Function

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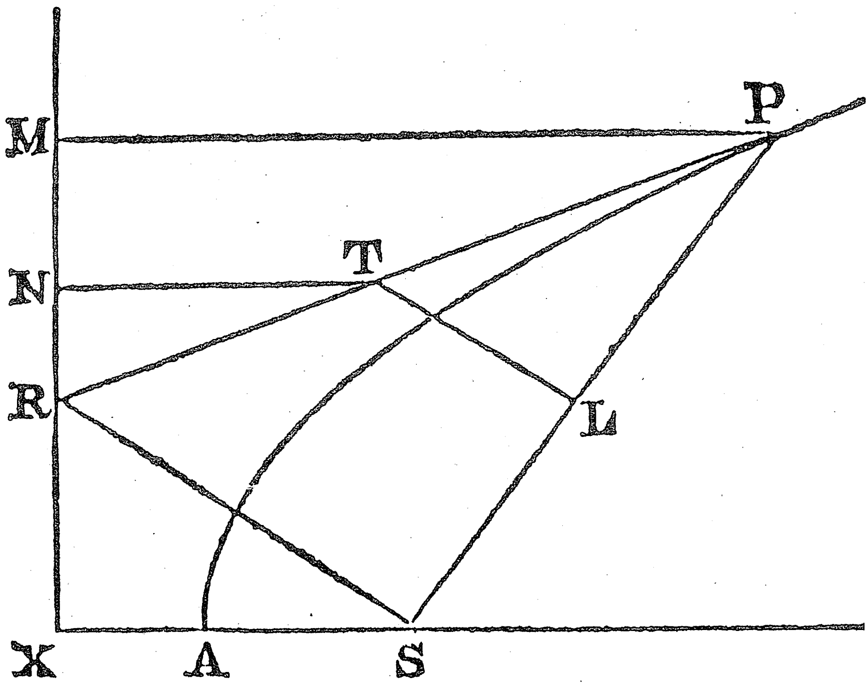

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Volume 14 *Part 2*

This issue carries as its main article a discussion by Robyn Arianrhod on symmetry ideas in modern Physics. Many of these find expression in the language of Group Theory, one of the most important and developed aspects of Abstract Algebra. It is interesting to reflect that a topic initially pursued for its own intrinsic interest and its connections to other branches of Mathematics should turn out to be of such importance in telling us the fundamental structure of the world in which we live. This is a perennial source of wonder to many — that mathematics is the language, as it has been put, in which the book of nature is written. There have been many theories as to why this should be so, but perhaps it's safest to say that we still don't really know.

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THE FRONT COVER

Reproduced opposite, for easy reference, is our front cover diagram. The curved line joining the points A, P is a *conic section* and it is formed in the following way. A fixed point S (called the *focus*) and a fixed line MX (called the *directrix*) are given. The *conic* section is the set of all points P for which the distance to the focus (SP) is a fixed ratio (called e, the eccentricity) times the distance (MP) to the directrix.

If $e = 0$, the directrix is at infinity and the conic section is a circle, for $0 < e < 1$, the conic section is an ellipse. The ellipse actually has two foci (and two directrices) and these are further away from the centre the larger e becomes. (Hence the name "eccentricity".) The ellipse lines up with its long axis perpendicular to the directrices. (Hence the name "directrix".) If $e = 1$, the curve is a parabola and if $e > 1$, a hyperbola (again with two foci and two directrices) results.

These are precisely the curves produced by the various cross-sections of a cone. Hence the name "conic section". For more on this, see *Function, Vol.* 10, *Part* 2.

The theorem illustrated by the diagram imagines the following situation. PR is the tangent to the conic section at the point P. T is any point on this tangent. The line TN is drawn perpendicular to MX and TL is perpendicular to SP. The theorem states that

$$SL = e \cdot TN.$$

It may be proved as follows. By another theorem on conic sections it is known that $\angle RSP$ is a right angle. As $\angle TLP$ is also a right angle, RS and TL are parallel. Thus the triangles RSP, TLP are similar and hence

$$\frac{SL}{SP} = \frac{RT}{RP}.$$

But the triangles RNT, RMP are also similar. Hence

$$\frac{RT}{RP} = \frac{TN}{PM}.$$

Thus

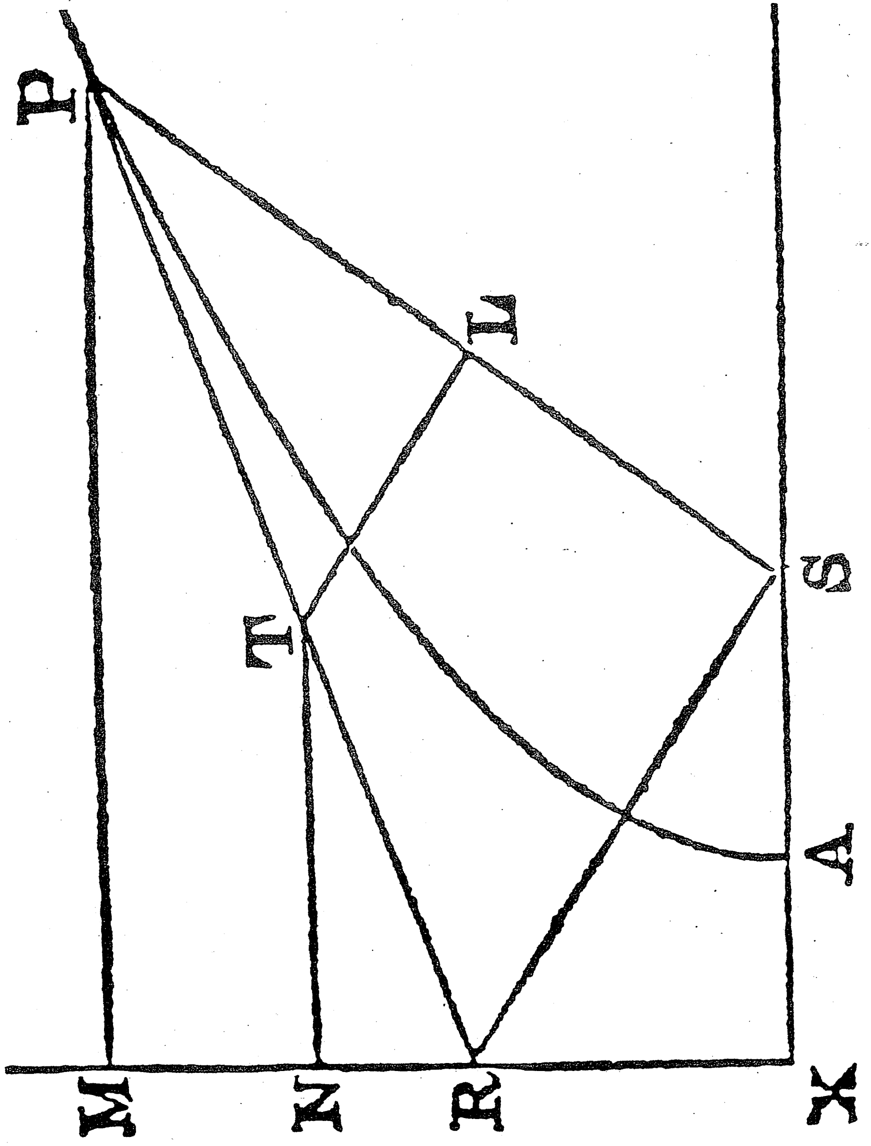
$$\frac{SL}{SP} = \frac{TN}{PM}.$$

Rearranging this equation, we have

$$\frac{SL}{TN} = \frac{SP}{PM}.$$

and, as this last ratio is equal to e, the result follows.

This theorem is taken from an 1881 book, *An Introduction to the Ancient and Modern Geometry of Conics*, by Charles Taylor, a Cambridge academic.


P
L
S
T
A
M
N
R
X

Taylor attributes it to a Professor Adams. Professor Adams was John Couch Adams (1819–1892) who in 1858 became the Lowndean Professor of Astronomy at Cambridge. He developed this and other theorems on conics in the course of a series of lectures on the lunar orbit delivered in 1861.

