Function

Founder Editor G. B. Preston

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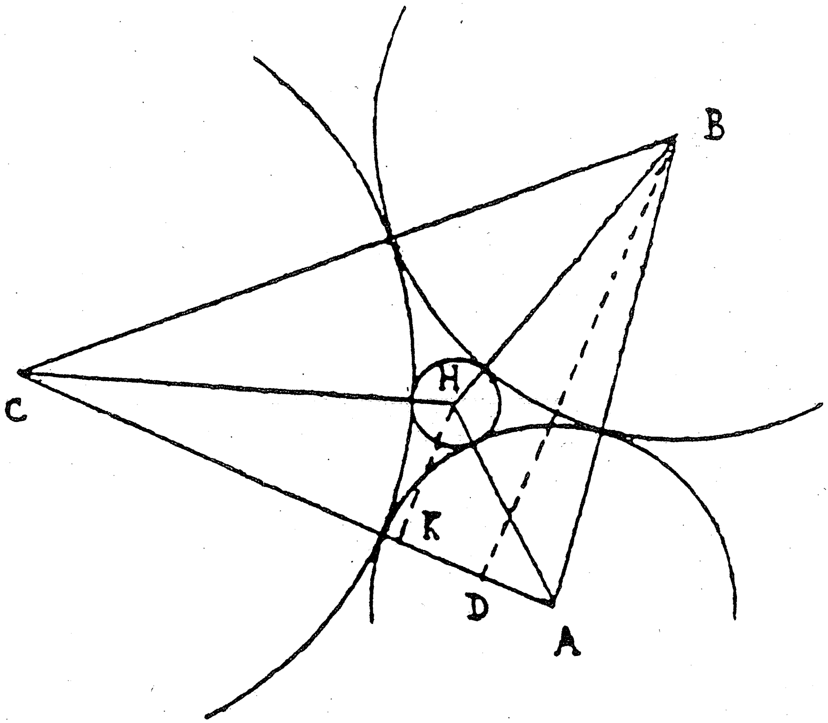

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

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FUNCTION

Volume 15 *Part 4*

(Founder editor: G.B. Preston)

THE FRONT COVER

The diagram reproduced on the front cover is a version of that used by René Descartes to prove his "Four Circles Theorem". See page 120.

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FUNCTION welcomes submissions and queries from its readers. For details, see the information opposite.

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Published by the department of Mathematics, Monash University

AVERAGES: SOME GEOMETRICAL AND PHYSICAL INTERPRETATIONS

K. McR. Evans, Dromana, Vic.

1. Arithmetic Mean

The arithmetic mean $A(x_1, x_2)$, of two numbers x_1, x_2 is defined by

$$A(x_1, x_2) = \frac{x_1 + x_2}{2}$$

This mean is the most commonly used average of two numbers. Two geometrical illustrations are given below.

In Figure 1, $BCDE$ is a trapezium with $\overline{EB}$ parallel to $\overline{DC}$. X is the mid-point of $\overline{DE}$ and $\overline{XY}$ is drawn parallel to $\overline{EB}$.

With lengths as shown, it can be proved that

$$x = \frac{x_1 + x_2}{2} = A(x_1, x_2).$$

Note that the result is true whether or not DEB is a right angle.

In Figure 2, if M is the mid-point of $\overline{BC}$, then

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = (A(x_1, x_2), A(y_1, y_2))$$

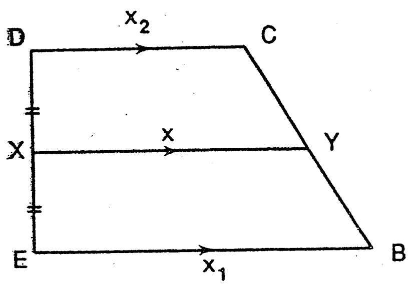


Figure 1

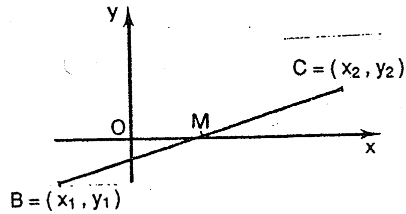


Figure 2

2. Geometric Mean

The geometric mean, $G(x_1, x_2)$, of two positive numbers x_1, x_2 is defined by

$$G(x_1, x_2) = \sqrt{x_1 x_2}.$$

Three geometrical illustrations follow.

In Figure 3, the lengths of intervals $\overline{AB}$, $\overline{BC}$ are given. Construct a semi-circle on diameter $\overline{AC}$ and draw $\overline{BD}$ perpendicular to $\overline{AC}$.

Using similar triangles (shown), it can be proved that

$$x = \sqrt{ab} = G(a,b).$$

Three squares are drawn as shown in Figure 4 with vertices A, B, C collinear. It can be proved that

$$b = \sqrt{ac} = G(a,c).$$

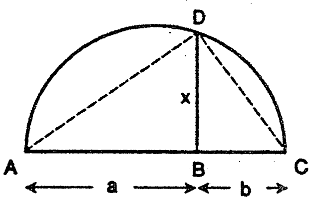


Figure 3

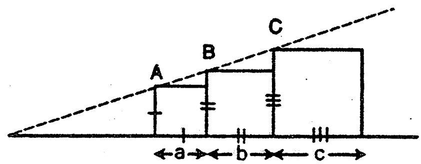


Figure 4

In Figure 5, $\overline{AP}$ and $\overline{PB}$ have given lengths. A circle is drawn through A, B and a tangent, $\overline{PT}$, is drawn from P.

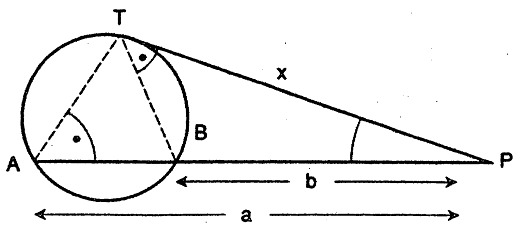


Figure 5

It can be proved that $x = \sqrt{ab} = G(a,b)$.

A physical illustration of the use of the geometric mean occurs in finding the weight, W (unknown), of an object with a balance having arms of unequal lengths a,b and scale pans of unequal weights w_a, w_b. (See Figure 6 opposite.)

Initially, the empty pans are in balance so

$$w_a a = w_b b \qquad (1)$$

Put the object in the right-hand pan, and put a standard weight, W_1, in the left-hand pan so that the pans are balanced. Then

$$(W_1 + w_a)a = (W + w_b)b.$$

Using (1) we obtain

$$W_1 a = Wb. \qquad (2)$$

Next put the object in the left-hand pan and put a standard weights, W_2, on the right-hand pan so that the two pans are again balanced. Then

$$(W_2 + w_b)b = (W + w_a)a.$$

Using (1) we obtain

$$W_2 b = Wa. \qquad (3)$$

Multiplying left-sides and right-sides of (2) and (3) gives

$$W_1 a W_2 b = WbWa$$

i.e. $\quad W_1 W_2 = W^2 \qquad$ (dividing by ab)

i.e. $\quad W = \sqrt{W_1 W_2} \qquad$ ($W > 0$)

