a number of mathematicians who were also active in the religious life of the Catholic church and who might one day become candidates for sainthood. The article was prompted by the news that Francesco Faà di Bruno, an Italian mathematician who lived from 1825 to 1888, had in fact progressed some distance down this road. In the course of the article, I made reference to a number of members of the Jesuit order who had achieved fame as mathematicians.

The purpose of this letter is to give details of two more Jesuit mathematicians. I inadvertently overlooked these when I wrote before and I now rectify the omission.

One name I should have mentioned is that of *Gregory St. Vincent* (1584–1667). (St. Vincent was his *surname*. He has not been canonised.) St. Vincent was a pupil of Christoph Clavius (1537–1612), who *was* mentioned in the earlier article (as the originator of our modern calendar). St. Vincent wrote on comets, mechanics and geometry and is best remembered for work which is now seen as contributing to early versions of the integral calculus. He was an influential teacher and indeed taught several other Jesuits who have some small place in the history of mathematics.

The second person I should have mentioned is *Matteo Ricci* (1552–1610). Ricci was not perhaps as creative or as influential a mathematician as St. Vincent, but he is a much better-known figure and his life was certainly a colourful one. The last 27 years of it were lived in China, where he worked as a missionary.

He is remembered, both within and outside the Catholic church, for what the *Dictionary of Scientific Biography* (DSB) calls "his complete adaptation to China". Indeed, he wrote many of the standard mathematics texts used in China at that time – often these were translations and adaptations of works by Clavius.

His major contribution, however, was the project he began and partially carried through: to translate the Geometry of Euclid into Chinese. This work was finally completed in 1865 by Alexander Wylie (an English Protestant missionary) and Li Shan-lan (a Chinese mathematician).

The DSB ends its account of Ricci by stating that "he was proposed for beatification in 1963 at the Second Ecumenical Vatican Council". Beatification, as I made clear in my earlier article, is one step removed from canonisation (the formal declaration of sainthood) and one step above the level of recognition accorded to Faà di Bruno.

The sentence is, however, puzzling. Ricci, as I learned by contacting the Jesuit order, has not been beatified, nor would it seem that even 26 years on has the process of his beatification made much headway. What the DSB may mean, and it can hardly mean more than this, is that in 1963, at the second Vatican council, some influential person suggested Ricci's beatification. The suggestion, however, appears not to have been acted upon.

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PERDIX

The Australian Mathematical Olympiad, competed in by high-school students from across Australia, took place on February 14th and 15th this year. On each day competitors had a four-hour paper, each consisting of 4 questions, to tackle. Here are the papers. Can you do the questions? Send me your comments and solutions.

Paper 1

Question 1

Let the number of different divisors of the integer n be $N(n)$, e.g. 24 has the divisors 1, 2, 3, 4, 6, 8, 12 and 24, so $N(24) = 8$. Determine whether the sum $N(1) + N(2) + \ldots + N(1989)$ is odd or even.

Question 2

Suppose BP and CQ are the bisectors of the angles B, C of triangle ABC and suppose AH, AK are the perpendiculars from A to BP, CQ. Prove that KH is parallel to BC.

Question 3

The integers† $u_1, u_2, u_3, \ldots$ satisfy the conditions

$u_1 = 1$, and

$$u_n = \frac{1}{u_1 + \ldots + u_{n-1}} \quad \text{for all integers } n > 1.$$

Prove that there exists a positive integer N such that $u_1 + u_2 + \ldots + u_N > 1989$.

Question 4

Let n be even. Four different numbers a, b, c, d are chosen from the integers $1, 2, \ldots, n$ in such a way that $a + c = b + d$.

Show that the number of such selections is $\dfrac{n(n-2)(2n-5)}{24}$.

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† "Integers" is wrong here, because the numbers $u_3, u_4, \ldots$ are not integers. So for "integers" read "numbers".

Paper 2

Question 5

Let n be a non-negative integer, $d_0, d_1, d_2, \ldots, d_n$ be each either equal to 0, 1 or 2 and $d_0 + 3d_1 + \ldots + 3^k d_k + \ldots + 3^n d_n$ be the square of a positive integer. Prove that $d_i = 1$ for at least one i, where $0 \le i \le n$.

Question 6

Four rods AB, BC, CD and DA are freely jointed at A, B, C and D and move in a plane so that the shape of the quadrilateral $ABCD$ can be varied. P, Q and R are the mid-points of AB, BC and CD respectively. In one position of the rods, the angle PQR is acute. Show that this angle remains acute no matter how the shape of $ABCD$ is changed.

Question 7

Let $f(n)$ be defined for positive integers n. It is known that

(*i*) $f(f(n)) = 4n + 9$ for each positive integer n, and

(*ii*) $f(2^k) = 2^{k+1} + 3$ for each non-negative integer k.

Determine $f(1789)$.

Question 8

Points X, Y and Z on sides BC, CA and AB respectively of triangle ABC are such that triangles ABC and XYZ are similar, the angles at X, Y and Z being equal to those at A, B and C respectively. Find X, Y and Z so that triangle XYZ has minimum area.

Asian Pacific Mathematical Olympiad

A new venture starts this year, the Asian Pacific Mathematical Olympiad. This first year is a trial run with just a small number of countries involved: Canada, Hong Kong, Singapore and Australia. I shall report on its results in the next issue of *Function*.

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