More recent books, such as Durell's *A Concise Geometrical Conics*, refer to the theorem as "Adams' property" of the conic. It has a number of corollaries (consequences). These may be proved by the use of Adams' property. One states that if from the point T we draw two tangents to the conic section, and these touch the conic section at P, Q (say), then either

$$\angle TSQ = \angle TSP$$

or

$$\angle TSQ + \angle TSP = 180^\circ.$$

Can you prove this result? And can you say under what circumstances the second possibility arises?

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GROUP THEORY AND SYMMETRY IN PHYSICS

Robyn Arianrhod, Monash University

A *binary* operation is a mathematical operation between *two* objects. For example, addition is a binary operation between two objects; most commonly, the objects on which addition operates are numbers, but we can also define the operation of addition so that it operates on objects called vectors, or matrices, or functions, say.

To generalise the idea of a binary operation on two objects, we can write

$$a * b$$

to mean any binary operation $*$ on any type of object a, b. With this notation, we can then generalise the theory of mathematical operations by speaking in *abstract* rather than particular terms. This, in fact, is the language of *group theory*, which is central to the mathematical study of such topics as mechanics (quantum and classical), atomic and molecular structure, and relativity.

A group is a set G of objects (called *elements* of the group G), together with a binary operation $*$, where for all $x,y,z \in G$,

$$x * y \in G$$

$$x * (y * z) = (x * y) * z$$

$$x * e = x = e * x,\ e \in G,\ e \text{ being a special element called the "identity element"},$$

$$x * x^{-1} = e = x^{-1} * x,\ x^{-1} \in G.$$

A group is said to be *abelian*[†] (commutative) if, in addition,

$$x * y = y * x, \ \forall x,y \in G.$$

For example, the set of vectors is an abelian group under the operation of addition (where $*$ has become $+$). The set of real numbers is an abelian group under the operation of addition (i.e. when $*$ becomes $+$), while the set of non-zero real numbers is an abelian group under multiplication, i.e. when $*$ becomes $\times$. The set of real, non-singular 2×2 matrices and the operation of matrix multiplication is a non-abelian group.

In Physics, one of the most important groups is that whose objects are *coordinate transformations*. A coordinate transformation is probably not the sort of object you had in mind when I spoke of generalising the notions of operations and objects! But this is part of the beauty of mathematics, that its structures *can* be so surprisingly abstracted. You will have been most likely to come across the idea of coordinate transformations in studying conics:

$$(x - a)^2 + (y - b)^2 = c^2$$

is the equation of a circle whose origin is at (a,b); however, we can transform the coordinates as follows:

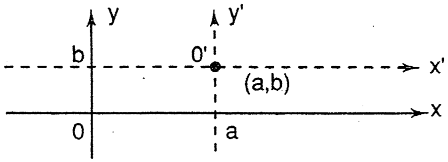
$$x' = x - a$$

$$y' = y - b$$

and write the equation for the above circle as

$$x'^2 + y'^2 = c^2.$$

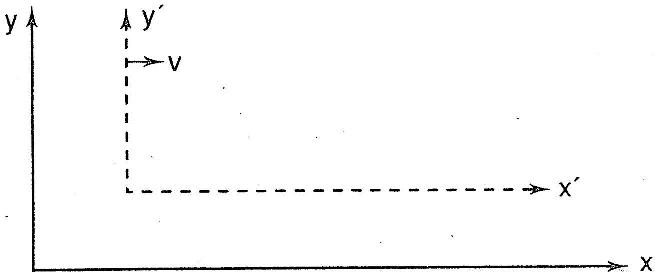
This is the equation of a circle with centre at the origin O' of the new coordinate system (x',y'):



This is an example of a *translation* in both the x and y directions: the coordinate axes are *translated* from their old positions to the new ones, in which the equation of the circle is much simpler. (The purpose of coordinate transformations *is* to make the equations of Physics simpler!)

You may also have seen the idea of coordinate transformations when considering the relative motion of two bodies, for example, a person on a railway station platform can be located with respect to axes painted on the ground, while a person in a moving train can be located by axes painted on the railway carriage floor.

[†] Named after the Norwegian mathematician Niels Abel (1802-1829).

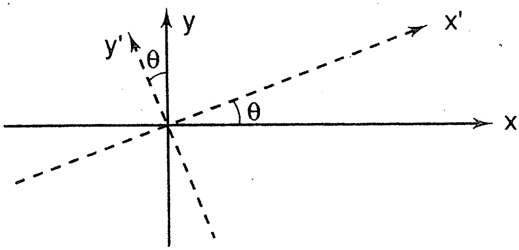


If the train is moving in the x direction with speed v, then at time t,

$$x' = x - vt \qquad (1)$$

if, initially, the y, y' axes coincided. This is another *translation*, but this time only in the x direction.

In addition to translating the coordinate axes, we can also *rotate* them, as, for example, when they are painted on a wheel.



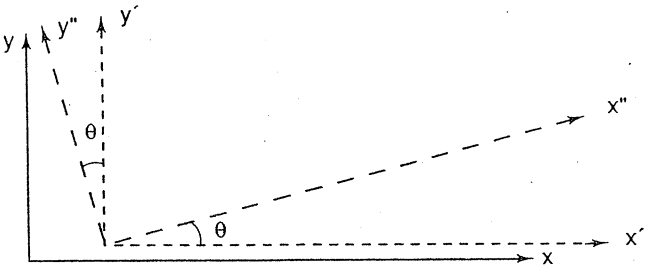
We can obviously extend these coordinate translations and rotations into three dimensions, translations of, and rotations about, all three axes being possible. Mathematically, we can extend these notions into any number of dimensions, for example, four for space-time.

Coordinate transformations can be expressed as *functions* of variables: for example, the coordinate transformation (1) above can be written as

$$x' = f(x,t) = x - vt.$$

Thus, we can consider the operation of composition of functions as a possible group operation for the objects called coordinate transformations: if we apply various coordinate transformations in succession — for instance, a translation in the x direction, followed by a rotation through θ —, then, mathematically speaking, we have performed the operation of "composition of functions". In the generalised notation of binary operations, we have this.

(x translation) * (θ rotation).



Note that this operation * is not abelian (show this). Does the set of two-dimensional (or three-dimensional) coordinate translations and rotations form a group? Yes, it does: can you find an identity and an inverse for each type of coordinate transformation?

The importance of groups of coordinate transformations in physics lies in their application to conserved, or "invariant", physical or geometrical quantities. The invariance of some aspect of a physical system under a group of coordinate transformations is called a *symmetry* of the system. For example, consider the physical system in which a particle moves in the x direction under the influence of a force F. In many important problems in Physics, the force on such a particle can be expressed as

$$F = -V'(x)$$

where $V(x)$ is a function of x and is called the *potential*. Then Newton's second law of motion can be written as

$$F = m\ddot{x} = -V'(x).$$

Here, the motion of the particle is one-dimensional — the particle moves along a single straight line, the x-axis, and so we will have only one-dimensional coordinate transformations, *viz.* spatial translations in the x direction, to consider as group elements in this problem. If the potential $V(x)$ is invariant under elements of this group, i.e.

$$V(x + x) = V(x), \text{ for all } c,$$

then clearly $V(x)$ must be independent of x. Then Newton's second law gives

$$m\ddot{x} = 0,$$

which, on antidifferentiation with respect to t, gives

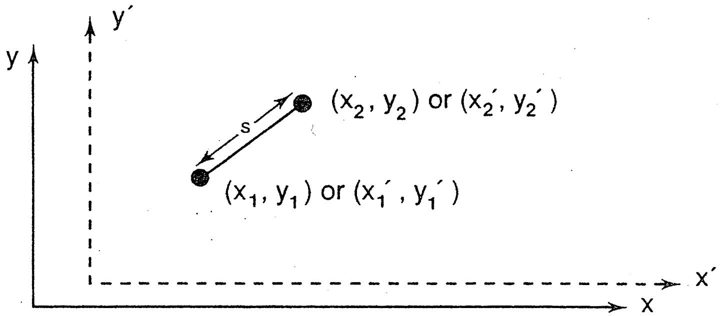
$$m\dot{x} = constant.$$

This is a mathematical derivation of the law of *conservation of momentum*, which has been established by physical experiment. The derivation shows that this law arises from an invariance of the physical system with respect to translations in space; this means that it doesn't matter where in space we do an experiment in this system. In other words, momentum is conserved because it is immaterial whether we observed the motion of such particles in Melbourne or in Perth.