$$= G(W_1, W_2).$$

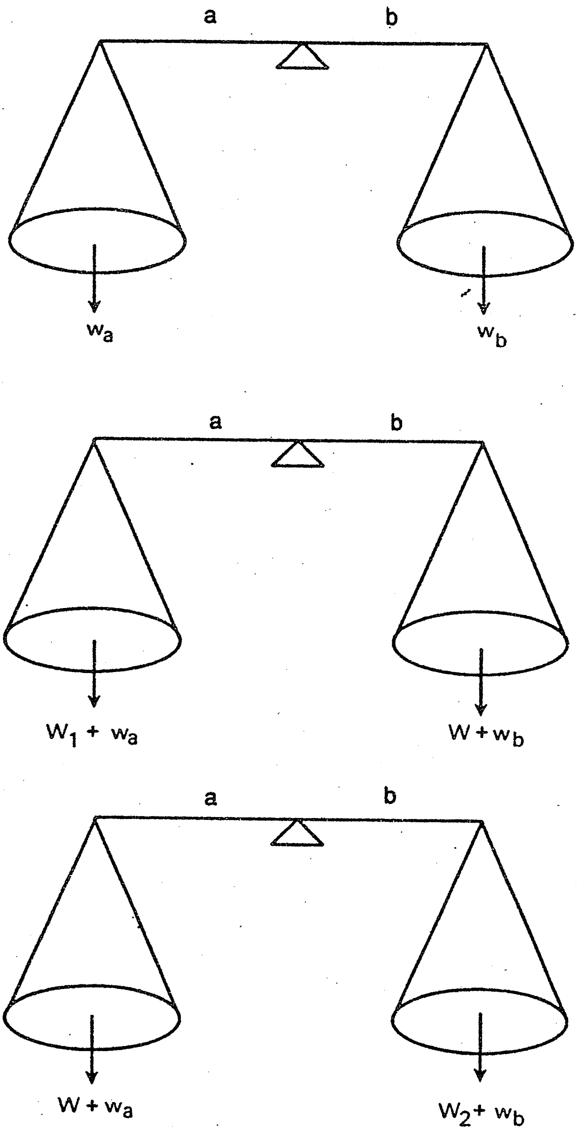


Figure 6

3. Harmonic Mean

The harmonic mean, $H(x_1, x_2)$ of two positive numbers x_1, x_2 is defined by

$$H(x_1, x_2) = 1/\{\tfrac{1}{2}(\tfrac{1}{x_1} + \tfrac{1}{x_2})\}.$$

The procedure for calculation is:

Find the reciprocals of x_1, x_2
Find the arithmetic mean of the reciprocals
Find the reciprocal of the arithmetic mean.

The expression for the harmonic mean can be simplified algebraically to

$$H(x_1, x_2) = \frac{2x_1 x_2}{x_1 + x_2}$$

and it can also be written simply using index notation

$$H(x_1, x_2) = \left(\frac{x_1^{-1} + x_2^{-1}}{2}\right)^{-1}.$$

A geometrical illustration is the following.

In Figure 7, ΔABC is right-angled at C. A square is inscribed as shown. It can be proved that

$$x = \frac{ab}{a + b}$$

and hence the semi-perimeter of the square is

$$2x = \frac{2ab}{a + b} = H(a, b).$$

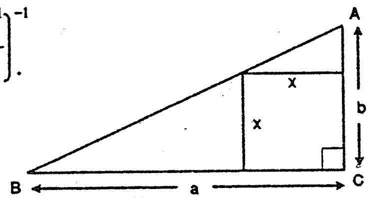


Figure 7

Notice that the result remains true in the more general case (Figure 8) where ΔABC is not necessarily right-angled, but the inscribed figure is a rhombus, i.e.

$$2x = H(a, b).$$

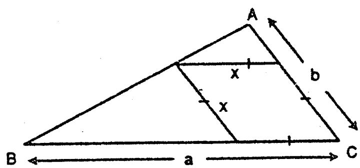


Figure 8

A physical illustration of the use of the harmonic mean is shown in Figure 9.

A car travels from A to B at v_1 km/h and then from B to A at v_2 km/h.

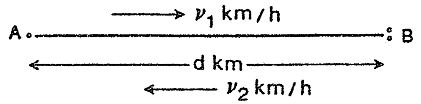


Figure 9

We assert that the average speed for the total journey is $H(v_1,v_2)$ km/h.

Proof Let $AB = d$ km

Let v km/h be the average speed

Let $t_1 h$ be the time taken from A to B

Let $t_2 h$ be the time taken from B to A.

Now

$$v = \frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{v_1} + \frac{d}{v_2}} = \frac{2}{\frac{1}{v_1} + \frac{1}{v_2}} = H(v_1,v_2) \ .$$

A second physical illustration occurs in optics. See Figure 10. An object A, in front of a spherical mirror centre C, has image B along the ray AC.

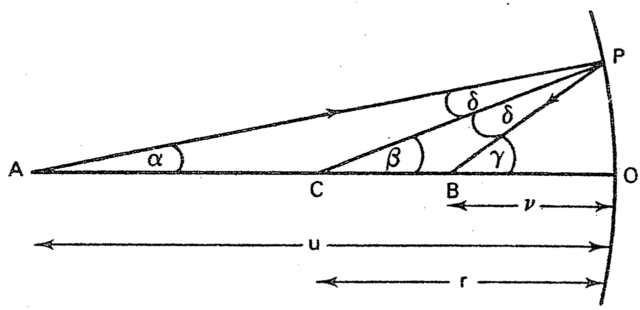


Figure 10

If the distances of A,B from the mirror are u, v respectively, and if r is the radius length of the mirror, we shall show that

$$r = H(u,v).$$

In the figure, an incident ray of light, AP, meets the mirror at P and is reflected back along PB. We use a law of optics which states that the magnitude of the angle of incidence, APC, is equal to the magnitude of the angle of reflection CPB. We shall also assume that the angle magnitudes α, β, γ, δ (see Figure 10) are small.

We first find some connections between α, β, γ, δ

From $\Delta\ APC$, $\qquad \beta = \alpha + \delta. \qquad (4)$

From $\Delta\ CPB$, $\qquad \gamma = \beta + \delta. \qquad (5)$

Elimination of δ (by subtraction) gives

$$\beta - \gamma = \alpha - \beta$$

i.e.

$$\beta = \frac{\alpha + \gamma}{2}$$

$$= A(\alpha, \gamma).$$

[This is an interesting but incidental result.]

From $\Delta\ APB$, $\qquad \gamma = \alpha + 2\delta. \qquad (6)$

(This may also be obtained by eliminating β from (4) and (5).)

Next we find some connections between u,v,r.

By the sine rule in $\Delta\ APC$

$$\frac{\sin\ \delta}{u - r} = \frac{\sin\ \alpha}{r}. \qquad (PC = CO = r)$$

Since the angle magnitudes are small and are measured in *radians*, $\sin\ \delta \simeq \delta$, $\sin\ x \simeq \alpha$

$$\frac{\delta}{u - r} = \frac{\alpha}{r}. \qquad (7)$$

This is one connection between u, r but with the auxiliary variables α, δ which must be eliminated. We now seek another with v and the same auxiliary variables. By the sine rule in $\Delta\ BPC$

$$\frac{\sin\ \delta}{r - v} = \frac{\sin(\pi - \gamma)}{r}$$

$$= \frac{\sin\ \gamma}{r}$$

$$= \frac{\sin(\alpha + 2\delta)}{r} \qquad \text{(from (6))}$$

Again since δ, γ are small, we have

$$\frac{\delta}{r - v} = \frac{\alpha + 2\delta}{r}$$

$$= \frac{\alpha}{r} + \frac{2\delta}{r}$$

$$\therefore \quad \frac{\delta}{r - v} = \frac{\delta}{u - r} + \frac{2\delta}{r} \qquad \text{(using (7) to eliminate } \alpha\text{)}$$

$$\therefore \quad \frac{1}{r - v} = \frac{1}{u - r} + \frac{2}{r} \qquad \text{(dividing by } \delta\text{)}$$

Solving this equation for r gives

$$r = \frac{2uv}{u + v} = H(u,v) \; .$$

4. Root Mean Square

The root mean square, $R(x_1, x_2)$, of two numbers x_1, x_2 is defined by

$$R(x_1, x_2) = \sqrt{\{(x_1^2 + x_2^2)/2\}} = \left[\frac{x_1^2 + x_2^2}{2}\right]^{\frac{1}{2}}$$

The procedure for calculation is:

Find the squares of x_1, x_2

Find the arithmetic mean of the squares

Find the square root of the arithmetic mean.

A geometric illustration is shown in Figure 11 overleaf.

