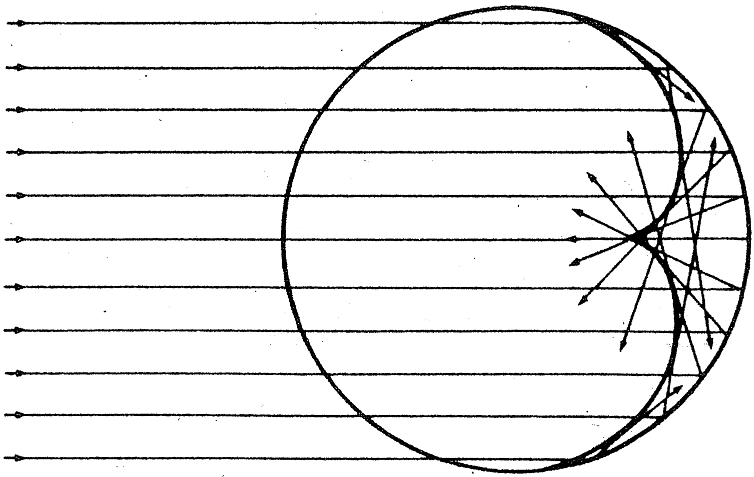
Function

Founder Editor G. B. Preston

Volume 16 Part 2 April 1991



A SCHOOL MATHEMATICS MAGAZINE

eg. by Aust. Post Publ. No. VBH0171

FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

* * * * *

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
FUNCTION,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

FUNCTION is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $17.00*; single issues $4.00. Payments should be sent to the Business Manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*$8.00 for *bona fide* secondary or tertiary students.

* * * * *

FUNCTION

Volume 16 *Part 2*

(Founder editor: G.B. Preston)

CONTENTS

* * * * *

THE FRONT COVER

If we sit drinking a cup of tea or coffee out in the bright sunshine we may sometimes see a pattern of bright curves on the surface of our drink. Figure 1 at right shows the sort of thing we're likely to observe.

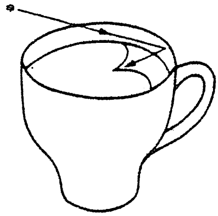


Figure 1

Such bright patterns are referred to as *optical caustics* and, once we start looking for them, we see them in all sorts of places: in windows, in reflections, in swimming pools, baths, all over. They take many shapes and the one shown here is set up by the circular outline of the cup.

Figure 2, which is also our front cover picture, shows the light rays from the caustic as they are reflected from the inside surface of the cup.

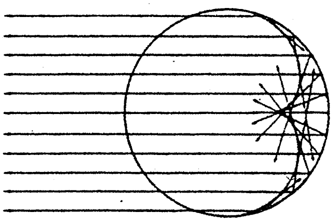


Figure 2

Using the law of reflection illustrated in Figure 3, it is possible to analyse the shape of the caustic. The details are not given here – in fact the detailed analysis is rather tedious and complicated.

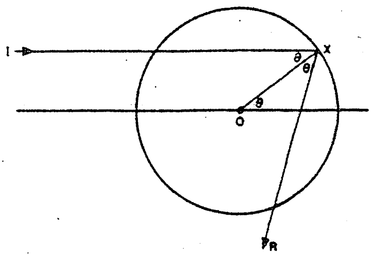


Figure 3

However, the result is a part (theoretically half, but usually not all is visible) of a curve known as the *nephroid*.

This is illustrated in Figure 4. [The word "nephroid" means "kidney-shaped curve" and it has, I suppose, very roughly the right shape.]

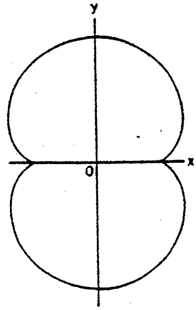


Figure 4

The nephroid has many other properties. See, for example, the account in *A Book of Curves* by E.H. Lockwood. Lockwood suggests placing "a dark-coloured cylindrical saucepan on the ground so that the rays from the sun or from a powerful electric lamp fall on it at an angle of about 60° to the horizontal".

You may care to plot a nephroid. This may be done from the equations

$$x = 3\cos\theta - \cos 3\theta$$

$$y = 3\sin\theta - \sin 3\theta$$

* * * * *

HERON QUADRATICS

K.R.S. Sastry, Addis Ababa, Ethiopia

Heronian triangles: Heron's formula for the area of a triangle in terms of the side lengths a, b, c and the semi-perimeter s

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \tag{*}$$

is well known. If the sides a, b, c of a triangle and its area Δ are all positive integers, the triangle will be said to be *Heronian*. Examples:

$$a = 3,\ b = 4,\ c = 5,\ \Delta = 6$$

$$a = 3,\ b = 25,\ c = 26,\ \Delta = 36.$$

One way to build a Heronian triangle is to use *Pythagorean* triangles. A Pythagorean triangle is an integer-sided right triangle. If the right angle is at C then we have $a^2 + b^2 = c^2$. It is known that Pythagorean triangles are given by†

$$(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)k \tag{**}$$

where m and n are positive integers, not both odd, $gcd(m, n) = 1$ and $k = 1, 2, \ldots$.

Now the area of a Pythagorean triangle is $\frac{1}{2}ab = k^2mn(m^2 - n^2)$ is a natural number. So by definition it is a Heronian triangle too. Here is a construction, suggested by the Hindu mathematician Brahmagupta (born 598 A.D.), of a Heronian triangle.

We first obtain two Pythagorean triangles from (**). To this end we first let $k = 1$, $m = 2$, $n = 1$ and obtain $(a_1, b_1, c_1) = (4, 3, 5)$. Next we let $k = 1$, $m = 3$, $n = 2$ and obtain $(a_2, b_2, c_2) = (12, 5, 13)$. These are illustrated in Figure 1.

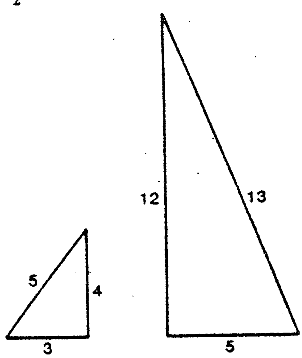


Figure 1.

† For a hint on how to derive (**) see Editor's note on page 116, *FUNCTION, Vol.* 14, *Part* 4 (August 1990).

We now produce two triangles similar to the ones in Figure 1 so that they have a common altitude. In this particular example, enlarging the first triangle by a factor of 3 is all that is required. This is illustrated in Figure 2.

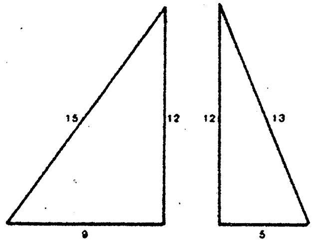


Figure 2.

Finally, we place the two Pythagorean triangles of Figure 2 along the common altitude to obtain a Heronian triangle. This can be done in two ways as Figure 3 shows.

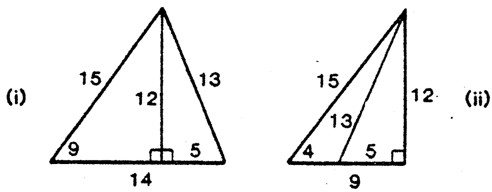


Figure 3.

In Figure 3(i) we have the Heronian triangle $a = 14$, $b = 13$, $c = 15$, $\Delta = 84$. In Figure 3(ii) we have the Heronian triangle $a' = 4$, $b' = 13$, $c' = 15$, $\Delta' = 24$. So much for the Heron part of the title, Heron Quadratics.

Curious Quadratics: Now look at the following quadratics:

$$\left.\begin{aligned} x^2 + 120x + 3600 &= (x + 60)^2 \\ x^2 + 169x + 3600 &= (x + 25)(x + 144) \\ x^2 + 218x + 3600 &= (x + 18)(x + 200) \end{aligned}\right\} \qquad (1)$$

All these three factorize over the integers, the middle coefficients have a constant difference: $169 - 120 = 218 - 169$. That much is obvious. What is *not* immediately obvious is that the factorizing quadratic triplets such as (1) generate Heronian triangles in which an altitude and the two sides all having a common vertex will be in arithmetic progression. In this example (1) these are respectively given by 120, 169, 218 – the middle coefficients themselves. The two Heronian triangles thus generated are exhibited in Figure 4.

