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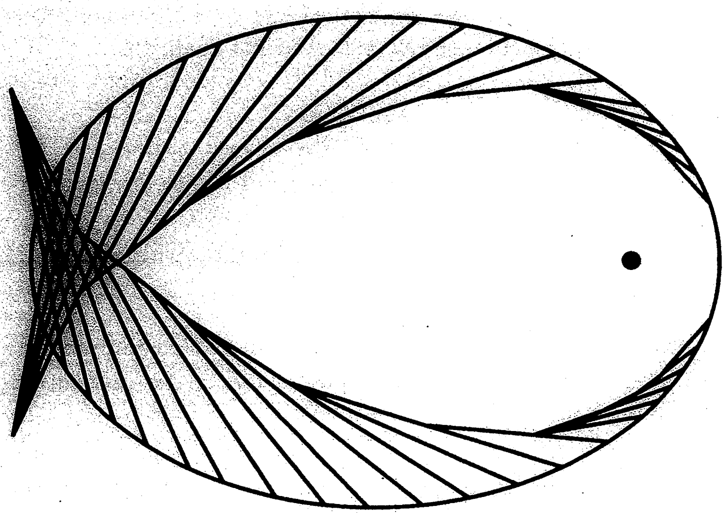

Mathematics Department - Monash University

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FUNCTION is a mathematics magazine produced by the Department of Mathematics at Monash University; it was founded in 1977 by Professor G.B. Preston. FUNCTION is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

FUNCTION deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of FUNCTION include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

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EDITORIAL

Welcome, old and new readers, to the new-look *Function*! We hope that you will find the new cover more lively and distinctive, and the new type-setting easier to read than previously. Along with these physical changes comes a change in editorial policy. Our aim is to make *Function* more lively, more readable and more interesting to our intended readership (of upper secondary school students). While *Function* will still cover a broad range of mathematical topics, there will be more emphasis on the human side of mathematics, and what it is to be a mathematician. We trust our readers will enjoy the changes being gradually introduced this year.

This issue of *Function* features a second article by Ravi Phatarfod on the statistics of elections and voting. In this article he shows, using fairly elementary probability methods, that a surprisingly *small* number of voters, voting together as a bloc, can determine the outcome of an election.

This issue includes our regular sections – *Problems and Solutions* (now called *Problem Corner*), *History of Mathematics* and *Computers and Computing*. Here you can read about Lewis Carroll, mathematician and author of *Alice in Wonderland*, and learn how to construct your own fractal.

We continue to welcome readers' contributions, whether they are letters, articles, new problems or solutions to earlier problems. Send them to the Editors of *Function* at the address on the inside back cover. We look forward to hearing from you.

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THE FRONT COVER:
LOCKWOOD'S GOLDFISH

Michael A. B. Deakin

Begin with the ellipse described by the equation

$$x^2 + 2y^2 = a^2, \tag{1}$$

whose long axis has length $2a$ and whose short axis has length $a\sqrt{2}$. This ellipse has its two "focal points" at $(\pm\frac{1}{2}a\sqrt{2}, 0)$. Let S be the point $(\frac{1}{2}a\sqrt{2}, 0)$, and (for future reference) let N be the point $(-\frac{1}{2}a\sqrt{2}, 0)$. Let P be any point on the ellipse. Now line up a set-square as shown in Figure 1, with the right angle at P and one of the shorter sides passing through S. Call the other of the two shorter sides PQ.

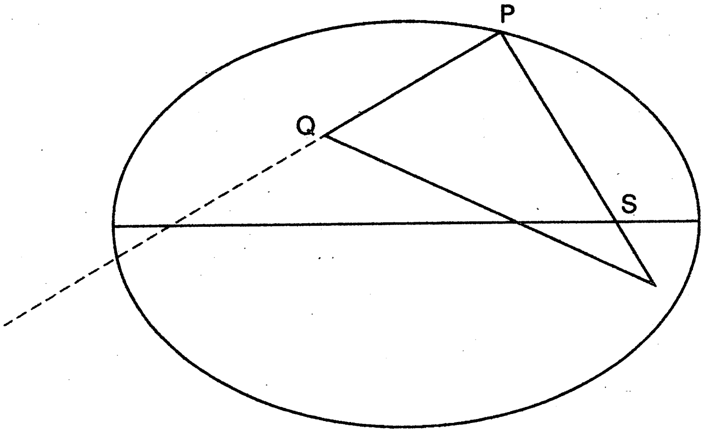

Figure 1

Now draw a line along the direction PQ. We may do this for many positions of the point P and the resulting straight lines will be tangent to a new curve, known as their *envelope*.[1] This is the situation shown on the front cover.