ABCD is a trapezium with $\overline{AB}$ parallel to $\overline{DC}$, $\overline{EF}$ is drawn parallel $\overline{AB}$ so that area *ABFE* = area *EFCD*

It can be proved that

$$x = \sqrt{\{\tfrac{1}{2}(p^2 + q^2)\}}.$$

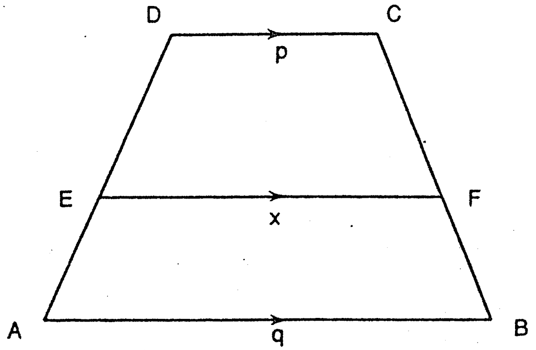


Figure 11

5. Some concluding remarks

(a) It can be shown that, for any two positive numbers x_1, x_2,

$$H(x_1, x_2) \le G(x_1, x_2) \le A(x_1, x_2) \le R(x_1, x_2).$$

(b) The index notation suggests a generalization from three of our definitions. We define the power mean, $M(n, x_1, x_2)$, of two positive numbers x_1, x_2 by

$$M(n, x_1, x_2) = \left(\frac{x_1^n + x_2^n}{2}\right)^{\frac{1}{n}} \qquad \text{for any real } n.$$

Thus

$$\begin{aligned} M(1, x_1, x_2) &= A(x_1, x_2) \\ M(-1, x_1, x_1) &= H(x_1, x_2) \\ M(2, x_1, x_2) &= R(x_1, x_2) . \end{aligned}$$

(c) As a further generalization, we leave it to the reader to define all the means for m positive numbers

$$x_1, x_2, x_3, \ldots, x_m .$$

(d) As an exercise we leave it to the reader to prove all the geometrical illustrations mentioned in this article.

* * * * *

SPIN OUT AND THE CHINESE RINGS

R. Cowban, 81 Martin St., Gardenvale

One of my grandsons has been given an "educational" puzzle called SPIN OUT†. It consists of a grooved frame in which another long, rectangular frame can slide back and forth. The first frame is open at its right hand end only and contains a circular indent in one of its sides. See Figures 1 – 3. The slide carries a set of seven "waisted" shapes, each pivoted to the sliding frame. These can be rotated (one at a time) by making use of the indent. Initially these shapes are so aligned as to make a pattern of "vertical" stripes (Figure 1). The object is to line them up so that (Figure 3) a horizontal stripe is produced. The slide may then be removed from the first frame.

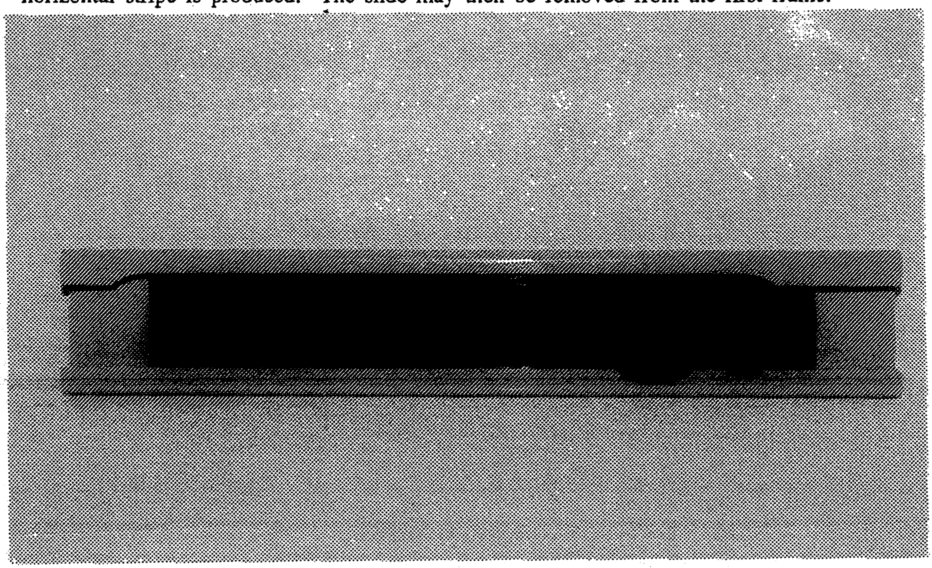


Figure 1

SPIN OUT: Initial Position

†Bill Ritchie, *SPIN OUT*, Binary Arts Corporation, 703 Timber Branch Drive, Alexandria, VA22302, USA; a mathematical puzzle, made of plastic, with an accompanying leaflet: available at toyshops.

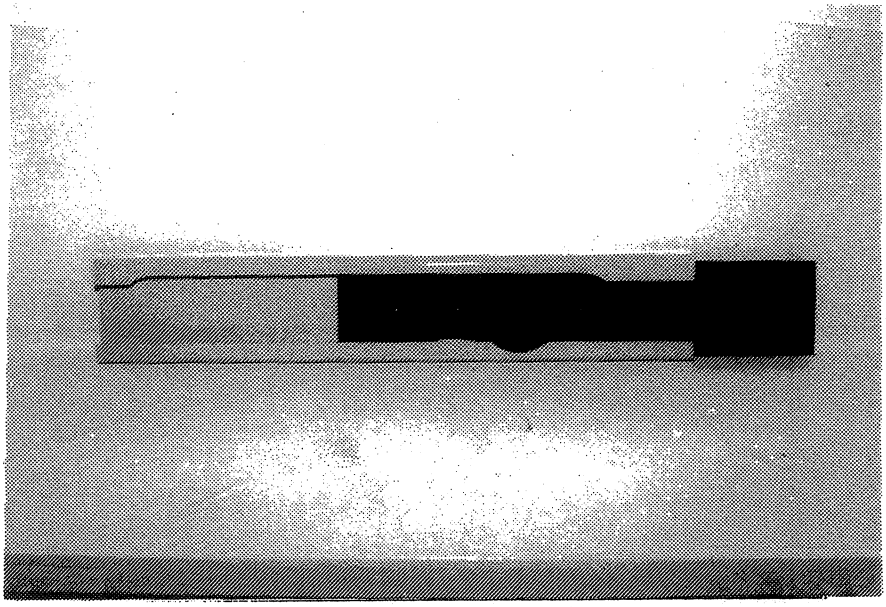


Figure 2

SPIN OUT: Intermediate Position

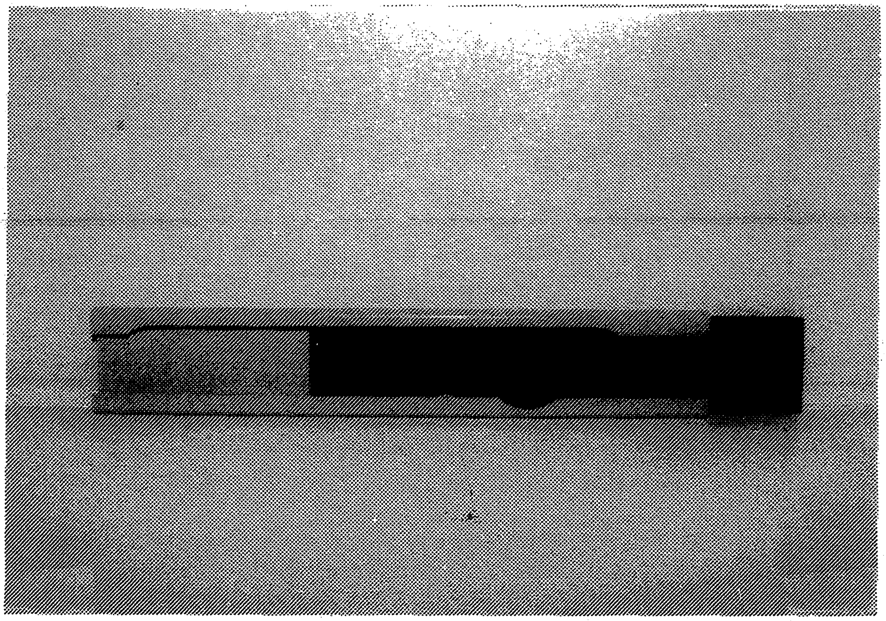


Figure 3

SPIN OUT: Final Position

My grandson found out how to solve the puzzle "sight unseen" in less than half an hour. It took me somewhat longer, but in the process I learned that the underlying strategy for solving this puzzle is the same as that for solving another famous puzzle, known as the *Chinese Rings*. These are illustrated in Figure 4.