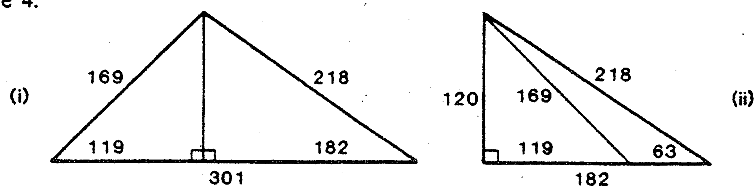


Figure 4.

So, the Heronian triangles associated with the quadratic triple (1) are $(a, b, c) = (301, 218, 169)$ and $(63, 218, 169)$. This leads us to define *Heron quadratics* as the triplet of quadratics

$$Ax^2 + Bx + c,\ Ax^2 + (B + D)x + c,\ Ax^2 + (B + 2D)x + C \tag{2}$$

in which:

(i) the parameters A, B, C, D are natural numbers,
(ii) all the three factorise over the integers,
(iii) the first of these has one zero.

Determination of Heron quadratics: It is quite a simple matter to determine the Heron quadratics (2) as an application of the formula for Pythagorean triples (**). Here we go.

Now $Ax^2 + Bx + C$ has one zero and the other expressions of (2) factor over the integers if and only if there are natural numbers u_1 and u_2 such that

$$B^2 - 4AC = 0,\ (B + D)^2 - 4AC = u_1^2 \text{ and } (B + 2D)^2 - 4AC = u_2^2 \tag{3}$$

From (3) we deduce that

$$2BD + D^2 = u_1^2 \text{ and } 4BD + 4D^2 = u_2^2 = 4u_2'^2, \text{ say.}$$

These yield

$$BD = u_1^2 - u_2'^2,\ D^2 = 2u_2'^2 - u_1^2. \tag{4}$$

The second of the above equations (4) may be written as

$$D^2 + u_1^2 = 2u_2'^2 \tag{5}$$

Put $D^2 = (d - e)^2$, $u_1 = d + e$ in (5) and obtain

$$d^2 + e^2 = u_2'^2. \tag{6}$$

Lo and behold! (6) is the Pythagorean relation (**). Its solution is

$$d = 2kmn,\ e = k(m^2 - n^2),\ u_2' = k(m^2 + n^2).$$

This in turn gives

$D^2 = (d - e)^2 = k^2(m^2 - 2mn - n^2)^2$ and choose D to be the positive square root of this

$u_1 = d + e = k(m^2 + 2mn - n^2)$

$u_2 = 2u_2' = 2k(m^2 + n^2)$.

Furthermore, from (4), $BD = u_1^2 - u_2'^2 = (u_1 + u_2')(u_1 - u_2')$. So

$B = \dfrac{4k\,mn(m^2-n^2)}{|m^2-2mn-n^2|}$, where $|\ \ldots\ |$ refers to the positive square root (above) and this has to be a natural number. For simplicity we choose $k = |m^2 - 2mn - n^2|$. This yields the parameters

$$B = 4mn(m^2 - n^2),\ D = (m^2 - 2m - n^2)^2,\ 4AC = B^2. \tag{7}$$

Here is a numerical construction of the Heron quadratics (2). Put $m = 2, n = 1$ in (7). This gives

$$B = 24,\ D = 1,\ 4AC = 576, \text{ i.e. } AC = 144.$$

The natural numbers A and C are to be chosen so that $AC = 144$. This can be done in several ways. If $A = 2, C = 72$ then we obtain the Heron quadratics

$$2x^2 + 24x + 72 = 2(x + 6)^2$$

$$2x^2 + 25x + 72 = (x + 8)(2x + 9)$$

$$2x^2 + 26x + 72 = 2(x + 4)(x + 9).$$

We then have the associated Heronian triangles with the altitude and the sides, all having a common vertex, given by 24, 25, 26 respectively. These are shown in Figure 5.

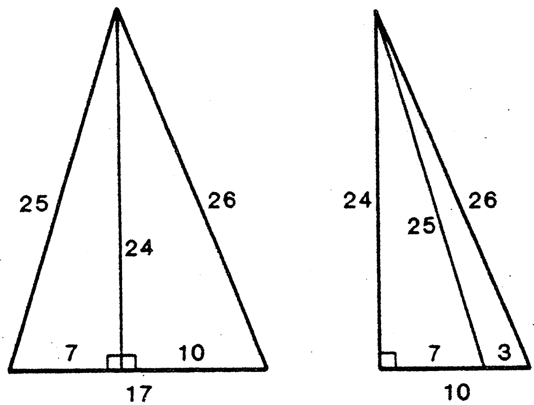


Figure 5.

$a = 17,\ b = 26,\ c = 25,\ \Delta = 204;\ a' = 3,\ b' = 26,\ c' = 25,\ \Delta' = 36.$

Further Questions: Quadratics and Pythagorean triples can provide a number of opportunities for discovery. Such is the case with the study of Heronian triangles. Here are some examples to get started.

1. Look at the factorizations $x^2 + 4x + 3 = (x + 1)(x + 3)$ and $x^2 + 4x + 4 = (x + 2)^2$. These suggest the problem of determining the coefficients B and C so that $x^2 + Bx + C$ and $x^2 + Bx + C + 1$ both factor over the integers. The corresponding harder variety is to determine the coefficients A, B, C, D so that both quadratics $Ax^2 + Bx + C$ and $Ax^2 + Bx + C + D$ factor over the integers. Even harder is this: Determine the polynomials of degree $n, n > 2$, so that the polynomials $P(x)$ and $P(x) + D$, D an integer, both factor over the integers. My consideration of this problem is the subject of a forthcoming article in *Mathematical Spectrum*†.

2. Look at the factorizations $x^2 + 5x + 6 = (x + 2)(x + 3)$ and $x^2 + 5x - 6 = (x - 1)(x + 6)$. These suggest the problem of determining the non-zero coefficients A, B, C so that both $Ax^2 + Bx + C$ and $Ax^2 + Bx - C$ factor over the integers.

3. The side lengths of a Heronian triangle are in arithmetic progression with a common difference of 13. There is an infinity of such Heronian triangles. Determine such a Heronian triangle having minimum area.

4. Look at the Heronian triangle sequence (3, 4, 5); (13, 14, 15). Determine the next member triangle.

5. Suppose a, b, c, d are the side lengths of a cyclic quadrilateral with semi-perimeter s.

   (i) Derive Brahmagupta's formula $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ for its area.

   (ii) Brahmagupta also gave formulae for the diagonals of a cyclic quadrilateral in terms of its side lengths. Rediscover these formulae.

* * * * *

One Complex Function in Several Variables

"Quality of Teaching of mathematics is a function of several variables *viz.* $T = T(S, B, C, M, R, E, \ldots)$, where S is suitability of syllabus, B is books selected for reading, C is competence of teachers, M is method of Teaching, R is receptivity of students, E is examination system and so on. To bring about all round improvement in T, all the elements on which T depends should be suitably improved."

A.C. Banerjee

† A U.K. counterpart of *Function*.

HANGING CHAINS, SAMURAI SWORDS AND SQUARE WHEELS

Michael A.B. Deakin, Monash University

If you have a scientific calculator, you will have no difficulty generating values of the power function e^x, where e (= 2.7182818284 ...) is an important mathematical constant. e^{-x} (= $1/e^x$) may also be similarly generated and so it is not difficult to calculate values of

$$y = \tfrac{1}{2}(e^x + e^{-x}). \tag{1}$$

This expression has another name. It is termed the *hyperbolic cosine* of x (for reasons I won't go into here) and is abbreviated to $\cosh x$. Thus we have

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}). \tag{2}$$

My CASIO fx-570 allows direct evaluation of the function $\cosh x$. You enter a value and then press in order buttons marked **hyp** and **cos**.

Figure 1 shows the graph of $y = \cosh x$. It is called a *catenary*, a word which derives from the Latin *catena* meaning "chain". The name derives from the fact that a heavy chain, suspended by its ends, adopts the shape of an arc, or part, of a catenary.