If we take any curve C_1 and any point S we may take a tangent to the curve and find the point P where this tangent meets a perpendicular

[1]See *Function, Vol.* 4, *Part* 3, pp. 14-21 and *Vol.* 5, *Part* 1, pp. 2-8.

passing through S. This point P lies on a curve c_2, known as the *pedal* of c_1 with respect to S. Thus our original ellipse is the pedal with respect to S of the envelope we have just constructed.

Conversely, this envelope is called the *negative pedal* of the ellipse (again with respect to S). Negative pedals of various curves with respect to various points are discussed in a number of texts on Geometry, notably in E.H. Lockwood's *A Book of Curves* (Cambridge University Press, 1971), Chapter 19.

This discussion includes an account of the particular negative pedal shown on the cover and also in Figure 2.

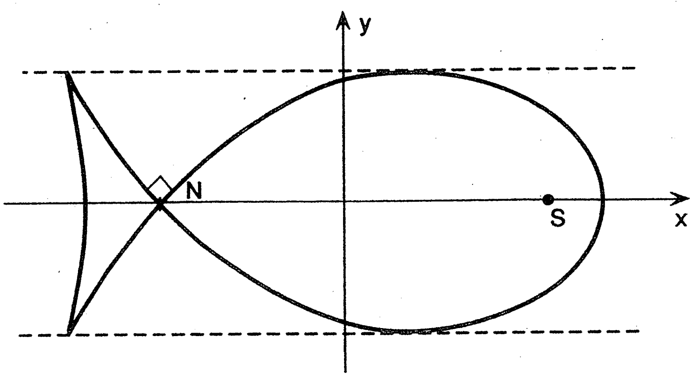

Figure 2

Lockwood's account there is based on an earlier and fuller one he gave in *The Mathematical Gazette, Vol.* 41 (1957), pp. 254-257. The curve set up – i.e. the one displayed in Figure 2 – he rather generously christened "Burleigh's Oval" after one of his students who first brought it to his attention. The name, however, is not particularly appropriate as the curve is not really an oval at all. Ovals, as indeed their name implies, are egg-shaped curves; see the cover story of *Function, Vol.* 10, *Part* 5. Perhaps a better name, certainly one giving a better description of the shape, would be *Lockwood's Goldfish.*

An equation may be written down for Lockwood's Goldfish. This form

$$\begin{aligned}(x^2 + y^2)^2 + 2(x^2+y^2)(x^2+a\sqrt{2}x-a^2)\\ -\frac{1}{2}(4x^4+2a\sqrt{2}x^3-3a^2x^2+2a^3\sqrt{2}x+a^4)=0 \qquad (2)\end{aligned}$$

is derived (after correcting a misprint) from a more general equation on p. 107 of an 1879 text: Salmon's *Higher Plane Curves.* Lockwood, in his *Mathematical Gazette* article, moves the origin to the point N, the other focus of the original ellipse, and the point where the goldfish's "tail" joins its "body". This gives the slightly simpler form

$$(2x^2 + y^2)^2 - 2ax\sqrt{2}(2x^2 - 3y^2) - 2a^2(x^2 - y^2) = 0. \qquad (3)$$

The curve has a number of interesting properties. The tail joins the body, as just mentioned, at the second focus of the original ellipse and the curve makes a right angle with itself at the intersection N. These properties are not hard to prove and you may care to look into this. (Don't use the equations given above!)

As indicated in Figure 2, the maximum width of the tail is equal to the maximum width of the body, each of these distances being equal to $\frac{1}{2}a$. The two points at the end of the tail correspond to the choices of P (in the coordinate system of Equation (1) and Figure 2) as $(-\frac{1}{2}a\sqrt{2}, \pm\frac{1}{2}a)$. The total length measured around the curve has been calculated and comes out to be $a(\frac{1}{2}\pi + 3)\sqrt{2}$ and the total enclosed area is $4a^2/3$.