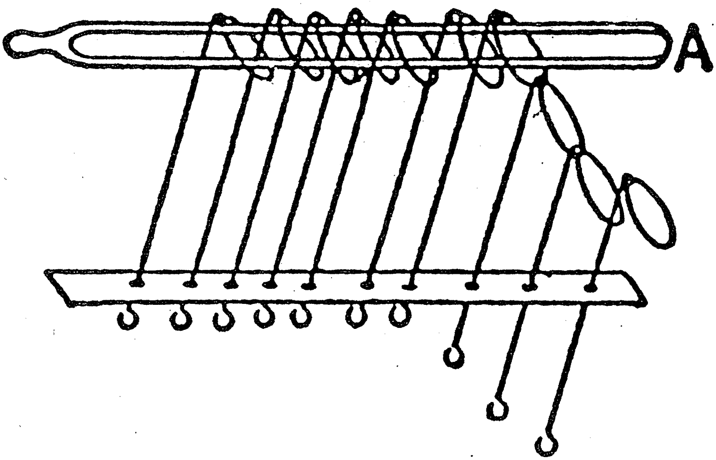


Figure 4: The Chinese Rings

There is what mathematicians call an *isomorphism*, or complete correspondence, between the two puzzles†. Although they *look* different they may both be analysed mathematically in the same way, using binary arithmetic, as we shall see.

A description of the Chinese Rings puzzle is given in several books, of which perhaps the best-known is Maurice Kraitchik's *Mathematical Recreations* (published by Dover, New York). On page 89 of that book, we read:

†This statement requires one very minor qualification. In the case of the Chinese Rings, it is *physically* possible, in certain special circumstances, to remove two rings at once – in other words, to make two moves together. This is not possible with SPIN OUT. However, the analysis of the Chinese Rings given in this article neglects this feature, and so the isomorphism is complete.

> The Chinese Rings are a toy, consisting of a fixed number of rings (usually from 5 to 8) hung on a bar in such a way that the ring on the right hand end can be taken off or put on at pleasure, but any other can be taken off or put on only when the next one on its right is on, and all the others to its right are off. The order of the rings is fixed. Only one ring can be taken off or put on at a time, except that the two rings at the extreme right can be put on or taken off together.†

Figure 4 shows the case of ten rings but I shall here concentrate on the case of seven rings, corresponding to the seven shapes in SPIN OUT. Denote the configuration of the ring by a set of asterisks (*) and zeros (0) in order: * if the ring is on the bar, 0 if it is off.

Figure 5 shows, in its second column, the sequence of moves beginning with the initial position (all rings on the bar) and ending with the final position (all rings off the bar).

Sequence in the positions of the rings	Locations of the rings (* = ON, 0 = OFF).	Binary notation
Initial position	* * * * * * *	1010101
2nd "	* * * * * * 0	1010100
3rd "	* * * * 0 * 0	1010011
4th "	* * * * 0 * *	1010010
....	....	....
23rd "	0 * 0 0 0 0 0	0111111
....	....	....
29th "	0 * 0 0 * 0 *	0111001
....	....	....
84th "	0 0 0 0 0 0 *	0000001
last "	0 0 0 0 0 0 0	0000000.

Figure 5

† This last clause is the detail referred to in the previous footnote. It is neglected in the analysis given below.

The same sequence of moves will free the two frames from one another in SPIN OUT.

According to whether the number of rings (or disks for SPIN OUT) is even or odd the number of moves required to solve the puzzle is either $\frac{1}{3}(2^{n+1} - 2)$ or $\frac{1}{3}(2^{n+1} - 1)$, where n is the number of rings. Thus in the case of $n = 7$, we need 85 moves, but if $n = 10$, we need 682. (Can you prove that the formulae quoted always give whole numbers?)

These formulae are proved in Kraitchik's book (pages 90, 91), where they are attributed to a certain M. Cros. Here is how it is done.

Reading from the left at each particular stage of the puzzle, allot a 1 to the *first* ring still *on* the bar. Then allot a 0 to the *next* ring still on the bar. Continue in this way assigning 1's and 0's alternately to the rings still *on* the bar. Next complete the pattern by allocating to each of the remaining places (corresponding to rings *off* the bar) going from left to right, the digit 1 or 0 that is *already immediately to its left.* If the ring on the extreme left is *off* the *bar*, the first digit is 0.

Thus if we have

0 * 0 0 * 0 *

we first assign 1's and 0's to the three asterisks

0 * 0 0 * 0 *
1 0 1

and then fill in the pattern

0 * 0 0 * 0 *
1 1 1 0 0 1

and finally, because the left hand ring is off, reach the string of digits: 0111001. In this way, we construct the third column of Figure 5.

The entries in Figure 5 thus formed are binary numbers, i.e. numbers expressed in base two. Thus the initial number 1010101 means

$$(1 \times 2^6) + (0 \times 2^5) + (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$$

i.e. 85. Now check the various binary numbers shown in Figure 5. Each is one less than its predecessors, until the final position is represented by the number zero. Thus 85 moves are required to complete the task of removing all the rings.

This proof generalises to n rings and you may care to construct it for yourself.

Ritchie, who procduces SPIN OUT, tell us that it was invented by an American, William Keister "a pioneer of switching theory and design at Bell laboratories". During the late 1930s he sought electrical analogues of the Chinese Ring puzzle and "stumbled onto a general series of binary code sequence puzzles. ... He designed a set of code sequence puzzles on paper which he then solved mathematically, using Boolean algebra, a precursor of today's computer languages".

Continued on p. 123.

BOOK REVIEW

Gender and Mathematics: An International Perspective, Edited by Leone Burton (Cassell Education, 1990, $55.95, 162 pages).

REVIEWED BY:

ROBYN ARIANRHOD, Monash University

One of the most exciting things about recent research into gender and mathematics is that it challenges women and men to re-examine both the nature of mathematics itself, and the standard methods of teaching and learning it. The international perspective of this provocative and informative collection of research articles adds a cross-cultural dimension to such an examination.

The articles are generally both readable and scholarly (no mean feat!) and offer a variety of anecdotal, statistical and speculative material which I can only briefly describe here. Let me say, however, that my feeling on reading the book is one of excitement at the possibilities for perceiving, describing and doing mathematics. Not all the findings are in agreement, but that only adds to the challenge of trying to sort out what *I* really think about mathematics. I am sure that teachers would benefit similarly from pondering the challenges offered here, and that they could guide their students in some useful and fascinating group discussions on *their* perceptions of and needs from their mathematics classes.

Several of the articles discuss the stereotypical image of mathematics as closed and complete, and suggest ways of involving students in an open-ended *process* of doing mathematics in an effort to avoid teaching students to merely generalize memorized techniques. As reported in the articles by Helen Verhage (the Netherlands), Evangelie Tressou-Milonas (Greece), Mary Barnes and Mary Coupland (Sydney) and Beth Marr and Sue Helme (Melbourne), such approaches have been overwhelmingly successful in encouraging girls and mature-age women students to enjoy maths and to feel good about both the subject and their ability to succeed in it. (This is a real achievement, as the evidence presented in this book and elsewhere does point clearly to the fact that boys (at least in Western cultures) are much more confident about their mathematics ability than are girls. It certainly seems true of my peers; what about yours?) These articles describe innovative approaches to teaching methods – group work, individually-paced work, computer-aided learning, and "hands-on" approaches such as both boys and girls doing embroidery in order to later examine the symmetry involved in the stitches – and to curriculum design: culture- and gender-inclusive topics such as studying symmetry by examining embroidery†, Islamic and Celtic art, Chinese tangrams, etc.; using female and non-Anglo names in text books; books; and making maths more relevant to girls.