Two other symmetries of physical systems which can be shown mathematically to lead to conservation laws are invariance of the system with respect to time, which implies that the energy of the system is conserved, and invariance of the system with respect to rotation, which implies that angular momentum is conserved.

Another, very different, application of this same group theory — that is, of the use of group theory in expressing invariant properties of a system from which can be derived physical conservation laws for the system — is that of relativity theory, both classical and Einsteinian.

The physical distance between two points in a plane is the same in each of the coordinate systems shown here.



$$s^2 = (distance)^2 = (y_2' - y_1')^2 + (x_2' - x_1')^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2.$$

The quantity "$(distance)^2$" is said to be *invariant* under the coordinate transformations

$$y \to y', \; x \to x'$$

shown in the diagram, and, in fact, s^2 is invariant under any of the coordinate transformations in the *group* of coordinate transformations in the plane — that is, the group of translations and rotations of the x and y axes.

Since distance is one of the fundamental quantities of (classical) physics, and is at the heart of all our physical laws of motion, this invariance of distance under the group of coordinate translations and rotations means that we would get the same physical results if we performed an experiment, involving measurement of distance, both in Melbourne and in New York, or on a railway platform and in a moving train. These results may seem obvious; why bother about group theory to deduce something so obvious?

Well, one answer is that, once the mathematical machinery is in place in a simple or obvious situation, it can be *generalised* to a less obvious one. Einstein did this in his theory of relativity, where he generalised three-dimensional space to four-dimensional space-time, with coordinates x,y,z,t; here, the laws of physics are invariant under the group of four-dimensional coordinate transformations, which is an obvious step to take mathematically, but not at all obvious physically as it requires such surprising assumptions as that the speed of light is constant (invariant) in all coordinate systems. (A velocity, including that of light, can be expressed as s/t, and we have just seen that the square of the distance s is invariant under coordinate transformations; in Einstein's theory, the equation of motion for light is, in four dimensions, $s^2 = constant$, and since s^2 is invariant under four-dimensional coordinate transformations, then the speed of light is so invariant.) This means that the speed of light is independent of the speed of its source, or, in practical terms, the speed of light is the same whether you are on a railway platform or in a speeding train — or a rocket speeding toward the sun. Think about this — it is not obvious, and it is not true for sound, which needs a medium through which to travel; since this medium (e.g. air) would be carried along by a moving frame, the sound it carries would have a different speed (frequency/sound) relative to an observer outside the moving frame (e.g. a train blowing its whistle) than it would to a person travelling with the frame and to whom the medium appears stationary.

Despite the fact that it is so hard to imagine, this idea of the constancy of the speed of light was actually an *experimental* result obtained in 1887 by Michelson and Morley. It was so counter-intuitive, and so contradicted the existing mathematical descriptions of motion, that physicists' confidence in their ability to understand nature was very shaken. The fact that this result can be expressed mathematically in terms of the theory of groups of coordinate transformations of a four-dimensional space-*time*, analogously to the easily visualizable group theory of three-dimensional or two-dimensional transformations, makes mathematicians believe that we *can* find a language for talking about, and thus understanding, the way the universe is constructed.

The language of group theory allows us to *talk* about certain physical laws in a special way: a way which captures the mathematical essence of what is happening in a number of very different physical situations, so that we can reduce the complexity of the world by looking at those aspects of nature which are the *same*, rather than being confused by all the differences. Mathematicians believe that this *sameness* expresses a fundamental truth about the way nature operates, a belief which is based on the philosophical assumption that the *simpler* a mathematical description of nature's processes is, the closer to the truth it is. Do you agree with this philosophy?

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Getting to First Base

It isn't that they can't see the solution.
It is that they can't see the problem.

G.K. Chesterton,
The Scandal of Father Brown

WE, THE LIVING

Michael A.B. Deakin, Monash University

I went to school in Tasmania and there, when I was in 5th grade, a religion teacher asked us how we knew that we would die. Eventually, one kid stuck his hand up and said, when called on to speak: "Well, er, most people have so far". This answer was seen by the teacher as a piece of typical Tasmanian understatement and the reply was accepted as correct.

But is it?

The world's population is exploding – i.e. growing at an alarmingly rapid rate. I have seen it written that "Of all the people who have ever lived, half are alive at this moment". This statement (call it Statement S), if true, denies the Tasmanian boy's reasoning.

I set out to see how true Statement S actually is. In what follows, I develop a simple mathematical model. It does accept that we will all die – we know this on grounds other than the Tasmanian boy's answer. There will be a lot of simplifying assumptions in the model but they will not be so wild as to invalidate the general conclusions to be drawn from it.

It will be convenient to measure *time* in 5-year periods. Imagine that every five years we take a census of the total human population of planet Earth. To make matters clear, suppose we are about to count all the people alive as we reach 0:00 hours Greenwich Mean Time on January 1, 1990. Call this number $X(0)$.

Now suppose that a similar exercise had been carried out precisely five years previously on January 1, 1985. Call the count then $X(1)$. And for the similar count on January 1, 1980, write $X(2)$. And so on.

As a (very good) approximation, we will suppose that the census counts

$$X(0),\ X(1),\ X(2),\ \ldots$$

form a decreasing geometric sequence. (This is known as Malthus's Law, after the Reverend Thomas Malthus, who first formulated it; see *Function Vol. 2, Part 3*, p.21.) We thus write

$$\begin{aligned} X(1) &= rX(0) \\ X(2) &= rX(1) \\ X(3) &= rX(2) \end{aligned}$$

etc., and in general

$$X(n+1) = rX(n). \qquad (1)$$

To relate our model to reality, we first need an estimate of the value of r. Malthus, back in 1798, took the doubling time – the period that it took for a population to double its size – as 27 years. More recently it has become somewhat less. A quite reasonable estimate is 25 years, that is to say, 5 census periods.

From this, we can conclude that

$$X(5) = r^5 X(0) = \tfrac{1}{2}X(0), \tag{2}$$

and so

$$r^5 = \tfrac{1}{2}$$

or

$$r = 2^{-1/5} \approx 0.87, \tag{3}$$

to a good approximation.

At first it might be thought that all we need do now to estimate the total number of people who have ever lived is to add up the series

$$X(0) + X(1) + X(2) + \ldots .$$

This, however, would give an overestimate, as many people will appear in more than one census. Some could end up being counted 20 times or more.

To overcome this problem we need to look at the Malthusian model in more detail.

At the next census there will be $X(0)$ people. Of these, there will be the $X(1)$ people from the last census, along with those born since, but less those who have died in the meantime. Let $BX(1)$ be the number born and $DX(1)$ the number who have died. B, D are referred to respectively as the *birth* and the *death rates*. We will treat them as constant: a reasonably good approximation.

We thus have

$$X(0) = X(1) + BX(1) - DX(1),$$

and similarly

$$X(1) = X(2) + BX(2) - DX(2),$$

and, in general,

$$X(n) = X(n+1) + BX(n+1) - DX(n+1).$$

I.e.

$$X(n) = (1+B-D)X(n+1). \tag{4}$$

Compare Equation (4) with Equation (1). This gives

$$1 + B - D = \frac{1}{r}. \tag{5}$$

Then at the next census $BX(1)$ previously uncounted people will be included. At the last $BX(2)$ were counted for the first time. And so on. The total number (T, say) of people who have ever lived is thus

$$BX(1) + BX(2) + BX(3) + \dots .$$

Thus

$$\begin{aligned} T &= B[X(1) + X(2) + X(3) + \dots] \\ &= B[rX(0) + r^2X(0) + r^3X(0) + \dots] \\ &= BrX(0)[1 + r + r^2 + \dots]. \end{aligned}$$

So

$$T = \frac{BrX(0)}{1-r}. \qquad (6)$$

To proceed further we need to know a value for B. The simplest way to estimate this is indirectly, via Equation (5).