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THE SQUARE ROOT LAW OF THE RESOLUTE MINORITY

Ravi Phatarfod, Monash University

In democratic societies most decisions are made by collections of people – cabinets, parliaments, committees, electorates, juries, etc.. With the exception of juries for some kinds of trials, the decision of the collection as a whole (in favour of, or against, a given proposal) is determined by majority vote in that collection.

This article looks at some elementary statistical laws governing majority decisions amongst a collection of people. For definiteness, we shall, from now on, call such a collection a committee. Usually, members of such committees can be divided into three groups – the resolute, the indifferent, and the flippant. The resolute are those who passionately believe in some proposal, and who vote together as a bloc; the indifferent are those who are amenable to persuasion and who vote independently of each other; the flippants are those who flip a coin to decide the issue. These group descriptions are only relative to an issue in question, and on different issues the membership of the groups may well be different.

This article looks at the power wielded by the vote(s) of a resolute person or a group of persons, and is based on an article on this theme by Penrose (1946).

Suppose you, singly or in a group, passionately believe in some proposal and are in a committee with a much larger group of indifferent members. Let's see exactly what power your vote has. We will assume that the rest of the committee members make up their mind individually, independent of each other, each member having the same probability p of having the same views on the proposal as yours. If the committee is fairly large and p is less than 0.5, your cause is a hopeless one; if, on the other hand, p is greater than 0.5, then the majority of the rest of the committee is favourable to you, and your vote is of limited value. It's only when p is close to 0.5 that your vote has any power. We shall assume that each person has, in fact, a probability of 0.5 of having the same views as yours. (Note, for this case, probabilistically the problem is the same as when each member is a flippant.) Also, to avoid ambiguities resulting from a tied vote, we shall assume, at least when we are dealing with small numbers, that the number of people on the committee is odd.

Power of a single vote.

If you are a member of a committee of three people, then the decision of the committee would go against you only when both the other members vote against you. Hence the probability, $P(W)$, that you will be on the winning side is

$$P(W) = 1 - \frac{1}{2} \times \frac{1}{2} = 0.75.$$

Similarly, in a committee of five people,

$$\begin{aligned} P(W) &= 1 - Pr\,[\text{all 4 vote against you}] \\ &\quad -Pr\,[\text{3 out of 4 vote against you}] \qquad (1) \\ &= 1 - \left(\frac{1}{2}\right)^4 - \binom{4}{3}\left(\frac{1}{2}\right)^4 = \frac{11}{16} = 0.6875, \end{aligned}$$

noting that under our assumptions, the number of persons voting against you follows the Binomial distribution, with $n = 4$ and $p = 0.5$.

The corresponding probabilities for committees of 7, 9, 11 and 17 are 0.65625, 0.6367, 0.6230 and 0.5982 respectively.

A measure of the power of your vote is the amount by which your chance of being on the winning side exceeds 0.5. Denoting this probability by $P(V)$, we have $P(V) = P(W) - 0.5$; for committees of sizes 3, 5, 7, 9, 11 and 17, the values of $P(V)$ are therefore 0.25, 0.1875, 0.15625, 0.1367, 0.1230 and 0.0982 respectively.

How does the power of your vote, $P(V)$, vary with the number of people on the committee? We can answer this using the following result.

Result. $P(V)$ *is half the probability that the rest of the committee is equally divided on the issue at stake, i.e., half the probability that your vote is decisive.*

To prove this result, consider a committee of yourself and $2n$ other people. Let X be the random variable representing the number of people out of the $2n$ who vote similarly to you. Then

$$\begin{aligned} P(V) &= P(W) - 0.5 \\ &= Pr[X \geq n] - 0.5 \\ &= Pr[X = n] + Pr[X \geq n+1] - 0.5 \\ &= Pr[X = n] + Pr[X \leq n-1] - 0.5, \text{ by symmetry} \\ &= \frac{1}{2}\{2Pr[X = n] + Pr[X \leq n-1] + Pr[X \geq n+1] - 1\} \\ &= \frac{1}{2}Pr[X = n], \qquad (2) \end{aligned}$$

since $Pr[X = n] + Pr[X \leq n-1] + Pr[X \geq n+1] = 1$.