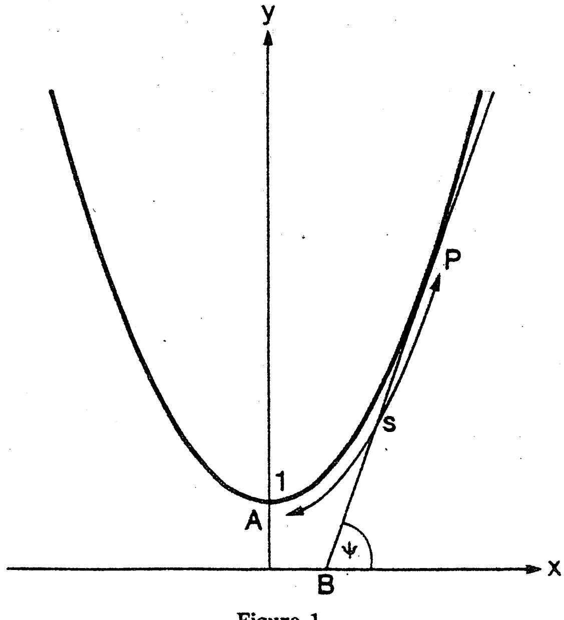


Figure 1

At first glance, the catenary looks rather like a parabola, and it is true that for *small* values of x,

$$\cosh x \approx 1 + \tfrac{1}{2}x^2. \tag{3}$$

However, for large values of x, this approximation breaks down very badly.

Look again at Figure 1. The vertex A is the point $(0, 1)$. Let P be a typical point on the curve. The arc-length s (i.e. the distance AP measured along the curve, taken negative for points P to the left of A) is readily available on your scientific calculator also. It can be shown that

$$s = \tfrac{1}{2}(e^x - e^{-x}). \tag{4}$$

The right-hand side of this equation is termed the *hyperbolic sine* of x, abbreviated to $\sinh x$ and pronounced "shine x". So†

$$s = \sinh x. \tag{5}$$

Now draw a tangent to the curve at the point P. Let this intersect the x-axis at B and make an angle ψ with that axis. The *slope* of the line BP is also $\sinh x$. (Prove this, if you know enough calculus, as an exercise.) But this slope is, of course, $\tan \psi$. Thus

$$s = \tan \psi. \tag{6}$$

This is a very important property of the catenary. Indeed, it is referred to as the *intrinsic equation* of the catenary.

The equation given for the catenary so far has been

$$y = \cosh x, \tag{7}$$

to rewrite Equation (2). For what follows we need a very minor generalisation. Suppose both x, y to be scaled by the same factor c. Then in the new units

$$\frac{y}{c} = \cosh\left[\frac{x}{c}\right]$$

or

$$y = c\cosh\left[\frac{x}{c}\right]. \tag{8}$$

In these units, Equation (6) becomes

$$s = c\tan \psi. \tag{9}$$

(Can you see why this should be so?) We are now in a position to show that a hanging chain takes up the shape of a catenary.

---

† $\sinh x$ and $\cosh x$ are very like the familiar $\sin x$ and $\cos x$ in many of their properties. E.g.

$$\cosh^2 x - \sinh^2 x = 1.$$

You may like to prove this and examine other analogues of trigonometric formulae.

Figure 2 (below) shows the arc AP of the chain and the forces acting on it. There are three of these:

[1] The weight of the chain in AP; this acts vertically downwards and has magnitude ws, where w is the weight per unit length;

[2] The tension in the chain supporting it at the point P; this acts along the chain (i.e. at an angle ψ to the horizontal) and will be denoted by T;

[3] The tension at the special point A; here the chain is horizontal and thus the force is horizontal; it will be denoted by T_0.

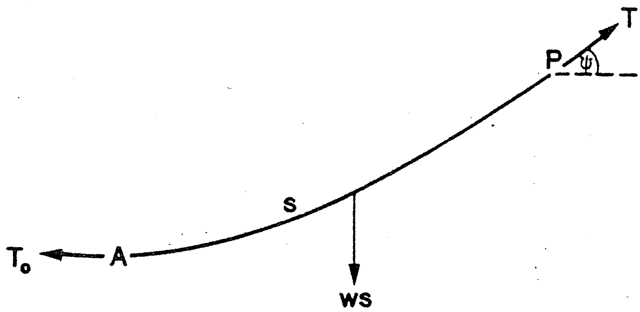


Figure 2

The vertical component of the support force [2] is $T \sin \psi$ and this must exactly balance the weight [1]. Thus

$$ws = T \sin \psi. \tag{10}$$

But similarly, the horizontal forces must balance, so

$$T_0 = T \cos \psi. \tag{11}$$

It now follows by dividing (10) by (11) that

$$s = (T_0/w)\tan \psi, \tag{12}$$

which is Equation (9), if we set $T_0/w = c$. So the curve adopted by the chain is indeed a catenary.

There are two practical consequences of this deduction. First, surveyors, until recently when optical devices became the norm, used chains or tapes for measurement of distance. If these were "unsupported" they hung in catenaries and so recorded arclength (the s of Equation (5), instead of the required measurement, x, in that equation); thus a correction based on Equation (5) had to be applied. Secondly, overhead power-lines or telegraph cables are supported only at the posts that are provided to that end. So they hand in catenaries and this fact must be taken into account when we calculate the amount of cable required.

The history of the catenary may be worth a brief digression. It was Galileo who first speculated on what shape a hanging chain might assume. He hypothesised that it would be parabolic, but this answer was disproved by the German scientist Jungius in 1669. The mathematician James Bernoulli in 1690 issued a challenge to his fellow mathematicians to find the correct answer and this was forthcoming from three sources: John Bernoulli (his brother), the Dutch physicist Huyghens, and Leibniz (one of the founders of calculus). The name "catenary" is due to Huyghens.

Twice in my professional life as a mathematician, I have needed to look at the properties of catenaries. The first of these was some four years ago and it was all very "hush-hush" at the time. The need for secrecy is, however, long past and I have no qualms in telling the story today.

There appeared on the doorstep of our department an official of the Victorian Treasury. He was charged with the preparation of Victoria's bid for what is *now* termed the "Multi-Function Polis" (but just for the record, visions and terminology have altered; at the time when the subject was a hotly debated political issue, the "leaks" were correct; it was initially seen as a Japanese enclave and was referred to as a "Technopolis").

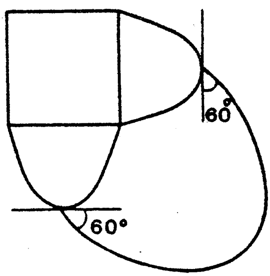
Anyhow, the fellow who turned up wanted to design a city whose streets were all arcs of catenaries. The Japanese would, he thought, fall about in rapture over this because (he said) the catenary was the traditional shape of the samurai sword.

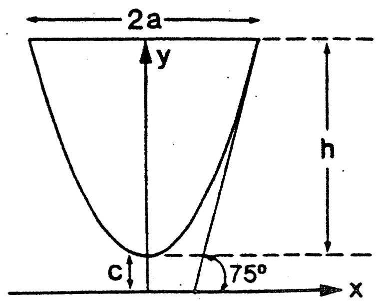
Where he got this alleged information, I don't know. I was deputed to check it and so read a very authoritative book on the samurai sword in the course of my duties – it nowhere even so much as mentioned the catenary. What is true, of course, is that any relatively short curved object can be approximated reasonably well by a suitably chosen "piece" of a catenary described by Equation (8), with suitable choice of the value of c.

(I also checked books on Japanese gardening looking for references to the catenary, but all in vain!)