How *does* one make maths relevant to girls without merely reinforcing traditional roles? Consider, for example, the work of Mary Barnes and Mary Coupland. In their

†Note that discussion is also offered as to whether such choices only reinforce traditional gender roles.

article, they quote previous work on the different ways in which men and women define their identity: men in terms of autonomy and separation, women in terms of relationships, responsibility and care; in response to this suggestion, they have designed a "humanized" introductory calculus course which seeks to include women's concerns in its motivation. Calculus, they say, is generally introduced in purely abstract terms concerning gradients of tangents to curves, "and no reason is given why this might be important or interesting". When applications *are* introduced, they can be summarized as being about "profits, weapons and machines". (The article by Máire Rodgers discusses girls' attitudes to mechanics.) Mary Barnes and Mary Coupland introduce "responsibility and care" into their introductory calculus course by using human-interest applications of calculus (with the help of computer programs to do the more complicated maths) to such topics as world population growth, populations of endangered species, disposal of radioactive waste, and the build-up of pollutants in lakes, rivers and the human bloodstream.

Despite the success of the pedagogical and curricular innovations described in these articles, some cautionary notes are also sounded. Frequently girls – especially those from lower social classes (Tressou-Milonas) – are unwilling to participate in new, open-ended approaches to mathematics. Often girls have been rewarded for being less assertive than boys, or perhaps they just feel insecure in maths classes, and if they are suddenly"plunged into open-ended situations which require higher risk-taking and uncertainty" than they are used to, they may find "the current move towards 'problem-solving and real-life mathematics' [leaves them in an even worse situation]" (Máire Rodgers).

It is also interesting to note that some previously-reported (Western) findings, such as the importance for girls' success of single sex maths classes and female maths teachers, were not born out in the cross-cultural study, involving 14 European, Pacific and Asian countries, carried out by Gila Hanna, Erika Kündiger and Christine Larouche. The authors suggest that perhaps these variables are significant when operating in conjunction with other (unknown) societal factors.

There are four articles which report various people's discussions on their experiences with and attitudes to mathematics: in those by Lyn Taylor and by Joanne Rossi Becker, male and female mathematicians and graduate students talk about how they actually *do* mathematics, and how they feel about it (humbled, for example!); secondary students discuss their motivations and interests in the articles by Zelda Isaacson and by Máire Rodgers. I found it very interesting to compare my experiences with those discussed here, and suggest that these articles could form the basis of useful discussions amongst male and female students.

Several articles include statistical data, so that in addition to being intrinsically interesting, they could provide excellent material for student projects or exercises in interpretation of data: those by Hann, Kündiger and Larouche, by Tressou-Milonas, by Gilah Leder (on gender and classroom practice), by Berinderjeet Kaur (on girls and maths in Singapore), by Prudence Purser and Helen Wily (on the numbers and professional destinations of male and female maths graduates in New Zealand), and by Giuseppina Fenaroli and her co-workers (on women and mathematical research in Italy).

There is plenty here that would interest students, although they may benefit more from teacher-guided group discussions than by individual reading. The articles are different in approach – some providing anecdotal evidence and others providing statistical data – and in direction; some are more readable or interesting or relevant than others. As a whole, however, they provide an excellent, accessible and international overview of current research in the field of gender and mathematics.

* * * * *

COMPUTER SECTION

EDITOR: R.T. WORLEY

In computer drawing packages one needs a method of drawing smooth curves. For example, the curve in Figure 1 could be drawn with a drawing package. To do this, one needs the ability to solve:

Problem 1. Given four points A, B, C and D, draw a smooth curve starting out from A and ending at D. The curve must leave A in the direction of B, and reach D from the direction of C.

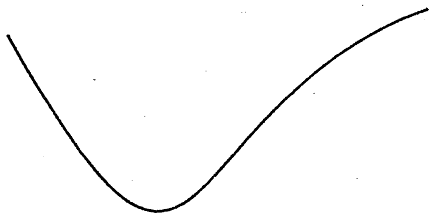


Figure 1

This problem is not really well posed, for there are many such curves, and we really haven't even said what we mean by the word smooth. The graph of a polynomial function $y = f(x)$, for example $y = 3x^2 - 2x + 1$, is one sort of curve we would call smooth. However not all curves can be the graph of a function – a function takes a single value at a given value of x, so any curve that intersects the y-axis, for example, in two or more places, is not the graph of a function. We can get around this by allowing both x and y to be polynomial functions of a variable t (thought of as the 'time' variable – at time 0 we are at A, and we progress along the curve from A to D as t increases from 0 to 1). In other words, we consider curves for which there are polynomials $p_x(t)$, $p_y(t)$ such that the point $P = (x, y)$ on the curve is given by

$$x = p_x(t), \quad y = p_y(t), \qquad 0 \le t \le 1.$$

The starting point is $A = (p_x(0), p_y(0))$, and the ending point is $D = (p_x(1), p_y(1))$. We call such a curve a polynomial curve.

The simplest polynomial curve is the straight line from A to D. This is given by

$$x = (1-t)x_A + tx_D, \qquad y = (1-t)y_A + ty_D$$

where $A = (x_A, y_A)$ and $D = (x_D, y_D)$. Since the rules for the x and y coordinates of points are similar, we write the rule as

$$P = (1-t)A + tD. \qquad (1)$$

If we put $t = 0$, we get the starting point $P = A$, and if we put $t = 1$ we get the ending point $P = D$. We generalise this idea to get a curve from A to D.

If we look closely at (1) we can regard it as a 'blending' of the points A and D. The point A has blending coefficient $(1-t)$ which is 1 when $t = 0$ and 0 when $t = 1$, and the point D has blending coefficient t which is 0 when $t = 0$ and 1 when $t = 1$.

Suppose we now take a third point C. The lines AC and CD are given by $(1-t)A + tC$, and $(1-t)C + tD$. We blend these using the same blending coefficients $(1-t)$ and t to obtain

$$P = (1-t)\{(1-t)A + tC\} + t\{(1-t)C + tD\}. \qquad (2)$$

This simplifies to the equation

$$P = (1-t)^2A + 2t(1-t)C + t^2D. \qquad (3)$$

This gives curves such as those illustrated in Figure 2. Notice that the curves always lie within the triangle ADC, and that the curve starts out from A in the direction of C, and the curve enters D from the direction of C.

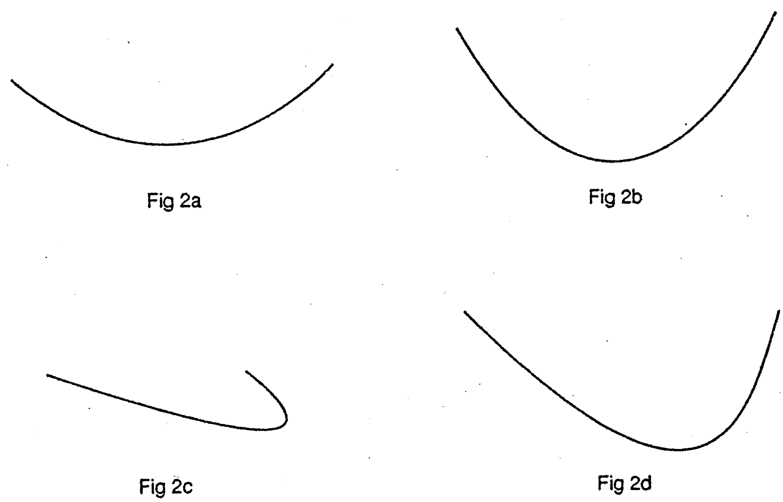


Figure 2

With two points B and C, as well as the starting and ending points A, D, we can blend the curves from A to C and from B to D of type given by equation (3), by writing

$$P = (1-t)\{(1-t)^2A + 2t(1-t)B + t^2C\} + t\{(1-t)^2B + 2t(1-t)C + t^2D\}$$

which simplifies to

$$P = (1-t)^3A + 3t(1-t)^2B + 3t^2(1-t)C + t^3D. \qquad (4)$$

If we draw some examples of these curves, experimenting with different positions of B and C, we can see (Figures 3, 4) that this curve has the properties required to solve Problem 1 (this fact can be proved mathematically). If we experiment (see Figure 3, in which AB is 1/3, 1 and 2 times CD in length, and Figure 4, in which AB is 1, 3/2 and 2 times CD in length) with various positions of B, while keeping the direction of AB unchanged, we discover that the length of AB has an effect on the curve – the longer AB, the further the curve goes in the direction of B before heading off towards D. In some sense, the points B, C exert a 'pull' on the curve. We see very clearly that we can draw many curves to solve problem 1, simply by varying the lengths of AB and CD while keeping their directions unchanged.