To see how $P(V)$ varies with n, we have from (2), taking n an even number,

$$P(V) = \binom{n}{n/2} / 2^{n+1} \tag{3}$$

Now, using Stirling's approximation,

$$n! \simeq \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n},$$

(3) reduces to, for n large,

$$P(V) \simeq 1/\sqrt{2\pi n}.$$

We can state, from the above, the following proposition.

Proposition 1. *The power of the individual vote is inversely proportional to the square root of the number of people in the committee.*

For example, an individual vote has half as much power in a committee of 100 as it has in a committee of 10. The above proposition holds (approximately) even for small values of n. Our previous calculations show, for example, that $P(V) = 0.1875$ for $n = 4$ and roughly half this, 0.0982, for $n = 16$.

The Power of a Bloc Vote

Of more practical importance is the power of a bloc vote. Suppose a committee or electorate consists of a small resolute group and a much larger indifferent group. First, let us see a few simple cases. Suppose we have a committee of 13, consisting of a resolute group of two and an indifferent group of 11 people. Then the resolute group would be in the winning side when five or more of the 11 indifferent people vote with them. Using the shorthand $X \sim -Bin(n, p)$ for "X has the Binomial distribution with n trials, with p as the probability of success at a trial", and denoting the probability of a proposition A when B is true by $Pr[A|B]$, we have

$$\begin{aligned} P(W) &= Pr[X \geq 5 | X \sim Bin(11, 0.5)] \\ &= 0.735. \end{aligned}$$

This means a small resolute group of 2 has a 74% chance of carrying the decision in a committee of 13. Similarly, for the case when we have 3 resolute people in a committee of 23, we have

$$\begin{aligned} P(W) &= Pr[X \geq 9 | X \sim Bin(20, 0.5)] \\ &= 0.748. \end{aligned}$$

To obtain a general relation for large committees and electorates, let us assume we have a resolute bloc of $x\sqrt{n}$ voters (where x is a positive number to be determined), and an indifferent group of n voters. Then the probability of the bloc of $x\sqrt{n}$ voters carrying the decision they desire is

$$P(W) = Pr[X \geq (n - x\sqrt{n})/2 | X \sim Bin(n, 0.5)] \tag{4}$$

For larger n, the Binomial distribution can be approximated by the normal distribution with mean $\mu = np$ and standard derivation $\sigma = \sqrt{np(1-p)}$. Using this approximation, with $p = 0.5$, (4) reduces, after some algebra, to

$$P(W) \simeq F(x) \tag{5}$$

where $F(x) = Pr(Z \leq x)$ is the distribution function of the standard normal variable Z.

From (5), we can determine the required size of the resolute bloc to ensure a given probability $P(W)$ of winning. For example, suppose we have a committee with $n = 100$ indifferent voters. How many resolute voters are needed to be 99.9% sure of carrying the decision? From (5) and tables of the standard normal distribution, we must have $x = 3$ to ensure that $P(W) = 0.999$. The size of the resolute bloc is then $x\sqrt{n} = 30$. If, instead, the "committee" has $n = 10,000$ indifferent voters, the size of the resolute bloc becomes 300, with the same probability $P(W)$ of winning the vote.

Notice how the size of the resolute bloc is proportional to the *square root* of the number n of indifferent members in the committee. The following table shows how the size of the absolute bloc varies with both n and $P(W)$.

	Probability of decision being carried by resolute bloc			
No. of indifferent members	0.75 (x=0.675)	0.841 (x=1)	0.977 (x=2)	0.999 (x=3)
25	3	5	10	15
64	5	8	16	24
100	7	10	20	30
400	14	20	40	60
900	20	30	60	90
10,000	68	100	200	300
1,000,000	675	1000	2000	3000

This result can be stated more formally as:

Proposition 2: *The same degree of control is obtained in two populations with blocs proportional in sizes to the square root of respective population numbers.*

Let us apply the preceding results to an Australian federal election. Suppose that in a particular electorate there is a resolute group of voters who vote as a bloc, say the Greenies. Typically, about 80% of the remaining voters would be committed voters (for Labor or Coalition). This would leave about 15,000 swinging (or indifferent) voters in an electorate of 75,000 voters.