Anyhow, we set out to design his city for him and here was how we went about it. In Figure 3 we have a square city centre and the first ring of catenaries (only two are shown). From the vertex of each of these sprout catenaries of the second ring (only one is shown), etc. Now we want the catenaries to make 60° angles as shown and we also want the big catenaries to be scaled-up versions of the small ones. These requirements give Figure 4, where $2a$ is the width of the square. From this we can work out (I omit the details) that

$$c = 0.4932a. \tag{13}$$





Figures 3, 4

The height h may now be calculated to be

$$h = c \cosh \frac{a}{c} - c = 1.412a. \tag{14}$$

Now I mentioned beforehand that catenaries may at times be approximated by parabolas. This is what we did here. Taking $a = 1$ and moving the axes gives Figure 5. The

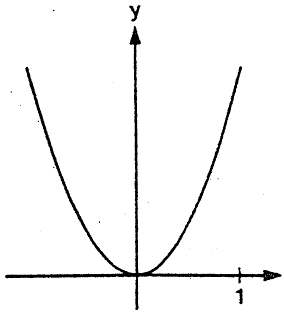


Figure 5

catenary has the equation in these coordinates

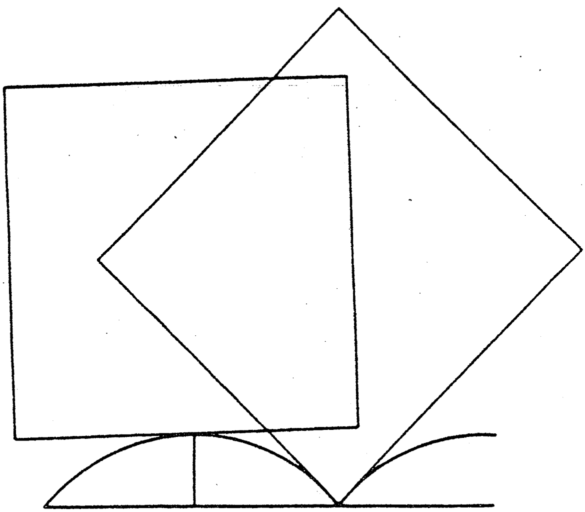


Figure 6

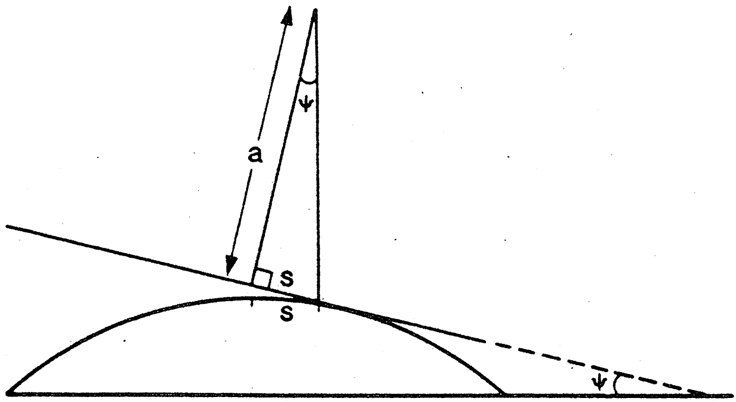


Figure 8

$$y = 0.4932 \cosh(2.0275x) - 0.4932 \tag{15}$$

but it may well be approximated by the parabola

$$y = 1.412x^2. \tag{16}$$

The discrepancy is everywhere less than about $2\frac{1}{2}\%$.

So we made lovely computer pictures of streets that were actually parabolas and coloured them in. The treasury official forgot all about samurai swords and said they looked like chrysanthemums (they didn't really) and the Japanese were sure to love them. They didn't, as history has shown. The four consultants were each paid a pittance but the department of Mathematics got a new colour plotter.

More recently, one of our lab managers at Monash came to me with a problem. His brother, a sculptor, had been asked to design an exhibition in which a vehicle with square wheels was able to roll evenly on a specially constructed road. Figure 6 opposite shows the problem. The centre of the square wheel (shown in more detail in Figure 7) must not ride up or down at all. The key to the analysis is Figure 8 (opposite).

Choose the angle ψ as shown and note that (because the wheel rolls without slipping on the humped roadway) the distance ≤ along the curve is the distance measured along the wheel. The angle ψ also turns up (prove this as an exercise) at the centre of the wheel. Thus

$$s = a \tan \psi \qquad (17)$$

which (with a slight difference of notation) is Equation (9) again.

It is a routine exercise to express Equation (17) in standard coordinates. The result is

$$y = a(\sqrt{2} - \cosh \frac{x}{a}) \qquad (18)$$

and x varies between $\pm 0.8814a$. We make replicas of this curve and line them up as indicated in Figure 6, and so make our roadway.

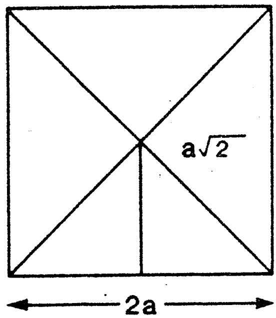


Figure 7

I mentioned this problem to a friend, a research chemist with CSIRO, and was surprised to find that he knew the answer! When he was a schoolboy in Auckland, he'd been placed in a class of extra Mathematics for gifted students – and this was one of the problems they'd studied.

* * * * *

Those Picking Foreigners!

"The mathematical talent is a very rare gift. It is very rarely identifiable, and it ought to be spotted in infancy. The language of mathematics is international, the only complete international language; a foreign mathematician can pick out a talent in our children as easily as in his own."

Charles Percy Snow

LETTERS TO THE EDITOR

In reference to the article on Hypatia, in the last edition of *Function*, the author refers to a letter from Synesius to Hypatia, requesting a hydroscope, or more probably a hydrometer. The author was perplexed as to why he may have required such an instrument.

May I suggest that the implied illness of Synesius may have affected the quantity and/or quality of his urine, a matter of grave concern of the physicians of that time (and modern doctors as well). For clinical purposes the measurement of the specific gravity (or density) is made by a hydrometer (urinometer) which is calibrated to read 1.000 at 16°C in water. It is therefore a measure of the quantity of solids in solution. In health, urea and sodium chloride are the main solutes contributing to the specific gravity of urine, which varies with the nature and quantity of food eaten as well as with the amount of water of other fluid taken. The normal range is 1.016 to 1.020. In diabetes, on the other hand, the quantity of glucose in the urine may far outweigh the total of all the other solutes present and has a high specific gravity due to the presence of as much as 10 per cent glucose.

The *Papyrus Ebers* which dates from approximately 1500 B.C. contains what is thought to be the first reference to diabetes:

"A medicine to drive away the passing of too much urine":

Prescription:

Branches of Qadet plant	1/4
Grapes	1/8
Honey	1/4
Berries from üan tree	1/32
Sweet beer	1/6

Cook: filter and take for 2 days.

The hydrometer may also be useful in brewing the beer!

Charles Hunter,
Department of Anatomy,
Monash University

* * * * *

WHICH IS MORE ACCURATE?

I recently heard from my erratic correspondent, the eccentric Welsh recluse, Dr. Dai Fwls ap Rhyll. Dr Fwls, you may remember, is a persistent and difficult critic of established concepts in Mathematics and Physics. His work has never really had the recognition it deserves; in part this is due to his retiring disposition and to his long having worked in isolation outside universities and established institutions. But it is my belief that the embarrassing and "heretical" nature of his conclusions has a lot to do with it.

Besides which, his paradoxes can all be put in very simple terms – he challenges established mathematics not in esoteric language, but with easy homespun examples that professionals may think beneath them. This quality, however, makes them especially suitable for me to transmit to *Function*.

This year he wrote to me about simultaneous equations. He asked me which was the better approximation to the true solution of the equations:

$$x + y = 1.99$$

$$100x + 101y = 200,$$

(1,1) or (2,0)?

This seemed a simple enough question, trivial indeed. So I substituted both answers into the equations he had given. The first gave:

$$1 + 1 = 2 \approx 1.99;\ 100 + 101 = 201 \neq 200.$$

The second yielded:

$$2 + 0 = 2 \approx 1.99;\ 200 + 0 = 200.$$

So it was pretty clear that the second answer was the better. But then I thought to try another check, and so I solved the equations exactly, and got

$$x = 0.99;\ y = 1.00$$

which is clearly much closer to (1,1) than to (2,0).

I can't find any mistake in my working, so I await Dr Fwls's elucidation. Probably in vain, I must fear. He raises these questions, but he never seems to get round to explaining the puzzles he poses.

Kim Dean
Union College
Windsor

* * * * *

HISTORY OF MATHEMATICS SECTION

EDITOR: M.A.B. DEAKIN

This issue will be devoted not to a single topic but to a collection of updates and miscellaneous items deriving from correspondence. My thanks in particular to K.R.S. Sastry of Addis Ababa (Ethiopia) who has been untiring in supplying me with interesting material.

We begin with two brief updates.

Maria Agnesi

The cover story for *Volume* 10, *Part* 4 concerned Maria Agnesi (1718-1799), an early woman mathematician. Recently a much better biography of Maria has appeared (in the *Archive for History of Exact Sciences*) than was previously available. The earlier brief biography by Edna Kramer states that her father was a professor of Mathematics. This information (which I repeated in my brief *Function* story) is incorrect. Clifford Truesdell, the author of the new biography, points out that although Maria was herself appointed as a professor of Mathematics at the University of Bologna, the reason that she taught no students and drew no pay was that she lived in Milan: the post was evidently an honorary one.