Let us suppose that the average life-span is 40 years. Indeed, to simplify the calculation, suppose everybody lives *exactly* 40 years. This is unrealistic, of course; we'll see later what might happen if we look at matters more realistically. But if we take the simple calculation first we find those dying between the 1985 and the 1990 census will be precisely those born between the 1945 and the 1960 censuses. The first figure is $DX(1)$, the second $BX(9)$. Thus

$$DX(1) = BX(9)$$

or

$$D = Br^8 \simeq 0.33B. \qquad (7)$$

Now combine Equation (7) with Equation (5) to get

$$1 + 0{\cdot}67B = 1{\cdot}15,$$

i.e.

$$B \simeq 0.22. \qquad (8)$$

We are now in a position to check Statement S. The fraction of all people who are alive of those who have ever lived is

$$\frac{X(0)}{T} = \frac{X(0)(1-r)}{BrX(0)} = \frac{1-r}{Br}. \qquad (9)$$

Substitute in the values for r, B from Equations (3), (8) to find

$$X(0) = 0{\cdot}68T. \qquad (10)$$

In other words, about 68% of all humans are still alive! The Tasmanian boy's answer was incorrect.

How much faith can we really put in this figure?

First let us see the effect of varying the assumed life-span. We supposed this to be 8 census periods. Now consider instead N census periods. Equation (7) is now altered, and when we combine it with Equation (5), we find that

$$B = \frac{1-r}{r(1-r^N)}, \tag{11}$$

so that

$$\frac{X(0)}{T} = 1 - r^N. \tag{12}$$

Now if $N = 1$ (a value typical of a very poor third-world country), this gives 0.13 and Statement S is clearly false. However, if $N = 15$ (a value about right for an affluent Western country), we find 0·87 and Statement S is more than fulfilled.

Statement S is most nearly true (i.e. $X(0) = \frac{1}{2}T$) if $N = 5$. So life expectancies (at birth) of less than 25 years do not yield the truth of Statement S. Those above 25 years do. This, of course, is precisely because we so chose r as to yield a doubling time of 25 years.

It is hard to find good estimates of N. I looked up some U.N. figures and was surprised to find that even in India and Bangladesh life expectancies at birth are claimed to exceed 50. I.e. $N > 10$. However, very poor countries simply do not have the relevant statistics. $N = 8$ is probably about right – as close as we can tell.

Another way to check our value of N is to look at U.N. figures for B. Our value corresponds to an annual crude birth-rate of 40·6 per thousand. (That is to say, $1{\cdot}0406^5 = 1{\cdot}22$, 0·22 being the value of B given by Equation (8).) This is high, though not impossibly so; you will find it in many third-world countries (again, however, these being the ones for which statistics are not particularly reliable). Probably it indicates that we have, if anything, underestimated N : that is to say, underestimated $X(0)/T$.

Our assumption that everyone died at the same age was, of course, quite unrealistic. Nevertheless, it probably doesn't affect the outcome of the calculation very much. There will be groups in the population for which N is small and these will be balanced by groups for which N is large. The different values of $X(0)/T$ would need, in a more realistic model, to be averaged. This might produce some discrepancy from the value calculated above, but it could hardly alter it too drastically. So Statement S seems safe.

Do I then believe Statement S?

Well, perhaps.

There is a real problem with Equation (1), the Malthusian model. The key to the doubt lies in Equation (6), where I summed an infinite series – although, of course, we have not been around for infinite time. This does not *directly* affect very much the accuracy of Equation (6) and the calculations based on it; but it does throw some doubt on Equation (1).

Right now, there are about 5.2 billion people on earth (a billion being 1,000,000,000). So, one census ago there were 4.5 billion or so; one census period before that there were 3.9 billion. Etc. And 156 census periods ago,on this logic, there would have been just two people. Now this is less than 800 years ago – and even the notorious Bishop Usher† dated Adam and Eve before that!

The Malthusian model is, in fact, a very good description of recent human history. But humankind has been around a lot longer than the few centuries over which it has been valid. For most of history, it is believed, the human population lived a hunter-gatherer existence, with the population essentially constant, $B = D$ and $r = 1$.

We don't know very accurately how long this period lasted. After consulting a number of authoritative texts, I found a (very rough) consensus that the origin of our species, *Homo sapiens*, was about 100,000 years ago.

Even more uncertain than this figure is the estimate of how many people there have been during this long period. I looked and looked but failed to find any reliable estimates. So eventually I pulled a figure out of the air: 1 million.

The other thing we need to know is how long people lived back then. Lacking information, I simply assumed 40 years – as in the previous analysis.

So we "guesstimate" that our value of T has left out something like

$$100{,}000 \times 1{,}000{,}000/40$$

people, i.e. 2.5 billion. Now our previous estimate of T was, from Equation (10), 6·75 billion. If we add to this the 2·5 billion we've just estimated, we get 9·25 billion: a little less than twice the current population. So Statement S seems true, but just.

But there are so many uncertainties. Statement S may well be true. Equally, it may well be false. Of course, if we wait another doubling period, it will almost certainly be true if population goes on increasing as it is now.

But, of course, supposing even it were false, it does not support the Tasmanian boy's logic. Say even 90% of all people had in fact died, would you accept an experiment that gave only 90% support for a supposed universal truth?

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† Bishop Usher attempted to date the creation by means of a very detailed study of Old Testament chronologies, which he took to be literally true in every slightest particular. He assigned a date of 4004 BC to the creation. Very few, if any, reputable biblical scholars would today agree with this methodology.

A THREE-DIMENSIONAL COSINE RULE

Garnet J. Greenbury, Brisbane

In *Function, Vol.* 13, *Part* 4 (August 1989), Aidan Sudbury outlined a proof that in the tetrahedron $0XYZ$ of Figure 1, the areas of the four sides obeyed the relation

$$(\text{Area } \Delta\ XYZ)^2 = (\text{Area } \Delta\ 0YZ)^2 + (\text{Area } \Delta\ 0ZX)^2 + (\text{Area } \Delta\ 0XY)^2.$$

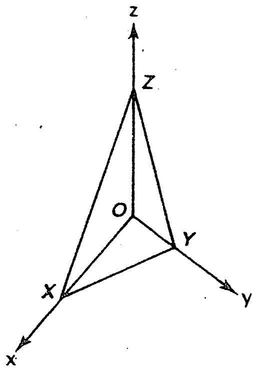

Figure 1

This follows because $\angle Y0Z$, $\angle Z0X$, $\angle X0Y$ are right angles. The equation is directly analogous to Pythagoras' theorem for right-angled triangles.†

In the case of non-right-angled triangles the more general cosine rule replaces Pythagoras' theorem and here I consider a three-dimensional version of that.

Consider Figure 2. $\angle A0B$, $\angle A0C$ are right angles but $\angle B0C = \theta$ and is not necessarily a right angle. Let the area of $A0B$ be A_1, that of $A0C$ be A_2, that of $B0C$ be A_3 and that of ABC be A_4. Then

$$A_4^2 = A_1^2 + A_3^2 - 2A_1A_2 \cos\theta.$$

The proof goes like this.

Label the axes as shown in the diagram. I.e. put $0A = l$, $0B = m$, $0C = n$. Let $0D$ be perpendicular to BC and put $0D = h$. Join AD and let $0E$ be perpendicular to AD. Put $0E = k$. We now calculate the various areas.

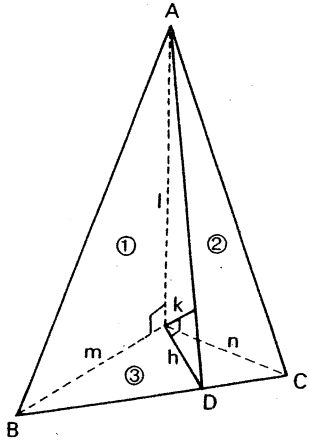

Figure 2

Clearly

$$A_1 = \tfrac{1}{2}lm$$

and

$$A_2 = \tfrac{1}{2}ln.$$

† For more on this, see *Function, Vol.*7, *Part* 1, p.8 and *Part* 2, p.24. (Eds.)