Suppose further that the election is a close contest, with the proportion of voters for either major party being about 0.5. We have here $n = 15,000$ and $3\sqrt{n} = 370$. Thus a bloc of only 370 resolute voters would effectively (with probability 99.9%) decide the election in that electorate.

Reference:

Penrose, L.S. (1946), Elementary statistics of majority voting, *J. Royal Statistical Society*, **109**, 53-57.

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HISTORY OF MATHEMATICS

EDITOR: Michael A. B. Deakin

Lewis Carroll – Mathematician?[1]

Lewis Carroll is best remembered as the creator of *Alice in Wonderland* and other works of fantasy. The name "Lewis Carroll" is actually a pseudonym derived from the first two names of the author, an Oxford cleric and academic, Charles Lutwidge Dodgson.

Dodgson was born in 1832, the third child of Charles Dodgson, a clergyman, and his wife and cousin (née Francis Jane Lutwidge). The young Charles first attended Richmond School, in Yorkshire, and came to the attention of his headmaster for the mathematical ability he displayed. From Richmond he went to Rugby, of *Tom Brown's Schooldays* fame, which he found less congenial. After three years, aged 19, he "escaped" (his own word) to Christ Church, Oxford, where he remained until his death in 1898.

He seems to have been extremely shy, stuttering, it is said, except in the presence of the various nymphets he befriended. Of these, the best-known was Alice Liddell, the original Alice; she was the daughter of a colleague, Dean Liddell, with whom Dodgson fell out when he requested permission to photograph the pre-adolescent Alice in the nude (albeit from behind).

Dean Liddell and almost an entire pre-Freudian generation took rather a dim view of this request, although we now know that Dodgson was puritanical to a fault. (He nurtured thoughts of expurgating further the already expurgated Shakespeare of Dr Bowdler, and expressed the hope that the illustrators of his books would not ply their trade on a Sunday.) His interest in photography was genuine and deep. The quality of his work in this area is high.

It is paradoxical that Dodgson, who by profession was a mathematician (he never practised as a parson, although he took holy orders in 1861), is deservedly better remembered for his creative writing and, indeed, his photography. The aim of this article, however, is to discuss his mathematical achievements.

[1]Dr Deakin is currently overseas. This article is reprinted, under a new title and with minor amendments, from *Function, Vol.* 7, *Part* 3.

His mathematical work lies in five main areas, only two or three of which he recognised as respectably mathematical. Most of his writing in mathematics was to do with Euclidean geometry, and the best-known of his strictly mathematical books, *Euclid and his Modern Rivals*, the only one still in use to any extent, lies in this area. However, when he was in his 30's he did also occupy himself with determinants, numbers arising in the study of matrix algebra. His other mathematical interest was the theory of tournaments and elections, of which more later. Beyond these three interests, he wrote mathematical recreations and works on Symbolic Logic under his pseudonym.

This distinction is important. Dodgson and his *alter ego*, Carroll, shared many concerns and their writing styles are similar. It has been said that Dodgson had two personalities, his own and Carroll's. This seems not to be so. Rather, he used the pseudonym for his works of fantasy, thus distinguishing them from his serious writing. The pseudonym gave him, shy as he was, some protection from the fame that *Alice* brought him.

Of his mathematical achievements, Carroll himself was wont to say that they lie "chiefly in the lower branches of mathematics". No "Dodgson's Theorem" exists, and few mathematicians would claim that he had initiated any lasting mathematical advance. He was a pedantic (i.e. tediously finicky) teacher, obsessed with his own idiosyncratic notations, such as m for *sine* and ⌓ for *cosine*. He attracted very few students to his lectures, which were, surprisingly perhaps to us, regarded as very dull. He published numerous, now forgotten, pamphlets, most of them divorced from the mainstream mathematics of his day, and even less relevant to us.

He probably approached that mainstream most closely in his work on determinants, which arise in the theory of matrix algebra. This was a relatively new branch of mathematics in 1866, when Dodgson published a brief note in the *Proceedings of the Royal Society.* A determinant associates a single number with a square array of numbers. Determinants arise particularly in the solution of simultaneous equations. Efficient methods for this evaluation needed to be developed, and it was to this question that Dodgson's paper addressed itself.