The Missing Nobel Prize

In *Volume* 11, *Part* 2 was a story on the Nobel Prize and the Fields Medal. The story goes that Nobel did not endow a Prize in Mathematics because of his hostility to the Swedish mathematician Gösta Mittag-Leffler. This explanation would seem to come from Mittag-Leffler himself. We deduce it from a letter by the American R.C. Archibald who had visited Mittag-Leffler. The letter was discovered by Sister Mary Thomas à Kempis in the archives of Brown University and published (in part) by her in *The Mathematics Teacher* (1966, pp. 667-668). Whether it's true or not is another matter. The information that the two men fell out over the affections of the mathematician Sonya Kovalevskaya would seem to be fanciful embroidery. Archibald seems to imply that their rift was caused by business rivalry. However, if Mittag-Leffler told the story to Archibald, he very likely told it to his friend Fields also and thus did help to set up the Fields Medal as a substitute for the missing Nobel Prize.

The Kiss Precise

Following the cover article to *Volume* 15, *Part* 4, we have had a letter from K.R.S. Sastry. The situation to be analysed is that of Figure 1.

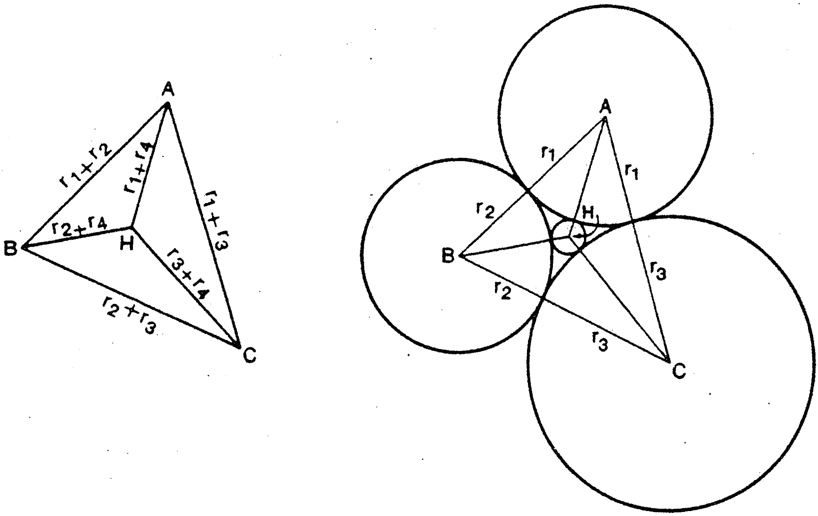


Figure 1

Four circles are each tangent to the other three. Suppose the radii to be r_1, r_2, r_3, r_4 and write

$$\varepsilon_1 = 1/r_1 \qquad \varepsilon_2 = 1/r_2 \qquad \varepsilon_3 = 1/r_3 \qquad \varepsilon_4 = 1/r_4.$$

Then the four circles theorem (popularly known as the "kiss precise") states that

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2. \tag{1}$$

We outlined two proofs of this result, but avoided details because they are complicated in the extreme. Indeed, the second of the two proofs outlined involved the use of computer algebra to set up and factorise a 16th-degree polynomial.

Mr. Sastry uses the cosine rule on ΔAHC (Figure 1) and also on ΔAHB. He then uses the area rule on the same triangles. This then gives him expressions for $\cos\alpha$, $\cos\beta$, $\sin\alpha$, $\sin\beta$. Use of the cosine rule on ΔABC then produces an expression for $\cos(\alpha+\beta)$. He then writes

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and inserts the various expressions previously found. Writing the resulting equation in terms of ε_1, ε_2, ε_3, ε_4 and simplifying produces (after a lot of work!) a quartic equation. Which is much simpler than one of degree 16.

There is a lot of tedious algebra involved, but a proof may be constructed along these lines. [Indeed, it is a minor variant of the original proof outlined by Descartes, the theorem's discoverer.) After much heavy algebra, we find

$$[(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) - \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2](\varepsilon_1 + \varepsilon_4)^2 = 0. \qquad (2)$$

The first factor gives Equation (1) which has two solutions for ε_4 in terms of ε_1, ε_2, ε_3. This corresponds to the fact that the fourth circle may nestle between the other three, as shown in Figure 1 or else may enclose them all. See the diagram in *Volume* 15, *Part* 4. But actually there are other solutions also.

For instance, the fourth circle might coincide with the first. In such a case, the fourth (coincident) circle touches, the second and third circles, and also of course the first circle (itself), so this is a possible, if "trick", solution. Furthermore, we may think of the first circle coinciding with itself on the inside, or on the outside. This is the interpretation of the other factor in Equation (2).

The point arises as to why we don't also have the possibilities $(\varepsilon_1 + \varepsilon_2)^2 = 0$, $(\varepsilon_1 + \varepsilon_3)^2 = 0$. Actually we do. The Descartes-Sastry proof treats the point A differently from the points B, C. But we could equally well have used B or C as the "special" vertex and these choices would have led respectively to the two cases mentioned above.

Thus, counting up all the possibilities for the fourth circle, we find:

(a) the nestling circle
(b) the enclosing circle
(c) the first circle, twice
(d) the second circle, twice
(e) the third circle, twice.

A total of eight.

Now this is as it should be. If we take any three circles in the plane and seek to draw a fourth circle tangent to all three, there are (in general) eight solutions. This is known as the *Problem of Apollonius*. For more detail, see Pedoe's book *Circles* (p. 23). The first of the proofs outlined in the earlier article is, incidentally, due to Pedoe and is based on such considerations, though I did not go into all this when I wrote previously.

" Napoleon's Theorem "

Let ABC (Figure 2) be any triangle and let ABR, BCP, CAQ be equilateral triangles constructed on its sides as shown. Let O_1 be the centre of ΔBCP, O_2 that of ΔCAQ and O_3 that of ΔABR. Then:

$$\Delta O_1O_2O_3 \textit{ is equilateral.}$$

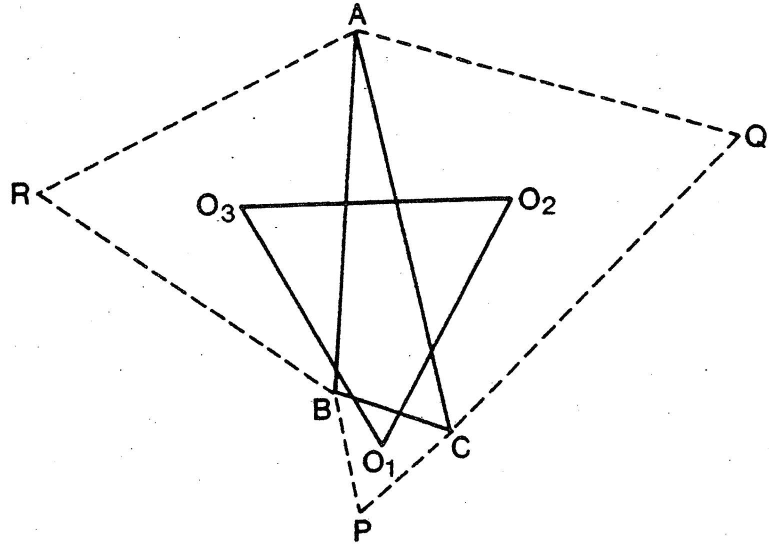


Figure 2

The true history of this theorem seems not to be known. However, the book *Geometry Revisited* by H.S.M. Coxeter and S.L. Greitzer, from which Figure 2 is adapted goes on to give some interesting stories (without, of course, attesting to their historical accuracy).

It is known that Napoleon Bonaparte was a bit of a mathematician with a great interest in geometry. In fact, there is a story that, before he made himself ruler of France, he engaged in a discussion with the great mathematicians Lagrange and Laplace until the latter told him, severely, "The last thing we want from you, general, is a lesson in geometry". Laplace became his chief military engineer.

The theorem stated above has been attributed to Napoleon, though the possibility of his knowing enough geometry for this feat is as questionable as the possibility of his knowing enough English to compose the famous palindrome

ABLE WAS I ERE I SAW ELBA.

Coxeter and Greitzer refer to the triangle $O_1O_2O_3$ as the *Napoleon Triangle* of the original triangle ABC – or more precisely as the *outer Napoleon triangle*. For we could put R on the other side of AB, P on the other side of BC and Q on the other side of AC and so form an *inner Napoleon triangle*. This triangle is also equilateral.