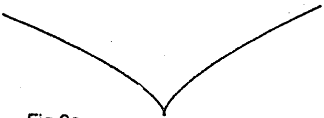


Fig 3a

Fig 3b

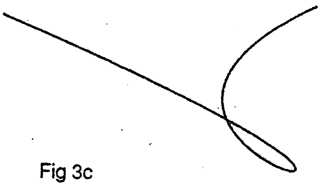


Fig 3c

Figure 3

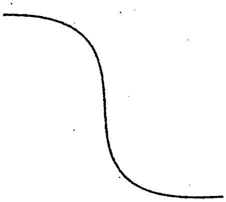


Fig 4a

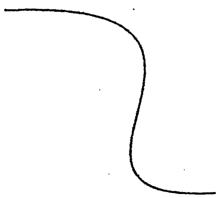


Fig 4b

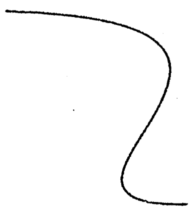


Fig 4c

Figure 4

One can take further points to exert a 'pull' on the curve, leading to equations similar to (4). The corresponding equations can be obtained by extending the blending idea. However the cubic curve (3) defined by two end points A, D and two control points B, C normally can give the type of curve we want to draw.

An interesting feature of the Bezier curves, as the curves described above are called, is that they can be drawn simply by plotting mid-points. The simplest example is the straight line (1). To draw the line we plot the point corresponding to $t = 1/2$. This is simply $N_1 = (1/2)A + (1/2)D$, the midpoint of AD. Next, we plot the points corresponding to $t = 1/4$ and $t = 3/4$. These are the points $M_2 = (3/4)A + (1/4)D$ and $M_3 = (1/4)A + (3/4)D$. A simple calculation shows that $M_2 = (1/2)A + (1/2)M_1$, the midpoint of AM_1, and that M_3 is the midpoint of M_1D. Likewise the points corresponding to $t = 1/8$, $3/8$, $5/8$, and $7/8$ are the midpoints of AM_2, M_2M_1, M_1M_3 and M_3D. This gives rise to the following algorithm for drawing a straight line.

```
procedure line(A,D:point)
var M:point;
begin
     if SufficientlyDifferent(A,D) then begin
          M := MidPoint(A,D);
          PlotPoint(M);
          line(A,M);
          line(M,D)
          end
     else begin
          PlotPoint(A);
          PlotPoint(D)
          end
end;
```

This algorithm requires the type 'point' to be defined, a function that gives the midpoint, a procedure that plots points, and a function that tests if two points are different enough that the midpoint will not be either of the two end points. It is interesting to watch this in action drawing a line. I should point out that this is not the best way of drawing a straight line – it is simply a curious method that leads to the method of drawing the Bezier curves given by equations (3) and (4).

If we take $t = 1/2$ in equation (3) we get the point

$$N_1 = (1/4)A + (1/2)C + (1/4)D$$

which I will call the mid3point of A, C and D. If we take $t = 1/4$ in equation (3) we get the point

$$N_2 = (9/16)A + (6/16)C + (1/16)D,$$

which (after some calculation) we discover can be written as

$$(1/4)A + (1/2)[(1/2)A + (1/2)C)] + (1/4)N_1$$

which is mid3point(A, midpoint(A, C), N_1). This leads to the following way of drawing such a curve.

```
procedure Bezier3(A,C,D:point)
var N:point;
begin
    if SufficientlyDifferent(A,C,D) then begin
        N := mid3point(A,C,D);
        PlotPoint(N);
        Bezier3(A,midpoint(A,C),N);
        Bezier3(N,midpoint(C,D),D)
        end
    else begin
        PlotPoint(A);
        PlotPoint(D)
        end
end;
```

Similar analysis can be done for equation (4), leading to the definition of the mid4point of A, B, C, D as $(1/8)A + (3/8)B + (3/8)C + (1/8)D$ and a plotting routine

```
procedure Bezier4(A,B,C,D:point)
var S:point;
begin
    if SufficientlyDifferent(A,B,C,D) then begin
        S := mid4point(A,B,C,D);
        PlotPoint(S);
        Bezier4(A,midpoint(A,B),mid3point(A,B,C),S);
        Bezier4(S,mid3point(B,C,D),midpoint(C,D),D)
        end
    else begin
        PlotPoint(A);
        PlotPoint(D)
        end

end;
```

One of the prime advantages of the above methods of drawing Bezier curves is that the basic operations used are just addition and division by 2. Both of these operations are fast operations for a computer to perform, so the drawing can be done very quickly. As presented, the algorithms are recursive, and although easy to write in Pascal, they are difficult to write in standard BASIC.

The Bezier curve given by equation (4) can be used to draw a curve which is extremely close to a quarter circle (and hence, using four such curves, a full circle). The Bezier curve starting at $A = (1, 0)$ in the direction of $B = (1, 4(\sqrt{2}-1)/3)$ and entering $D = (0, 1)$ from the direction of $C = (4(\sqrt{2}-1)/3, 1)$ lies between the circle of radius 1 and the circle of radius 1.0003. It is, to within an error of .03%, a quarter circle.

* * * * *

HISTORY OF MATHEMATICS SECTION

EDITOR: M.A.B. DEAKIN

The Four Circles Theorem

The issue of *Nature* (the scientific periodical) for June 20, 1936 contained the following verse.

The Kiss Precise

For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. Soddy.

Its author, Frederick Soddy, is best remembered as a chemist. He won the Nobel Prize for Chemistry in 1921 for his discovery of isotopes, whose name "isotope" is also due to him. Soddy's interests extended well beyond Chemistry, however, and embraced Mathematics, Economics and Social Responsibility in Science.

This is an example of his interest in Mathematics and the verse, or rather its first two stanzas, describes the theorem depicted on the cover. To see quite what the theorem says, look at Figures 1 and 2 overleaf. The three circles centred at A, B, C each touch (kiss) the other two, thus generating three distinct points of mutual tangency. We now seek to introduce a fourth circle, tangent to each of the other three.

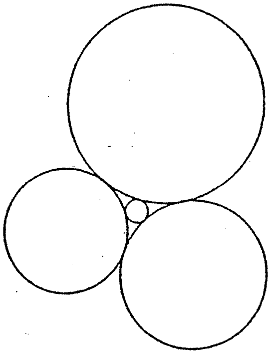


Figure 1

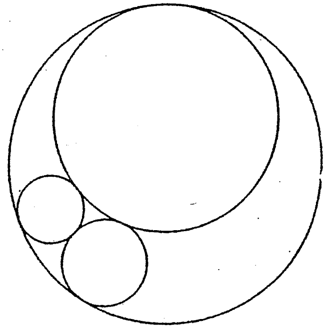


Figure 2

This fourth circle may be drawn in either of two ways: it may nestle between the other three as in Figure 1 ("one in three") or else encircle the others as in Figure 2 ("three in one"). These are the only two configurations possible and the first stanza of Soddy's verse describes the situation in detail.

The second stanza gives the relation that must hold between the radii of the various circles if they are to be mutually tangent in either of these ways. The smaller a circle is, the more violently it curves. Its curvature (bend) is defined as $1/r$, where r is the radius ("the inverse of the distance from the centre").

If we let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ be the curvatures of the four circles, then we have the Four Circles Theorem:

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2. \qquad (1)$$

This formula applies directly to Figure 1 and it also applies to Figure 2 if we make ε_4 negative ("concave bends have a minus sign"); the "concavity" here refers to the mode of tangency.

The third stanza gives an extension to spheres. The final two lines give the theorem in this case.

The material was suggested to us by Professor Ian Rae, the Dean of Monash University's Faculty of Science, and by Professor Bert Bolton, recently retired from the Chair of Theoretical Physics. Their interest in the topic inspired me to find out more about the theorem. I had in fact seen it before, in Martin Gardner's *Scientific American* column for April 1961. Gardner used these columns as the basis for a number of books of popular Mathematics and the book that includes the Four Circles Theorem is called *Mathematical Circus*.