Furthermore

$$A_3 = \tfrac{1}{2}mn \sin \theta.$$

But, by the ordinary cosine rule, the length of BC is

$$\sqrt{m^2+n^2-2mn \cos \theta},$$

so that we also have

$$A_3 = \tfrac{1}{2}h \sqrt{m^2+n^2-2mn \cos \theta}.$$

Thus

$$h = \frac{mn \sin \theta}{\sqrt{m^2=n^2-2mn \cos \theta}}. \qquad (1)$$

Next consider the area of the triangle AOD. This is $\tfrac{1}{2}hl$, but it can also be computed as $\tfrac{1}{2}k \times$ the length AD, i.e. $\tfrac{1}{2}k\sqrt{(l^2+h^2)}$. So we have

$$\tfrac{1}{2}hl = \tfrac{1}{2}k \sqrt{l^2+h^2},$$

from which it follows that

$$k = \frac{h\,l}{\sqrt{(l^2+h^2)}}.$$

Now the volume of the tetrahedron is $\tfrac{1}{3}kA_4$, by a standard formula, but it can also be computed as $\tfrac{1}{3}lA_3$. Then

$$\tfrac{1}{3}kA_4 = \tfrac{1}{3}kA_3,$$

so that

$$A_4 = \frac{l}{k}A_3 = \frac{l}{2k}mn \sin \theta$$

$$= \frac{lmn \sin \theta}{2hl} \sqrt{l^2+h^2}.$$

Thus we have

$$A_4^2 = \tfrac{1}{4}m^2n^2\sin^2\theta\left[\frac{l^2}{h^2} + 1\right]$$

$$= \tfrac{1}{4}m^2n^2\sin^2\theta\left[\frac{l^2(m^2+n^2-2mn \cos \theta)}{m^2 n^2 \sin^2 \theta} + 1\right], \text{ by Equation (1).}$$

But this expression now simplifies. We find

$$A_4^2 = \tfrac{1}{4}(l^2m^2 + l^2n^2 - 2l^2mn\cos\theta + m^2n^2\sin^2\theta)$$

$$= (\tfrac{1}{2}lm)^2 + (\tfrac{1}{2}ln)^2 + (\tfrac{1}{2}mn\sin\theta)^2 - \tfrac{1}{2}(lm)(ln)\cos\theta$$

$$= A_1^2 + A_2^2 + A_3^2 - 2A_1A_2\cos\theta.$$

More generally, if $\angle AOB = \phi$ (not a right angle) and $\angle AOC = \psi$ (also not a right angle), then we have

$$A_4^2 = A_1^2 + A_2^2 + A_3^2 - 2A_1A_2\cos\theta - 2A_2A_3\cos\phi - 2A_3A_1\cos\psi.$$

Perhaps a reader can supply a proof of this more general result.

* * * * *

GENERATING FUNCTIONS

Marta Sved, University of Adelaide

Introduction

The story begins at Euclid Park Primary School where the parents' association is very active. All fathers and mothers of the 200 children always attend every meeting. (Note that these children are very lucky, each having two parents in the home and no brothers or sisters at the school.) However, decision-making becomes very difficult, so it is now moved that a working committee of *six* parents should be elected: there should be two kinds of membership on this committee for the parents. A single membership for a couple should mean that *either* the father *or* the mother is to attend any of the committee meetings, while a double membership involves both parents at each meeting.

The motion is passed. However, the meeting is still in session, trying to appoint the committee. Of course, the number of possibilities is somewhat large.

Solution of the problem

Let us formulate a more general problem. There are n couples, that is $2n$ persons in the association. A committee of r members is to be formed; membership being awarded to couples. A *single membership* entitles *one representative* of the couple to attend any one meeting, while a *double membership* means that *both* husband and wife are to attend.

Denote by E_r^n (E for Euclid Park) the number of possible committees. In our case $n = 200$, $r = 6$. One natural way to approach the problem of evaluating E_r^n is by the method of recursion, which will now be described.

Assume that the solution to the problem is known for the case of n couples. The parents, Mr and Mrs Newberry, of a new student appear on the scene. We want to find the number of committees of r members when there are $n + 1$ couples. There are three possibilities to be considered.

(a) Mr and Mrs Newberry are both appointed (double membership).

(b) The Newberry couple is given a single membership.

(c) The Newberrys are ignored.

In case (a) there are $r - 2$ members to be found out of the original n couples.
In case (b) there are $r - 1$ members to be found out of the original n couples.
In case (c) there are r members to be found out of the original n couples.

Hence

$$E^{n+1}_{r} = E^{n}_{r-2} + E^{n}_{r-1} + E^{n}_{r} \qquad (R)$$

We may now evaluate E^{n}_{r} for any values of n and r. Clearly, there is only one way to choose a committee of 0 members; also there is only one way each to choose a committee of 1 or 2 members from a pool of 1 couple.

Thus

$$E^{1}_{0} = E^{1}_{1} = E^{1}_{2} = 1.$$

Now we may use the relation (R) to calculate further terms. We have

$$E^{2}_{0} = E^{1}_{-2} + E^{1}_{-1} + E^{1}_{0}$$

$$= 0 + 0 + 1 = 1.$$

($E^{1}_{-2} = 0$ as it is clear that there is no way to form a committee of -2 persons! Also $E^{n}_{r} = 0$ if $r > 2n$; the committee can't contain more people than available.)

Similarly

$$E^{2}_{1} = 2,\ E^{2}_{2} = 3,\ E^{2}_{3} = 3,\ E^{2}_{4} = 1.$$

Now use (R) to obtain successively the other values. Since in our case $n = 200$, this is a somewhat cumbersome procedure. However, there is an easier method, now to be illustrated, using what are called "Generating Functions".

Define a function, connected with the values $E^{n}_{0}, E^{n}_{1}, \ldots$ in the following way:

$$G^{n}(x) = E^{n}_{0} + E^{n}_{1}x + E^{n}_{2}x^{2} + \ldots + E^{n}_{2n}x^{2n}. \qquad (1)$$

Remember that for $r > 2n$, $E^n_r = 0$. Consider now the product

$$G^n(x)\left[1 + x + x^2\right]$$
$$= G^n(x) + G^n(x)x + G^n(x)x^2$$
$$= E^n_0 + E^n_1 x + E^n_2 x^2 + \dots + E^n_{2n} x^{2n}$$
$$+ E^n_0 x + E^n_1 x^2 + \dots + E^n_{2n-1} x^{2n} + E^2_{2n} x^{2n+1}$$
$$+ E^n_0 x^2 + \dots + E^n_{2n-2} x^{2n} + E^n_{2n-1} x^{2n+1} + E^n_{2n} x^{2n+2}$$
$$= E^n_0 + (E^n_0 + E^n_1)x + (E^n_0 + E^n_1 + E^n_2)x^2 + \dots + (E^n_{r-2} + E^n_{r-1} + E^n_r)x^r +$$
$$\dots + E^n_{2n} x^{2n+2}.$$

If we look at (R) and also check the first and last two terms, we find that the product becomes

$$E^{n+1}_0 + E^{n+1}_1 x + E^{n+1}_2 x^2 + \dots + E^{n+1}_r x^r + \dots + E^{n+1}_{2n+2} x^{2n+2}.$$

This means that by defining $G^{n+1}(x)$ by replacing n by $n + 1$ in equation (1), we obtain:

$$G^{n+1}(x) = G^n(x)(1 + x + x^2).$$

Since

$$G^1(x) = E^1_0 + E^1_1 x + E^1_2 x^2 = (1 + x + x^2),$$

it follows that

$$G^2(x) = G^1(x)(1 + x + x^2) = (1 + x + x^2)^2$$

and similarly

$$G^n(x) = (1 + x + x^2)^n.$$

Hence our problem is solved if we find the coefficient of x^r in $(1 + x + x^2)$.

Put $1 + x = a$, $x^2 = b$. Then

$$(1 + x + x^2) = (a + b)^n.$$

We have, in our case, $n = 200$. Thus, using the binomial theorem,

$$G^{200}(x) = (a + b)^{200}$$
$$= a^{200} + \begin{bmatrix}200\\1\end{bmatrix} a^{199}b + \begin{bmatrix}200\\2\end{bmatrix} a^{198}b^2 + \begin{bmatrix}200\\3\end{bmatrix} a^{197}b^3 + \dots$$
$$= (1+x)^{200} + \begin{bmatrix}200\\1\end{bmatrix}(1+x)^{199}x^2$$
$$+ \begin{bmatrix}200\\2\end{bmatrix}(1+x)^{198}x^4 + \begin{bmatrix}200\\3\end{bmatrix}(1+x)^{197}x^6 + \dots .$$

And now we can expand $(1+x)^{200}$, etc. also by the binomial theorem.

$$(1+x)^{200} = 1 + \binom{200}{1}x + \binom{200}{2}x^2 + \binom{200}{3}x^3 + \binom{200}{4}x^4 + \binom{200}{5}x^5 + \binom{200}{6}x^6 + \dots .$$

The coefficient of x^6 is thus $\binom{200}{6}$ in this term. We may similarly find x^6 in the terms

$$\binom{200}{1}(1+x)^{199}x^2, \quad \binom{200}{2}(1+x)^{198}x^4, \quad \binom{200}{3}(1+x)^{197}x^6.$$

The result is

$$\binom{200}{6} + \binom{200}{1}\binom{199}{4} + \binom{200}{2}\binom{198}{2} + \binom{200}{3},$$

a number which works out to be 95 968 824 600. (Try finding this *without* using generating functions!)