The history of these and related theorems is rather obscure. Mr. Sastry has written to us on this subject also. Notice that the angles ACO_2, CAO_2, ABO_3, etc., were we to construct them, would all be equal and have the measure $30°$.

What Mr. Sastry looked at was the question of what happens when we begin with not necessarily a triangle, but a general polygon of n sides. On each side construct triangles with sides making equal angles α_n to the sides of the original polygon. Thus Napoleon's theorem covers the case $n = 3$ and tells us that $\alpha_3 = 30°$.

Next Mr. Sastry analysed the case $n = 4$ and found that in order for the second quadrilateral (the analogue of the Napoleon triangle) to be a square we must start with not a general quadrilateral, but a parallelogram. In this case, $\alpha_4 = 45°$.

He then looked at the case $n = 5$ and found that one had to begin with a special pentagon in which each diagonal was parallel to one of the sides. In this case $\alpha_5 = 54°$. He then conjectured that in general

$$\alpha_n = \left[1 - \frac{2}{n} \right] 90°.$$

At this stage, Mr. Sastry wrote to Professor Coxeter who referred him to a paper by Leon Gerber in which the general result is proved. This appeared in 1980, but Mr. Sastry has since learned that it was proved earlier by the Italian mathematician Barlotti (in 1955).

The earliest proof of Napoleon's Theorem that Gerber was able to find appeared in 1863 in an obscure German publication, no copy of which exists in Australia. The case $n = 4$ was proved by Victor Thébault in 1937 and is thus known as Thébault's Theorem. We may state it thus:

> Construct squares outwardly on the sides of any parallelogram. Their centres form the vertices of a square.

The general case proved by Barlotti is also in an obscure journal of which there is no copy in Australia.

Gerber, however, points out that the original n-gon must in each case be "affinely regular" – that is to say, a perspective view of a regular n-gon. E.g. a square drawn from the right perspective appears as a parallelogram, etc. This is the basis for Gerber's proof of the general result.

* * * * *

Napoleon did it, too

> "Mathematicians are like Frenchmen; whatever you say to them they translate into their own language, and forthwith it is something entirely new."

Johann Wolfgang Goethe (1749-1832)

PROBLEMS SECTION

EDITOR: H. LAUSCH

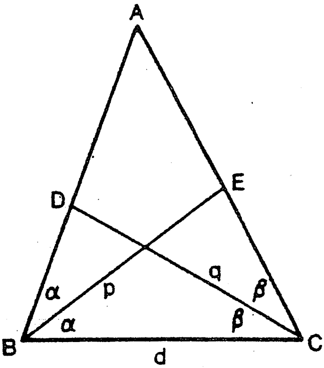
SOLUTIONS

The Steiner-Lehmus Theorem (*Function*, Problem 15.1.5) continued

Readers have supplied Function *with more proofs of the theorem which states:* If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

The first solution is due to R.E. Nelder, Toowoomba, Queensland. It consists of a short trigonometric argument:

Solution. For the notation in this proof the reader is referred to the diagram below.



Let the bisectors of the two angles 2α and 2β be equal and assume that $\alpha > \beta$, i.e. $AC > AB$. Then $\dfrac{\text{area of } \Delta BEC}{\text{area of } DBDC} = \dfrac{\frac{1}{2}pd \sin\alpha}{\frac{1}{2}qd \sin\beta} = \dfrac{\sin\alpha}{\sin\beta} > 1$, i.e.

$$\text{area } \Delta BEC > \text{area } \Delta DC. \tag{1}$$

Furthermore, $\dfrac{\text{area } \Delta BAE}{\text{area } DCAD} =$

$$= \frac{\frac{1}{2}p.AB.\sin\alpha}{\frac{1}{2}q.AC.\sin\beta}$$

$$= \frac{AB}{AC} \cdot \frac{\sin\alpha}{\sin\beta}$$

$$= \frac{\sin 2\beta}{\sin 2\alpha} \cdot \frac{\sin\alpha}{\sin\beta}$$

$$= \frac{2 \sin \beta \cos \beta \sin \beta}{2 \sin \alpha \cos \alpha \sin \beta}$$

$$= \frac{\cos \beta}{\cos \alpha}.$$

Since by assumption $\beta < \alpha$, we have $\frac{\cos \beta}{\cos \alpha} > 1$ and so

$$\text{area } \Delta BAE > \text{area } \Delta CAD. \qquad (2)$$

Adding (1) and (2), we obtain:

$$\text{area } \Delta BEC + \text{area } \Delta BAE > \text{area } \Delta BDC + \text{area } \Delta\ CAD,$$

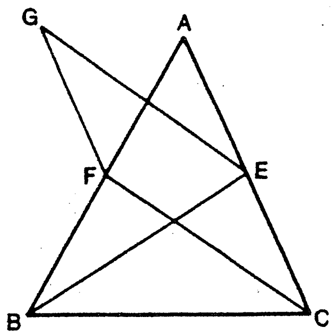
i.e. area $\Delta ABC >$ area ΔABC. This is not a known property of triangles.

For a while the suspicion raised by J.J. Sylvester in Philosophical Magazine, *Vol. 4, 1852, that a direct proof of the Steiner-Lehmus theorem was impossible had gained currency. J.A. Deakin, Shepparton, sent* Function *one proof by contradiction, which he had found in his archives (source unknown).*

Solution. The reader is referred to the diagram below for notations used in the proof; BE and CF are the bisectors of B and C of the triangle ABC, and $BE = CF$.
The proof is by contradiction.

Assume that $\angle C > \angle B$. Since the angles are bisected and $BE = CF$, we have $BF > CE$. Draw the parallelogram $CEGF$ and join BG.

Since $EG = CF = BE$, ΔEGB is isosceles. Hence $\angle EGB = \angle EBG$. But $\angle EGF = \angle ECF$. Hence $\angle EGF > \angle EBF$ and $\angle FGB < \angle FBG$, so that $FG > FB$ and hence $EC > FB$. But this contradicts the assumption that $BF > CE$. Therefore the original assumption is untrue, so that $\angle B = \angle C$ and ΔABC is isosceles.



There is more to come on the Steiner-Lehmus Theorem in subsequent issues of Function.

Solutions to some other problems

In Function, *Volume 15, Part 3, on page 93, a proof provided by John Barton, North Carlton, of Problem 15.1.6 was reproduced. Andy Liu, Edmonton, Alberta, Canada, sent two comments to* Function. *We first repeat the problem and the solution communicated by John Barton and then present Andy Liu's remarks.*

Problem 15.1.6. Let O and I be the circumcentre and incentre, respectively, of a triangle with circumradius R and inradius r; let d be the distance OI. Show that $d^2 = R^2 - 2rR$.

Solution. Let P be the intersection of AI with the circumcircle of $DABC$. Then

$$R^2 - OI^2 = (R - OI)(R + OI)$$

$$= AI \cdot IP \quad \text{(intersecting chord theorem)}$$

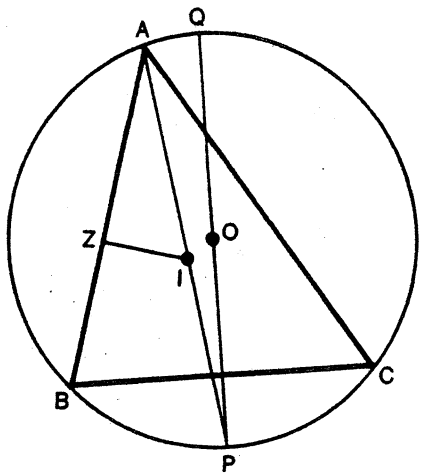
$$= AI \cdot PB$$ by a theorem based on the fact that the triangle of the ex-centres of ΔABC has ΔABC as its pedal triangle.

Let Z be the point on AB such that AB and IZ are perpendicular, and Q be the point on the circumcircle of ΔABC opposite to P. Then

$$AI = IZ \ \operatorname{cosec} \frac{A}{2} = r \operatorname{cosec} \frac{A}{2} \ ;$$

$$PB = PQ \sin \frac{A}{2} = 2R \sin \frac{A}{2} \qquad (\angle BQP = \angle BAP = \frac{A}{2}).$$

Hence $d^2 = R^2 - 2rR$.



Andy Liu's comments:

1. To show that $PB = PI$, it is not necessary to quote theorems as above. We have $\angle BIP = \angle BAI + \angle ABI = \angle CAI + \angle CBI = \angle CBP + \angle CBI = \angle IBP$. Hence $PB = PI$.