Sadly perhaps, Soddy's discovery of the theorem is not the first. The honour of discovery goes to René Descartes, the originator of coordinate geometry. It occurs in his correspondence with Princess Elizabeth of Bohemia, who was one of his pupils. In a letter dated November, 1643 a formula equivalent to Equation (1) is given, together with an indication of its proof. It uses the diagram reproduced on the cover of this issue of *Function* and gives formulae for AD, AK in terms of the radii of the various circles. From this, and some heroic algebra which is not reproduced in the letter, Descartes was able to derive his result.

The theorem was rediscovered in 1826 by Jacob Steiner, a major figure in the revitalisation of Geometry around that time. Later still in 1842, an English amateur, Philip Beecroft, also rediscovered it and gave a somewhat simpler proof. He seems not to have published this, but it is given in H.S.M. Coxeter's *Introduction to Geometry*. Soddy, if I understand Gardner aright, did not have a proof of the result, though he may have been the first person to state the corresponding theorem for spheres.

A number of proofs of the theorem are now known. All are somewhat involved for presentation in *Function*, but we give here an argument (short of a complete proof) devised by the 20th Century geometer Daniel Pedoe, but incorporating an improvement due to Coxeter.

If we assume that a simple algebraic relation gives ε_4 in terms of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, then (because there are *two* possibilities for ε_4) it seems likely that this relation is quadratic. Moreover, by symmetry, it must be that this relation will not alter if we rename the various circles. One algebraic expression with this property is $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, a second is $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2$. All other symmetric functions of the curvatures either involve cubic (or higher) terms, or else depend on these two.

So the required formula must connect

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 \quad \text{with} \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4.$$

Moreover it must be consistent in the units in which it is expressed. (See *Function*, *Vol.* 10, *Part* 1, p.14.) Thus we require

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = k(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2, \qquad (2)$$

where k is a constant. There is no other possibility meeting all the conditions imposed. This is because the left-hand side must be measured in units of length^{-2} and so also, therefore, must be right. Thus $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, which is measured in units of length^{-1} needs to be squared.

In order to evaluate k, choose a very simple case: the simplest possible, illustrated in Figure 3. Here two of the circles have degenerated into parallel straight lines with zero curvature ("zero bend's a dead straight line") whose point of contact is at ∞. We now have $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 = \varepsilon_4 = \varepsilon$ (say). Substituting these values into Equation (2) produces the value $k = 1/2$ as claimed by Soddy and the others.

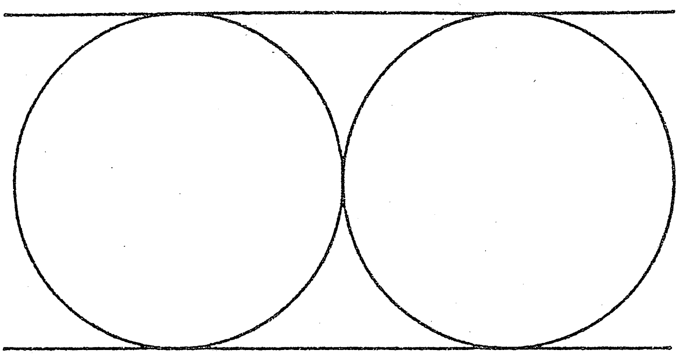


Figure 3

The extension to spheres soon led to versions of the theorem for four and higher dimensions. By January 9, 1937 various people (at least three of them) had produced announcements of n-dimensional versions of the theorem, all of them couched in verse and all stating, but not proving, the general case. *Nature* on the date just mentioned published one of these. It took the form of a fourth stanza to Soddy's verse. The author was a Mr. Thorold Gosset and his stanza ran as follows.

And let us not confine our cares
To simple circles, planes and spheres
But rise to hyper flats and bends
Where kissing multiple appears.
In n-ic space the kissing pairs
Are hyperspheres, and Truth declares—
As $n + 2$ such osculate
Each with an $n + 1$ fold mate
The square of the sum of all the bends
Is n times the sum of their squares.

There is some reason to beleive that Gosset derived a proof of this result prior to Soddy's announcement of the result in the case $n = 3$. However if he did his proof remained unpublished and it was Coxeter who first published a rigorous mathematical proof.

We thought to look at the case $n = 2$ by other means. By use of Heron's formula (which gives the area of a triangle when the lengths of its sides are known), it is possible to deduce the formula

$$\pm \varepsilon_4\sqrt{\varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2} \pm \varepsilon_1\sqrt{\varepsilon_3\varepsilon_4 + \varepsilon_4\varepsilon_2 + \varepsilon_2\varepsilon_3} \pm \varepsilon_2\sqrt{\varepsilon_4\varepsilon_1 + \varepsilon_1\varepsilon_3 + \varepsilon_3\varepsilon_4} \pm \varepsilon_3\sqrt{\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_4 + \varepsilon_4\varepsilon_1} = 0 \quad (3)$$

The object is to deduce Equation (1) from this. Colin McIntosh, who used computer algebra packages to analyse the Beijing Theorem (*Function*, *Vol.*15, *Part* 2, p.48), also looked at this problem. Getting rid of the square roots in Equation (3) produces a polynomial equation of 16th degree and this he succeeded in factorising. One factor gives Equation (1), another gives a result that is readily seen not to be applicable, the third and final possibility is very difficult and still has not been analysed.

"The Kiss Precise" was not Soddy's only venture into mathematical verse. Shortly after publishing this, he produced a second such effort, entitled "The Hexlet". This also generated interesting geometrical work and Professor Bolton writes that it may have practical applications. However, that work, though related, is another story.

Evidently the use of verse as a medium for scientific publication was more acceptable in the 1930s than in more recent years. In 1967, Pedoe noted that Sir Alexander Oppenheim a few years previously had discovered "an attractive theorem in geometry" which Pedoe's wife had put into poetic form, "but *Nature* rejected the joint offering".

* * * * *

From p. 111.

He further writes:

> SPIN OUT's solution code sequence is exactly the same as the Gray binary code, a binary basic counting system named after Frank Gray a colleague of Keister's at Bell Lab. Gray created this code *in the 1930's* to provide an error correcting technique for electronics communication. It is still used today as a basis for electronic switching and for computer logic.

It is interesting to reflect that, long before the modern era of electronic digital computers, the groundwork for their design and study was being laid by engineers and mathematicians – by the application of Boolean algebra and binary codes to the design of communication networks, to problems in logic and to the creating of mathematical toys and puzzles.

* * * * *

CORRECTION

On p. 81 of *Vol.* 15, *Part* 3 a number of formulae were misprinted. Correct is

$$ln(1 + y) = y - (1/2)y^2 + (1/3)y^3 - \dots + (-1)^{n+1}\, y^n + E$$

with consequent change to the two succeeding formulae.

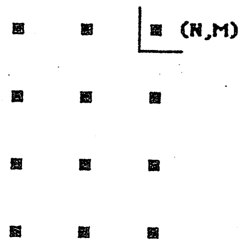
PROBLEMS AND SOLUTIONS

EDITOR: H. LAUSCH

SOLUTIONS

Problem 12.4.2. (Communicated by M.A.B. Deakin, from Chiang Mai, Thailand). The game NORTH–EAST is played on the rectangular array of points in the plane with integral coordinates (n, m), where $0 \leq n \leq N$, $0 \leq m \leq M$. Player A selects a point (p, q) and removes all those points for which $n \geq p$, $m \geq q$. Player B then selects a point (r, s) and removes all those points still left for which $n \geq r$, $m \geq s$, etc. The loser is the player who takes $(0, 0)$. The problem is to show that A has a winning strategy.

Solution (by the proposer). Let A move first. Suppose A takes the point (N, M)



Either (a) this is a winning strategy, in which case we are done;

or (b) it is a losing strategy, and now B can win by choosing some $n(\leq N)$, $m(\leq M)$.

But in case (b) A could have made this move in the first place and so forced the win.