About other problems

Generating functions, like in the above example, are widely used in combinatorics, the science of arrangements.

These arrangements depend on variables which are natural numbers such as n or r in the problem discussed. So we are interested in sequences of the type

$$u_0, u_1, u_2, \dots, u_n, \dots \qquad (S)$$

With this sequence we associate the function

$$G(x) = u_0 + u_1x + u_2x^2 + \dots + u_nx^n + \dots$$

where the coefficient of the nth power of x is the nth term of the sequence (S).

We call $G(x)$ the generating function of the sequence $\{u_n\}$.

A well-known example for a class of generating functions is:

$$B^n(x) = (1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n,$$

where the sequence is that of the *binomial coefficients*

$$\binom{n}{r}$$

and where n is fixed and r ranges from 0 to n.

* * * * *

LETTERS TO THE EDITOR

More News from Wales

It had been some time since I had heard from my friend and correspondent, the retiring, indeed self-effacing Welsh physicist Dai Fwls ap Rhyll. He used to write to me, not frequently, but regularly, once every one or two years; but recently I had heard nothing and had come to fear the worst.

I was therefore greatly pleased to get a (very brief) note from him. He said nothing to explain his years of silence and I trust that the brevity of his present missive is not indicative of ill-health.

He has, however, been researching the philosophical nature of negative numbers. "How can we envisage -1 apple?" he asks, "Except perhaps as some sort of hole in the fabric of the universe. 1 apple, yes, 2 apples, yes, even 1/2 an apple, even $\sqrt{3}/\pi$ of an apple, but -1?"

His note goes on to say that Euler and d'Alembert, in their work on logarithms of negative numbers, referred to the negative numbers as "imaginary numbers". They did not restrict this usage to the square roots of such numbers as we do today.

"In a very real sense," he writes, "the negative numbers do not exist. Or we might say, they *are* the positive numbers. The number line does not extend to the left of the zero. It stops there; but the number 0 acts as a mirror and so we *seem* to see numbers to its left. However, what we really see are merely reflections of the positive numbers. -1 is really only 1, seen in this mirror."

Here is Dr Fwls' proof that $-1 = 1$.

$$-1 = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1)\times(-1)} = \sqrt{1} = 1.$$

Quite what this new discovery will mean for mathematics, I can't say. Dr Fwls has not yet received the credit to which his earlier discoveries entitle him.

Kim Dean,
Union College,
Windsor

More on Cubics

Following on D.F. Charles' article "Pleasant Cubics" (*Function, Vol.* 13, *Part* 5), suppose a cubic to have the equation

$$y = x(x - A)(x - B).$$

If the cubic has three real roots, we may always achieve this form by shifting the origin in the x-direction.

By differentiating and setting $y' = 0$, we may show that there are two real turning points at $x = x_1$, $x = x_2$, where

$$x_1 = \frac{1}{3}\left\{(A + B) + \sqrt{(A+B)^2 - 3AB}\right\}$$

and

$$x_2 = \frac{1}{3}\left\{(A + B) - \sqrt{(A+B)^2 - 3AB}\right\}.$$

There is also a single point of inflection, where $y'' = 0$, and this occurs where $x = (x_1 + x_2)/2$. Thus the point of inflection occurs halfway between the two turning points.

The proof is easily adapted to the case where the cubic has one root but two turning points. In this case, also move the origin in the y-direction.

Garnet J. Greenbury,
Brisbane

And Yet More

I read with interest D.F. Charles' article "Pleasant Cubics" (*Function, Vol.* 13, *Part* 5).

This problem was neatly solved by M. Chapple in 1960. Chapple's article and one by me (also on this topic) will be published in the first issue of *Australian Senior Mathematics Journal* (1990).

For a related article, see the discussion by G. Graham and C. Roberts, "A diophantine equation from calculus", in the (U.S.) *Mathematics Magazine, Vol.* 62, *Part* 2 (1989), pp. 97-101.

W.P. Galvin,
University of Newcastle

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Would It Were So!

A graph in *The Bulletin* (28/2/1990) shows Australia's debt as being approximately –100 billion dollars.

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HISTORY OF MATHEMATICS SECTION

EDITOR: M.A.B. DEAKIN

Coin-Tossing and Other Matters

Suppose *A* and *B* play the following game: A coin is tossed twice; if it comes down tails on both occasions, *B* wins; otherwise *A* wins.

Our first thought about this game is that it seems rather unfair to B, and we might like to ask how unfair. What, in other words, is the probability that *A* wins?

This game was first discussed and this question first asked by the French mathematician Jean le Rond d'Alembert in 1754. d'Alembert was a member of the French intelligentsia of his day, a group known as the *Encyclopédistes*. Under the leadership of Denis Diderot (1713-1784), this group, which included Voltaire, Rousseau, Montesquieu and others, produced a vast encyclopedia of knowledge. This appeared over the years 1751-1765.

It was in the article entitled "Croix ou pile", which for our present purposes we can translate as "Heads or tails", that d'Alembert asked the question posed at the start of this article. He proceeded then to give two separate solutions.

First, he considered an analysis along lines we might very well use today. The two tosses of the coin might yield: *HH*, *HT*, *TH* or *TT*. Of these four possibilities only the last is favourable to *B*. Thus *A*'s probability of winning is 3/4.

This, I might as well say at once, is the correct answer, assuming the coin to be a fair one. (The case in which we have no knowledge at all about possible bias in the coin is also interesting; see *Function, Vol.* 13, *Part* 2, p.52. But here we stick with fair coins.) However, d'Alembert didn't like this answer. He rejected it in favour of another which he reached by arguing as follows.

Let the coin be tossed for the first time. Either it comes down heads, or it comes down tails. If heads, the game is already over. Only if the first throw is tails need the coin be tossed a second time. There are thus three possible outcomes: *H–*, *TH*, *TT* and of these three only the last is favourable to *B*. Thus *A*'s probability of winning is 2/3.

As long as the coin is fair, this answer is wrong. The error consists in assuming that the three outcomes are equally probable, when in fact they are not. It was Diderot who pointed out the error. Here is his argument presented in a somewhat more modern form than he himself used.

To find the probability of an event, we can proceed by first writing down all the ways it can come about. Next we insert the probability of each event in each way. Third we replace every "and" by a multiplication and every "or" by an addition. This gives the result.

If we do this here we find that A wins if:

(1) There is a head on the first throw

OR

(2) There is a tail on the first throw AND a head on the second throw.

Now on each throw, because the coin is fair, the probability of a head is 1/2, while the probability of a tail is also 1/2.

So here we have the probability that A will win being given by

$$\frac{1}{2} + \frac{1}{2} \times \frac{1}{2},$$

which is to say 3/4. The three outcomes considered by d'Alembert are not equally probable at 1/3 each as he assumed. Their probabilities are, in order, 1/2, 1/4, 1/4.

Now d'Alembert was not, as this story might indicate, a poor mathematician. Quite the reverse; he was a very good one. It is this that makes his error important enough to dwell on. The moral of the story is that probability theory can be a tricky thing and even very good mathematicians can be confused.

It also raises another point. d'Alembert was in fact the specialist mathematical consultant for the Encyclopedia project, and Diderot its general editor. d'Alembert was clearly the greater mathematician. Diderot's biographer notes, however, that Diderot's mathematical abilities are too often underrated today. This is a case in point.

G.J. Tee, of the University of Auckland, wrote to us to point out that if Diderot was able to correct his own mathematical consultant, he was hardly a slouch at mathematics.

There is a quite well-known story about Diderot, and told to his disadvantage. It's rather fun, so here it is. The very great mathematician Leonhard Euler was a contemporary of Diderot's. Euler worked in St. Petersburg (now Leningrad) for much of his life, his patron being Catherine the Great.