2. To show that $AI.PB = 2Rr = PQ.IZ$, it is not necessary to use trigonometry. We have $\angle ZAI = \angle BQP$ while $\angle AZI = 90^\circ = \angle QBP$. Hence triangles AZI and QBP are similar. It follows that $\frac{AI}{ZI} = \frac{QP}{BP}$ or $AI.PB = PQ.IZ$.

Problem 15.1.7 (a year-twelve problem from Great Britain). Show that if $\frac{m}{n}$ is an approximation of $\sqrt{2}$, then $\frac{m+2n}{m+n}$ is a better one.

The first solution is a contribution by John Barton:

Solution. Let $\frac{m}{n} - \sqrt{2} = h$. Then

$$\frac{m+2n}{m+n} - \sqrt{2} = \frac{(\sqrt{2}+h)+2}{(\sqrt{2}+h)+1} - \sqrt{2}$$

$$= \frac{(1-\sqrt{2})h}{1+\sqrt{2}+h}. \qquad (1)$$

In this problem we take m, n to be positive integers, noting that the quotient of two rational numbers can be written as the quotient of two integers.

(i) Suppose $h > 0$, so $h = |h|$. Then

$$\left| \frac{m+2n}{m+n} - \sqrt{2} \right| = \frac{(\sqrt{2}-1)|h|}{\sqrt{2}+1+h} < \frac{(\sqrt{2}-1)|h|}{\sqrt{2}+1} < |h|,$$

which proves the proposition for this case.

(ii) Suppose $h < 0$. Then $-h = |h|$ and $|h| < \sqrt{2}$, since $\frac{m}{n} > 0$. We now have

$$\left| \frac{m+2n}{m+n} - \sqrt{2} \right| = \frac{(\sqrt{2}-1)|h|}{\sqrt{2}+1+h}$$

$$< (\sqrt{2}-1)|h|, \text{ since } \sqrt{2} - |h| > 0,$$

$$< |h|,$$

and this proves the proposition for this case. Thus the proposition is generally true.

Note that (1) shows that $\frac{m}{n} - \sqrt{2}$ and $\frac{m+2n}{m+n} - \sqrt{2}$ have opposite signs, so that successive approximations are alternately greater or less than $\sqrt{2}$.

Seung-Jin Bang, Seoul, Republic of Korea, wrote down a different solution:

Solution. We prove the inequality $\left|\frac{m+2n}{m+n} - \sqrt{2}\right| < \left|\frac{m}{n} - \sqrt{2}\right|$. In fact, the inequality is equivalent to $\{(2\sqrt{2} - 1)n^2 + 2(\sqrt{2} - 1)mn - m^2\}(2n^2 - m^2) > 0$, and

$$2(\sqrt{2} - 1)(n + (1 + \frac{1}{\sqrt{2}})m)(n - \frac{1}{\sqrt{2}}m)(2n^2 - m^2) =$$

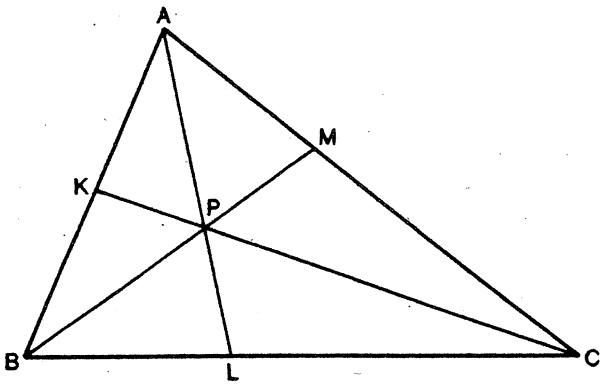
$$= 4(\sqrt{2} - 1)(n + (1 + \frac{1}{\sqrt{2}})m)(n - \frac{1}{\sqrt{2}}m)^2(n + \frac{1}{\sqrt{2}}m) > 0.$$

This completes the proof.

PROBLEMS

P. de Gail, Windsor, has a problem the solution of which, so he writes, would facilitate some design work he is doing on mandalas:

Problem 16.2.1. Given an arbitrary triangle ABC and three cevians APL, BPM, CPK (see diagram below) such that $PK = PL = PM$. Find trigonometric equations expressing $\angle LAC$, $\angle MBA$, $\angle KCB$ in terms of $\angle BAC$, $\angle CBA$, $\angle ACB$ and/or in terms of BC, CA, AB. Alternatively, find expressions for the partition points K, L, M.



Ed.: This problem admits the following variation:

Problem 16.2.2. Given a triangle ABC. Find a construction with compass and ruler only of a point P which has the following property: if CP (extended), AP (extended), BP (extended) intersect AB in K, BC in L, AC in M, respectively, then $PK = PL = PM$.

Problem 16.2.3 (submitted by David Shaw, Geelong). If p is a prime number, s the sum of the digits when N is expressed as a base-p numeral and h is the highest power of p contained (as a factor) in $N!$, prove that $h = \frac{N-s}{p-1}$.

Problem 16.2.4 (submitted by Juan Bosco Romero Marquez, Valladolid, Spain). Let T and T' be two right-angled triangles. Let R, r and R', r' be circumradius and inradius of T and T' respectively. Prove that if $\frac{R}{R'} = \frac{r}{r'}$, then T and T' are similar. Does this theorem hold for a larger class of triangles?

And, before I forget, here is a problem for forgetful people:

Problem 16.2.5 (Mathematical Spectrum, Volume 24, Number 2). A bath takes 3 minutes to fill and 4 minutes to empty. How long does it take to fill the bath with the plug out?

Mathematics Challenge for Young Australians

This is a new national programme, designed and conducted by the Australian Mathematical Olympiad Committee, which is a sub-committee of the Australian Academy of Science, to help teachers develop the potential of top mathematics students in Years 8 to 10. It has two stages: 1. a two-week Mathematics Challenge in March consisting of six problems for the solution of which students are given two weeks; 2. an eight to twelve-week Mathematics Challenge Enrichment Programme between April and June, with comprehensive support for teachers on methodologies and techniques. The whole project integrates with the Australian Mathematical Olympiad Committee's Three-Year Problem-solving Programme and opens a world of opportunity for teachers and talented students. In 1992 more than 8800 students have registered for stage 1 of the Challenge. Here are its problems:

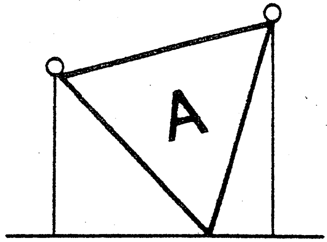
Problem One to Four must be attempted individually.

Problem One

17560 44356 41892 25731 78697 22171 90389 79500 53970 86075	Each of the ten numbers shown contains exactly one of the digits of Ann's telephone number in its correct position. For example, from the first number on the list, Ann's number could be 12345 or 17912, but NOT be 18537. What is Ann's telephone number? Explain why there is only one possible answer.

Problem Two

A large flag in the shape of an equilateral triangle is suspended by two of its corners from the tops of two vertical poles, one 4 m tall, the other 3 m tall. The third corner of the flag just touches the ground. What are the exact dimensions of the flag?



Problem Three

Let m be a positive integer. Integers a, b are defined as follows:

a consists of m digits, all equal to 1, and $b = 100 \ldots 05$ where there are $(m-1)$ zeros.

(a) Copy and complete the following table:

m	a	b	$ab+1$	$\sqrt{ab+1}$
1				
2				
3				

(b) Guess a formula for $\sqrt{ab+1}$ for any positive integer value of m.

(c) Show that for *any* positive integer value of m, $(ab+1)$ is the square of an integer, and find its square root.

Problem Four

I play chess with my partner. We often have a game and finish in a draw with just our kings remaining on the board. How many different such end-positions are there?

Explanatory Note: The two kings, one black, one white, are only restricted from being in squares which share a side or a vertex, or from being in the same square.

Problems Five and Six may be discussed with a partner who has also entered the Challenge. Separate solutions must be submitted. Write your partner's name clearly on your solutions for these problems.

Problem Five

When pirates go ashore to dig up buried treasure, each pirate in the digging party receives a different sized share of the plunder. The treasure is always completely shared out with the captain getting the largest share, the first mate the next largest and so on down to the cabin boy, if he is with the digging party.