Note that this argument uses the winning strategy theorem (see *Function*, *Vol.5*, *Part* 4, p.9).

Problem 15.1.3 A person sits for an examination in which there are four papers with a maximum of m marks for each paper; show that the number of ways in which a total of $2m$ marks may be obtained is $\frac{1}{3}(m + 1)(2m^2 + 4m + 3)$.

Solution (John Barton, North Carlton) We count the number of ways of getting a total of $m + r$ $(0 \leq r \leq m)$ on three papers, the fourth paper then providing the remaining $m - r$. There are two distinct "regimes" shown below separated by a line.

Paper I	Paper II	Paper III	
0	r	m	
	r + 1	m - 1	
	...	...	
	m	r	number = m - r + 1
...	...	...	
s	r - s	m	
	r - s + 1	m - 1	
	...	...	
	m	r - s	number = m - r + s + 1
...	...	...	
r	0	m	
	...	...	
	m	0	number = m + 1
r + 1	0	m - 1	
	...	...	
	m - 1	0	number = m
...	...	...	
r + t	0	m - t	
	...	...	
	m - t	0	number = m - t + 1
m	0	r	
	...	...	
	r	0	number = r + 1

The total for the first "regime" is

$$(m - r + 0 + 1) + (m - r + 1 + 1) + (m - r + 2 + 1) + \dots + (m - r + r + 1)$$
$$= (r + 1)(m - r + 1) + \tfrac{1}{2}r(r + 1)(2m - r)(m + r + 1).$$

The total for the second "regime" is

$$(m - 1 + 1) + (m - 2 + 1) + \dots + (m - (m - r) + 1)$$
$$= (m - r)(m - 1) - \tfrac{1}{2}(m - r + 1) = \tfrac{1}{2}(m - r)(m + r + 1).$$

Adding these, we get $\frac{1}{2}\{m^2 - 2r^2 + 2mr + 3m + 2\}$, and this has to be summed with respect to r for $0 \le r \le m$. We get

$$\tfrac{1}{2}(m^2 + 3m + 2)(m + 1) - m(0 + 1 + 2 + \dots + m) - (0 + 1^2 + 2^2 + \dots + m^2)$$
$$= \tfrac{1}{2}(m^2 + 3m + 2)(m + 1) - \tfrac{1}{2}m^2(m + 1) - \tfrac{1}{3}m(m + 1)(m + \tfrac{1}{2})$$

by use of two standard results. We may now simplify the result to give the expression quoted in the problem statement.

PROBLEMS

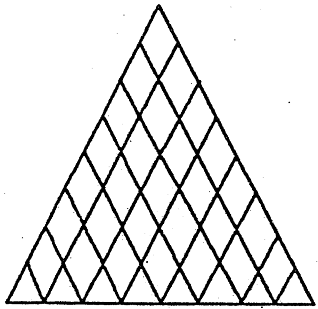
Problem 15.4.1 (from *Mathematical Spectrum*). A man has 3 sons. The age of the youngest times the sum of the ages of the other two is 1495; the age of the second son times the sum of the ages of the other two is 1767. How old are the sons?

Problem 15.4.2 (K.R.S. Sastry, Addis Ababa, Ethiopia). A parallelogram $ABCD$ (with diagonals AC and BD) is called self-diagonal if the sides are proportional to the diagonals, i.e. $AB : AC = BC : BD$. Prove that the parallelogram $ABCD$ is self-diagonal if and only if $AC + BD = \sqrt{2}(AB + BC)$.

Draga's choice

Many other countries have school magazines devoted to mathematics. In the ancient city of Ljubljana, known as Emona in Roman times, the Association of Mathematicians, Physicists and Astronomers of Slovenia has been instrumental for the last 18 *years in producing the periodical* Presek. *It is, of course, written in Slovenian, a Central European Slavonic language. Draga Gelt, artist of the Department of Earth Sciences at Monash University, who in the past has made contributions to* Function *through her art work, has read a number of* Presek *issues and recommended that the following problems be put before the readers of* Function.

Problem 15.4.3 (submitted by Sandi Klavzar; *Presek* 18, *part* 1, 1990/91). How many triangles are there in the figure below?



Problems 15.4.4 and 15.4.5 (*Presek* 17, *part* 1, 1989/90) are from ancient China:

Problem 15.4.4

You do not know how many there are.
Putting groups of three together, two are left.
Five and five together, three are left;
grouping seven and seven, two are left again.
Tell how many there are.

The author of this problem is Mr. Sun, who recorded it in his book during the fourth century. Do not be satisfied with your first solution. There are many solutions.

Problem 15.4.5 A city has a circular wall. We do not know its circumference or its diameter. The city has four gates. Outside, a tall tree grows, 3 li north of the city. If we leave the city through its southern gate and then walk eastward, we have to walk 9 li before we can see the tree. Calculate the circumference and the diameter of the fortress. (1 li = 612 metres.)

YEAR TWELVE INTERNATIONAL

Here are problems from the Entrance Examination of the Polytechnic of Athens.

Problem 15.4.6 Determine those odd positive integers n which have the property that the common roots of $f(x) = (x + 1)^n - x^n - 1$ and $h(x) = (x + 1)^{n-1} - x^{n-1}$ contain the roots of $x^2 + x + 1$.

Note that for solving this problem, some basic knowledge about complex numbers is useful.

Problem 15.4.7 Prove that the polynomial

$$f_n(x) = x \sin a - x \sin(na) + \sin(n - 1)a$$

is exactly divisible by

$$h(x) = x^2 - 2x \cos a + 1 \ ,$$

where a is a real number and n is an integer greater than 1.

MATHEMATICAL OLYMPIADS

Members of the team that is scheduled to represent Australia at the 32nd International Mathematical Olympiad in Sigtuna, Sweden's first capital where the country's first coin was minted, have gone through intensive preparations since the 1991 Asian Pacific Mathematics Olympiad (APMO). Beside the APMO, there are a number of multinational competitions such as: the Nordic Mathematical Olympiad (Denmark, Finland, Iceland, Sweden), the Austrian–Polish Mathematics Competition, the Hungarian–Israeli Mathematics Competition, the Balkan Mathematical Olympiad (Albania, Bulgaria, Cyprus, Greece, Romania, Yugoslavia), the Maghrebine Mathematical Olympiad (Algeria, Tunisia, Morocco), the Gulf Mathematical Olympiad (in 1988 and 1990, with participating countries Bahrain, Iraq, Kuwait, Oman, Qatar, Saudi Arabia and the United Arab Emirates) and the Iberoamerican Mathematical Olympiad (since 1986). The latter competition is of special interest to Australia inasmuch as Colombia and Mexico are in it as well as in the APMO. Function, *for once, is keeping a good eye on the Iberoamerican competition and will gladly receive solutions to the problems posed at Valladolid (Spain) on 25 September 1990.*

Problem 1 (Argentina). Let f be a function, defined for non-negative integers and satisfying the conditions:

(i) if $n = 2^j - 1$, for $j = 0,1,2,\ldots$, then $f(n) = 0$;

(ii) if $n \neq 2^j - 1$, for $j = 0,1,2,\ldots$, then $f(n+1) = f(n) - 1$.

(a) Show that for all integers $n \geq 0$, there exists an integer $k \geq 0$ such that $f(n) + n = 2 - 1$.

(b) Find $f(2^{1990})$.

Problem 2 (Colombia). In a triangle ABC, let I be the incentre and D, E, F the points of tangency of the incircle with the sides BC, CA, AB respectively. Let P be the other point of intersection of the line AB with the incircle.

If M is the mid-point of EF, show that P, I, M, D are either concyclic or colinear.

Problem 3 (Spain). Let $f(n) = (x+b)^2 - c$ be a polynomial, b and c being integers.

(a) If p is a prime such that $p \mid c$ and $p^2 \nmid c$, show that, for all integers n, $p^2 \nmid f(n)$.

(b) Let $q \neq 2$ be a prime, and suppose that $q \nmid c$. If $q \mid f(n)$ for some integer n, show that for each positive integer r, there exists an integer n' such that $q^r \mid f(n')$.

Time allowed: 4h30min. Each problem is worth 10 points.

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