The story goes that Catherine arranged that Euler and Diderot debate the existence of God: Euler taking the affirmative side and Diderot the negative. Euler, according to the story, spoke first, saying:

$$\frac{a + b^n}{n} = x,$$

therefore God exists.

Diderot, so the story goes, was unable to reply and so lost the debate.

Mr Tee points out that the story is almost certainly false and we do a disservice to Diderot by remembering him for it. It comes from E.T. Bell's very popular book *Men of Mathematics*. Bell wrote a number of books on mathematics, especially on its history. He was a lively and entertaining writer and has certainly enriched the appreciation of mathematics for many of his readers. Nonetheless, and regrettably, he was not always careful of his facts, and he has tales that are of very doubtful authenticity in some cases, and in others just plain wrong.

The story of the Euler-Diderot debate Bell got from a book by the 19th Century English mathematician Augustus De Morgan, *A Budget of Paradoxes*. He in his turn could only quote a rather unreliable source. (*A Budget of Paradoxes* was in fact a posthumous collection of articles published by De Morgan's widow.)

So the trail ends and we don't know beyond this where the story comes from. It's fun, but highly unlikely. Apart from its intrinsic implausibility, Diderot knew more than enough mathematics to avoid being taken in like that.

But let us return to coin-tossing and to some very recent history. Consider a variant on the game between *A*, *B*. Suppose two coins are tossed. Suppose further that we are informed that there is at least one that has come down heads. What is the probability that there are in fact *two* heads?

To analyse this, first note that the coins could have fallen in any one of four possible ways: *HH*, *HT*, *TH* or TT. But this time, because of the further information, we know that *TT* cannot have occurred; there is at least one head. The possibilities are thus *HH*, *HT* or *TH*. Of these only one (*HH*) corresponds to two heads and thus the required probability is 1/3.

Some years ago, during a school's lecture at Monash, the lecturer asked this question. He tossed two coins and covered them where they fell: one with his right hand, one with his left. He then asked the question several times, challenging his audience to find the answer.

However, probably quite unconsciously, each time he said "at least one head", his eyes strayed subtly but noticeably toward his *right* hand. And this could have induced his audience, possibly also quite unconsciously, to believe that the right-hand coin had come down heads.

If this is so, then everything depends on the left-hand coin, and there are only two, equally probable, cases. Either it's heads or it's tails, and if it's heads, there are two heads, and the probability of this happening is 1/2. This was the most commonly given reply.

We thus see that if we're told that a *particular* coin (right or left) has come down heads, and not just *some* coin, then the probability of two heads is no longer 1/3 but 1/2.

For a probability to increase in this way, further information needs to be given. It could well be that this was imparted by slight involuntary gestures in this case. But we'll probably never know.

However, if probability theory can be as subtle as this, let us not blame d'Alembert for his mistake in the Encyclopedia.

* * * * *

COMPUTER SECTION

EDITOR: R.T. WORLEY

Primes and Computers

These days mathematicians are making use of powerful computer programs (or 'packages', as they tend to be called). Indeed, it is more common for me to use a package to find out if a number is prime than it is to look up a table of primes. Some questions that I might ask the computer are:-

1. What is the 10000th prime?
2. Is 1111111 a prime?
3. What is the factorisation of 111111111?

In this article I shall look at issues related to answering the first question.

To answer questions such as the first (the 'Mathematica' package on the Macintosh computer knows about all primes up to 10^8), the computer package would probably access a long list of primes and print out the 10000th entry. The questions that arise are:-

A. How quickly can a list of the first million (say) primes be produced?

B. How much room does it take to store such a list?

These days it is feasible to produce a list of the first million or so primes quite quickly on a personal computer, using a technique known as Eratosthenes' Sieve.[†] The idea behind this ancient (around 230 BC!) technique is the following. Put the numbers 2, 3, 4,..., N (for some upper bound N, say $N = 10^6$ if we want all primes less than 10^6) into a sieve.

Pick out the smallest (which is 2) and shake the sieve so the multiples of 2 (which are 4, 6, 8, ...) fall through the holes in the sieve. Again pick out the smallest (which is now 3) and again shake the sieve so the multiples of 3 (which are 6, 9, 12, ...) fall out. Repeat the picking out of the smallest and shaking out its multiples until the sieve is empty. The numbers picked out are precisely the primes as any non-prime will be shaken out when shaking out the multiples of its smallest prime factor.

A simple method to implement this on a computer would be as follows.

1. Fill an array Sieve[], which has subscripts running from 2 to N, with the number -1 (suppose we let '-1' denote "in the sieve").

[†] See *Function, Vol.* 6, *Part* 2, p. 18.

2. Repeat steps 2a to 2c until termination:

   2a. Select the smallest subscript i such that Sieve[i] = –1. (If there is no such i then terminate.)

   2b. Set Sieve[i] = 1 (suppose we use '1' to denote "picked out").

   2c. For each multiple of i between $2i$ and N set Sieve[i] = 0 (where we use 0 to denote "fallen through").

3. Print out the list of primes by printing those i for which Sieve[i] = 1.

In BASIC (versions of BASIC differ – this is for an IBM PC, and you may need to make changes for your machine) this could be written as follows.

```
100  N=10000
110  DIM SIEVE(N)
115  PRINT "This program will print all primes less than", N
119  REM FILL SIEVE
120  FOR I=2 TO N
130  SIEVE(I)=-1
140  NEXT I
149  REM PICK SMALLEST AND GOSUB SHAKE
150  FOR I=2 TO N
160  IF SIEVE(I)=-1 THEN GOSUB 9000
170  NEXT I
179  REM PRINT LIST OF PRIMES
180  FOR I=2 TO N
190  IF SIEVE(I)=1 THEN PRINT I;
200  NEXT I
210  STOP
9000 SIEVE(I)=1
9010 J=I+I
9020 IF J>N THEN RETURN
9030 SIEVE(J)=0
9040 J=J+I
9050 GOTO 9020
```

This algorithm can be speeded up in a number of ways. Firstly, you may have already observed that when shaking the sieve so that multiples of a prime p fall through, all the multiples less than p^2 have already fallen through on earlier shakes (so line 9020 could be J=I*I). Another observation is that apart from 2, all primes are odd, so we can find all primes other than 2 by only putting odd numbers into the sieve. If we let Sieve[i] = –1 denote the fact that the odd number $2i+1$ is in the sieve, then the array Sieve[] of a given size N can be used to calculate primes up to $2N+1$.

A further modification is necessary for computers with small memories, because it may not be possible to create a large enough array. If we want to find all primes less than 10^6, instead of putting the numbers 2, 3, ..., 10^6 all into the one large sieve, we could use a smaller sieve, at first putting the numbers 2,...,1000 in it. Using this sieve we find all primes less than 1000 as above. Now our sieve is empty, so we put the numbers 1001,1002,...,2000 in it. We then shake out all multiples of the primes we found before (so we should have saved them somewhere!) and start picking out and shaking out multiples as usual. We now have all primes up to 2000, and our sieve is again empty. We refill the sieve with the numbers 2001,...,3000, shake out the multiples of the primes† we have already found, and pick and shake again.

Using this idea a program to calculate the first million primes will easily fit into a personal computer such as the IBM PC. A printout of such a program (written in Turbo Pascal) is available from the author if you send a stamped self-addressed envelope. The program takes around 5 minutes to produce the first million primes on a fast machine and should take less than an hour on a PC/XT running at normal speed. A BASIC version will produce the primes much more slowly (if the BASIC is not compiled) and using normal single precision arithmetic will not handle primes beyond 6 decimal digits. This is because some versions of BASIC do not handle decent size integers except using single or double precision arithmetic (which is slow), and in some cases single precision will handle integers only up to $8*10^6$ accurately and only print out single precision numbers with at most 6 digits accuracy.

Having written a program to produce the first million primes, we then find problems when we come to save the results. Most of the primes will have 6 or 7 decimal†† digits, and if we put a space between the numbers we find we need to store some 7 million characters. It would be nice to find a more economical way.