As pirates are not smart, the fractions they use for sharing the treasure always have 1 as the numerator. (Pirates using fractions such as $\frac{2}{3}$ or $\frac{4}{7}$ are punished by being made to walk the plank!)

(a) Show that there is only one way to share out the treasure amongst a digging party of three.

(b) Find the six different ways which may be used to share out the treasure amongst a digging party of four.

(c) Show that there is at least one way to share out the treasure amongst a digging party of any size greater than two.

Problem Six

Let m be a positive integer. An integer P is the product of positive integers whose sum is m. For example, when m is 7, P could be $1 \times 6 = 6$ or $2 \times 2 \times 3 = 12$, among many others. Let Q be the largest value of P for a given m.

(a) Explain why $Q = 18$ when $m = 8$ and why $Q = 3^{33}$ when $m = 99$.

(b) Find a general formula for Q for an arbitrary m, and justify that your answer is correct.

Two Australian mathematics competitions

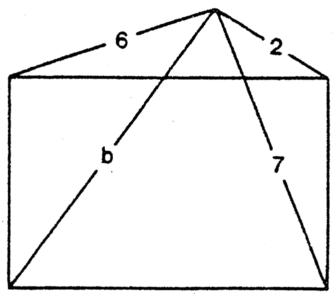
The TELECOM Mathematics Contests of 1991

On 13 August 1991 these two mathematics competitions were held in Australian schools. Here are the problems. Time allowed for either contest paper was four hours.

Junior Contest

(For students in Year 10 or less)

1. A surveyor measures the distance to each of three corners of a rectangular paddock as shown in the following diagram.



Determine if it is necessary to measure the distance b to the fourth corner.

2. Each of the numbers $1, 2, ..., n^2$ is placed in one of n^2 squares of an n-by-n grid paper in such a way that the numbers in each row, looked at from left to right, and in each column, looked at from top to bottom, are in an arithmetic progression. In how many ways can this be done?

Note: an arithmetic progression is a sequence of numbers $a_1, a_2, ..., a_n$ such that $a_{i+1} - a_i = a_i - a_{i-1}$, for $i = 2, ..., n-1$.

3. Let a, b and c be positive real numbers such that $a \geq b \geq c$ and $a + b + c \leq 1$. Prove that $a^2 + 3b^2 + 5c^2 \leq 1$.

4. Two circles C_1 and C_2 intersect at A and B. Let P be a point on C_1 and Q be a point on C_2, and suppose that the sum of the angles APB and AQB is $90°$.

Prove: if O_1 is the centre of C_1 and O_2 is the centre of C_2, then the triangles O_1AO_2 and O_1BO_2 are right-angled.

5. Determine all pairs (x, y) of positive integers x and y that satisfy:

   (i) $x \leq y$;

   (ii) $\sqrt{x} + \sqrt{y} = \sqrt{1992}$.

Senior Contest

(For students in Year 11)

1. Identical with Problem 4 of Junior Contest.

2. Identical with Problem 5 of Junior Contest.

3. Let a, b and c be positive odd integers. Prove that the equation $ax^2 + bx + c = 0$ has no solutions of the form $\frac{p}{q}$, where p and q are integers.

4. Let k be the circumcircle of triangle ABC and t its tangent at A. The points D and E are chosen such that

   (i) D is on AB and E is on AC;
   (ii) DE and t are parallel;
   (iii) $AD = 6$ cm, $AE = 5$ cm, $CE = 7$ cm.

   What is the distance between B and D ?

5. Determine the least positive integer n with the property: for every choice of n integers there exist at least two whose sum or difference is divisible by 1991.

The Junior Contest was entered by 153 students, whereas 119 took part in the Senior Contest. Certificates of Excellence were presented to 49 contestants.

The TELECOM 1992 Australian Mathematical Olympiad

The contest was held in Australian schools on February 11 and 12. On either day students had to sit a paper consisting of four problems, for which they were given four hours.

Paper 1

1. Let N be a regular nonagon, i.e. a regular polygon with nine edges, having O as the centre of its circumcircle, and let PQ and QR be adjacent edges of N. The midpoint of PQ is A and the midpoint of the radius perpendicular to QR is B. Determine the angle between AO and AB.

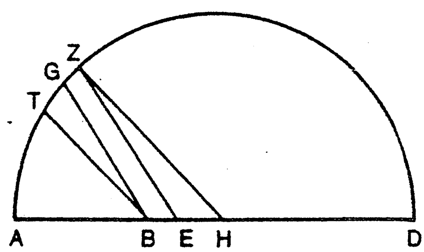
2. Let $ABCDE$ be a convex pentagon such that $AB = BC$ and $\angle BCD = \angle EAB = 90^\circ$. Let X be a point inside the pentagon such that AX is perpendicular to BE and CX is perpendicular to BD. Show that BX is perpendicular to DE.

4. Let K and L be positive integers. Prove that there exists a positive integer M such that, for all integers $n > M$, $\left(K + \frac{1}{2}\right)^n + \left(L + \frac{1}{2}\right)^n$ is not an integer.

Paper 2

5. Extend a given line segment AB in a straight line to D, where the length BD may be chosen arbitrarily (see diagram). Draw a semicircle with diameter AD, and let H be its centre. Let G be a point on the semicircle such that the angle ABG is acute. Draw EZ parallel to BG, where E is chosen such that $EH.ED = EZ^2$. Then draw ZH as well as the point T on the semicircle such that BT and ZH are parallel.

Prove: Angle TBG is one third of angle ABG.



6. Determine all functions f that

(i) take on real values;
(ii) are defined for all real numbers $x \neq \frac{2}{3}$;

(iii) satisfy the equation

$$498x - f(x) = \tfrac{1}{2}f\left(\frac{2x}{3x-2}\right) \text{ for all values of } x \text{ except } \frac{2}{3}.$$

7. Let $P_1, P_2, \ldots, P_{1992}$ be distinct points in 3-dimensional space such that every triangle $P_iP_jP_k$ of three different points has at least one side length less than 1 cm.

Prove that there are two spheres S_1, S_2, both of radius 1 cm, such that each of the given 1992 points lies in the interior of either S_1 or S_2 or both.

8. Let n be a positive integer. Show that

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n-1} \geq n\left(\sqrt[n]{2}-1\right).$$

There were 104 entrants. Gold Certificates were received by:

Lawrence Ip (year 12), Melbourne Church of England Grammar School, Victoria;
Anthony Henderson (11), Sydney Grammar School, NSW;
Adrian Banner (12), Sydney Grammar School, NSW;
Ben Burton (12), John Paul College, Queensland;
Martin Bush (12), Knox Grammar School, NSW;
Michael Bienstein (12), Melbourne High School, Victoria;
Geoffrey Brent (12), Canberra Grammar School, ACT;
Michael Russell (12), Collegiate School of St. Peter, South Australia;
Rupert McCallum (11), North Sydney Boys' High School, NSW;
Brett Pearce (12), St. Michael's Grammar, Victoria.

Congratulations to all! Twenty students, including all Gold Certificate winners, have been invited to represent Australia at the 1992 Asian Pacific Mathematics Olympiad. Students from about a dozen countries of the Asia-Pacific Region are expected to take part in the competition which was started in 1989.

* * * * *

A Flying Mystery

"*Philosophy*, that lean'd on Heav'n before,
Shrinks to her second cause, and is no more.
Physic of *Metaphysic* begs defence,
And *Metaphysic* calls for aid on *Sense* !
See *Mystery* to *Mathematics* fly !"

From *The Dunciad*
by Alexander Pope (1688-1744)

BOARD OF EDITORS

M.A.B. Deakin (Chairman) R. Arianrhod R.M. Clark P. Grossman H. Lausch R.T. Worley	Monash University
K. McR. Evans	formerly of Scotch College
J.B. Henry	Victoria College, Rusden
P.E. Kloeden	Deakin University
D. Easdown	University of Sydney, N.S.W.

* * * * *

BUSINESS MANAGER:	Mary Beal (03) 565-4445	Monash University, Clayton
TEXT PRODUCTION:	Anne-Marie Vandenberg	
ART WORK:	Jean Sheldon	

* * * * *

SPECIALIST EDITORS

Computers and Computing:	R.T. Worley
History of Mathematics:	M.A.B. Deakin
Problems and Solutions:	H. Lausch

* * * * *

Registered for posting as a periodical – "Category B"
ISSN 0313 – 6825

* * * * *

Published by Monash University Mathematics Department