If we examine the list of the first million primes, we discover that consecutive primes differ by at most 154, and clearly all differences are even numbers except the first (3–2). If, instead of storing the primes, we store half the differences between consecutive primes, we need at most two digits per prime difference. If we agree to use 2 digits in every case, which requires writing 1 as 01, we can avoid the need for a space between the numbers, and we only need to store 2 million decimal digits, compared with the 7 million required earlier. The only drawback with this method is that to get the 20th prime, say, we have to start with 3 and add twice the sum of the first 18 half-differences. At the cost of a little extra storage we can ensure we never have to add too many numbers.

Another way of storing the primes is to store (in a fairly compact form) the actual array sieve[] produced by the algorithm. To get the first million primes we need an array of about 16 million 0s and 1s ('fell through's and 'picked out's). However, if we observe that apart from the primes less than 30, all primes have remainder 1, 7, 11, 13, 17, 19, 23 or 29 on division by 30, we only need to store the 0s and 1s for these numbers. This requires only around 4 million 0s and 1s, which is comparable with the 2 million decimal digits needed by the previous method.

If one actually looks at the binary numbers used by computers, considering the storage unit 'byte' (which holds 8 binary digits), it takes 3 million bytes for the method storing the actual prime, 1 million bytes for storing the differences (although this can be approximately halved by some

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† We only need to use the primes less than 1000 if we eventually want the primes less than 10^6, for the reason noted in the first modification.

†† Computers normally work using binary arithmetic, but the point I want to make applies whether we use binary or decimal numbers. So I'll use decimal as this is probably more familiar.

clever tricks based on noticing that small differences occur more frequently), and half a million bytes to store the sieve array for the numbers with correct remainders on division by 30.

Given our list of primes we can ask some interesting questions – for example:

(i) How often does a prime end in 1 (or 3 or 7 or 9) ?
(ii) How often does each of the possible differences 2, 4, 6,... occur?
(iii) How many primes are there between 2 and N ?
(iv) How big is the Nth prime?

In the next issue, I will show graphs that are the result of searching a list of the first million primes to answer the above questions. It should be remarked that the answers to the questions are known (in a general sense) for all but the second question. Although it is known that approximately a quarter of the primes end in each of 1, 3, 7 and 9, that there are about $N/ln(N)$ primes between 2 and N, and that the Nth prime is approximately $Nln(N)$, it is not even known if the difference 2 between consecutive primes occurs infinitely often.

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PROBLEMS SECTION

EDITOR: H. LAUSCH

a. One starter problem

14.2.1. This slaying puzzle has been communicated to *FUNCTION* by the philosopher John Bigelow of La Trobe University at Bundoora, Victoria:

Powerful rats

They counted some rats in Mayfair;
The number was a third of a square,
If a quarter were slain
Just a cube would remain,
How many, at least, must be there?

b. Browsing old mathematics texts

Göttingen University, founded by the Hanoverian King Georg August [also known as King George II of England] in 1736 and manifesting his enlightened despotism, later became a great mathematical centre, thanks first to Carl Friedrich Gauss (1777–1855). As it happened, Gauss was a student of Abraham Gotthelf Kästner, a Göttingen professor of mathematics and prolific epigrammatist. The following problems are taken from an introductory algebra text of 1761 that was written by a much older student of Kästner's, the Slovakian mathematician, logician and physicist Matej Bučáni (1731–1796). Introducing one of his problem collections Bučáni explains: "§.30. Here I shall mention some problems without any solution at all, merely for exercise and for pleasure".

14.2.2. Someone was asked how much money he had. He gave as an answer: "I have loaned all my money; to Maevius I gave a third, to Sempronius a fourth of my money, and to Caius I gave 200 thalers [an old German currency unit], so all my money is gone". How much did the person lend?

14.2.3. On a clock the minute hand points to 60 minutes and the hour hand to 4 o'clock. One demands to know: (1) When will the minute hand catch up with the hour hand and cover it? (2) When will the two hands form a right angle? And (3) when will they form a straight line [i.e. form an angle of 180°]?

14.2.4. Someone is fallen upon by robbers. They inquire immediately about his money. Their victim answers: "I have just so much money on me that I can give each of you 5 thalers". They take his money and also his sabre. One of the robbers keeps the sabre and returns 13 thalers to the robbers' community chest whereby now each of the robbers receives 7 thalers. The number of thalers that was the value of the sabre was twice the number of the robbers. Now the question is: how many robbers were there? How much money did the victim carry? And what was the value of the sabre?

c. Aussie rules

14.2.5. (Communicated by Editor Michael Deakin). In an Australian Rules match, the Galahs beat the Goannas. One fan noticed that the Galahs scored as many goals as the Goannas scored behinds, and *vice versa*. He also noticed that the total points score of the Galahs (read from right to left) equalled that of the Goannas (but read from left to right). What were the scores registered by the teams?

d. Graphs: learn and solve

Consider a group of $n \geq 2$ people. Make a list of all $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of these people. Whenever two people in this group are friends, place a tick next to their entry in your list. You will get a much clearer idea about the "friendship network" in this group if you make a drawing in which each individual is depicted as a point and each friendship as a line that joins two points representing mutual friends. Any such configuration is called a *graph*, the points are referred to as *vertices* and the lines as *edges*. The number of vertices to which a given vertex is joined by a line is called the *degree* of this vertex (i.e. the number of friends a given individual has within the given group).

14.2.6 Show that every graph contains two vertices of the same degree (i.e. in every "friendship network" there are two individuals that have the same number of friends).

14.2.7. Show that the sum of the degrees of all vertices of a graph is equal to twice the number of edges.

14.2.8. Show that the number of vertices having odd degree is even.

A graph is called *regular* if all its vertices have the same degree (i.e. if each individual in the "friendship network" has the same number of friends).

14.2.9. If a regular graph has an odd number of vertices, then every vertex has even degree.

e. **It's 1990 ...**

... and *FUNCTION* has still not offered a problem that involves the number of the current year. Here is one, at last:

14.2.10. Determine all pairs (x,y) of positive integers which satisfy $xy + x + y = 1990$.

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DORDS AND GOONS

The Age (9/3/1989) told a story that bears both repeating and extending. In one edition of Webster's *Dictionary*, the entry "density" was to have been followed by "(D or d)", indicating the use of either a capital D or a lower case d as an abbreviation of the word. However, in one of the rare misprints in Webster the parenthetic entry appeared as "Dord". According to *The Age*, this gave rise to the belief that the *dord* was a unit of density (they didn't say which) and the numerical value of a density (in whichever unit) became referred to as the "dord rating" of the substance in question.

The Age also states that this history may be the origin of the belief that reference books deliberately include errors to guard against plagiarism. The idea is that if someone merely copies out, say, a mathematical table (rather than recalculating it), and then publishes the copy (including the error), there is then evidence on which to bring a court case for breach of copyright. *The Age* cast some doubt on this story but there is evidence that, in one case at least, it happened.

The work, *Handbook of Mathematical Functions*, by Abramowitz and Stegun, is an important reference book for mathematicians. It has been republished and reprinted many times. One edition (the 5th Dover) lists many corrections to entries in previous versions, but introduces a new and obvious error of its own, in that the spelling of Irene Stegun's name was altered to "Segun".

Mathematicians who are fans of the Goon Show (3AR Saturdays at 12 noon) tend to pronounce this as "Seagoon" (after Neddie, one of the show's characters) and so call the 5th Dover the "Seagoon edition". It has become something of a collectors' item.

It is widely believed that Dover introduced the error as a way of saying to potential plagiarists, "You have been warned!" I.e., though old booby-traps were gone, there might well be new ones.

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BERTRAND RUSSELL ON MATHEMATICS

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty — a beauty cold and austere, like that of sculpture".

"Mathematics may be defined as the subject in which we never know that we are talking about, nor whether what we are saying is true".

"Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true".

Bertrand Russell

AND ERNST MACH ...

"Strange as it may sound, the power of mathematics rests on its evasion of all unnecessary thought and on its wonderful saving of mental operations".

Ernst Mach

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CLEAR EXPOSITION!

"Whenever I meet in La Place with the words 'Thus it plainly appears', I am sure that hours, and perhaps days, of hard study will alone enable me to discover how it plainly appears".

N. Bowditch, quoted by H. Bowditch,
in his memoir on the life of
his father (N. Bowditch), *Celestial Mechanics*,
Vol. 4, 1839.

(Sent to us by I. Grattan-Guinness,
Middlesex Polytechnic, U.K.)

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