Unfortunately, it is almost incomprehensible, and to see what is meant, one does best to turn to his subsequent book: *Elementary Treatise on Determinants.*

[It was, incidentally, this book that followed most immediately on the heels of *Alice in Wonderland.* The story has it that Queen Victoria, enchanted by *Alice*, asked the publishers – Macmillan – for a copy of the author's next work, and was unamused to receive a copy of *Elementary Treatise on Determinants*. The story is probably apocryphal (i.e. of doubtful authenticity), but it makes a good yarn, and I follow convention in repeating it here.]

Elementary Treatise on Determinants is a reasonable introductory textbook, rather less original than its author supposed, but extremely systematic. Its eccentric notation is such that one would not recommend it to a modern reader. (For example, he refuses to use the word "Matrix", preferring "Block".) Some of the problems he set himself, but could not solve, now seem trivially easy as more advanced matrix algebra has become widely known. Appendix II of that work gives an expanded version of the *Royal Society* paper. It appears that what Dodgson had invented was a minor variant on what we now call *Gaussian elimination*, but its full details eluded him.

Dodgson's most extensive mathematical work lay in the field of euclidean geometry. It is in this field that his only mathematical work still reasonably available – *Euclid and his Modern Rivals* – is to be found.

One might imagine that the longevity of this book is due to the fact that, in the years preceding the book's publication, the non-euclidean geometries were developed,[2] but this is not so. Apart from knowing that they stem from a denial of Euclid's parallel postulate, Dodgson shows little acquaintance with these. What the book, a dialogue in five acts, attempts to do is to show the superiority of Euclid, as an expositor of *euclidean* geometry, over modern rivals such as, in particular, J.M. Wilson, a text-book writer of the time.

The following excerpt on the *pons asinorum* (i.e. the theory that the angles at the base of an isosceles triangle are of equal magnitude)[3] gives the flavour of Dodgson's writing. He is discussing Pappus' proof, which, in essence, proceeds by turning the triangle over and superimposing it on its previous position.

> **Minos:** It is proposed to prove 1.5 [i.e. the *pons asinorum*] by taking up the isosceles Triangle, turning it over, and then laying it down again upon itself.

[2]See *Function, Vol.* 3, *Parts* 2 and 4; *Vol.* 12, *Part* 4.

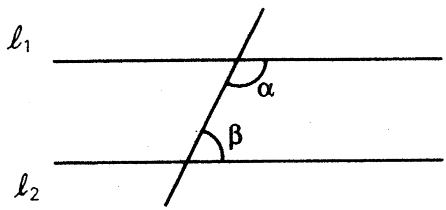
[3]See *Function, Vol.* 3, *Part* 3.

Euclid: Surely that has too much of the Irish Bull about it, and reminds one a little too vividly of the man who walked down his own throat, to deserve a place in a strictly philosophical treatise?

Minos: I suppose its defenders would say that it is conceived to leave a trace of itself behind, and that the reversed Triangle is laid down upon the trace so left.

Nowadays we dismiss such metaphysical questions from mathematics, omitting the actual motion from the argument. Dodgson's criticism does not apply to modern accounts at all.[4]

Dodgson did recognise that, when it came to the parallel postulate, Euclid's account might not be the best. Refer to the diagram. Euclid's version of the postulate is that ℓ_1, ℓ_2 are parallel if, and only if, $\alpha + \beta = \pi$; if $\alpha + \beta < \pi$ they meet when extended to the right, otherwise to the left.



By Dodgson's day, this rather cumbrous form had been replaced by the more illuminating *Playfair's Axiom*:

> *If P is a point not on a line ℓ, then exactly one line may be drawn through P parallel to ℓ.*

Dodgson resisted this new approach, though later, after he had retired on the proceeds of *Alice* (to devote his life to mathematical writing), he produced a convoluted alternative best passed over in silence.

In the field of symbolic logic, Dodgson wrote for publication under his pseudonym, which probably implies that he saw the subject as essentially recreational. His two books in the area are *Symbolic Logic* and *The Game of Logic*, both in print today. Opinions differ on their significance.

On the one hand, W.W. Bartley III (*Scientific American*, July 1972) can write: "his work on logic was highly original", but N.T. Gridgeman

[4]See *Function, Vol.* 3, *Part* 3.

(*Dictionary of Scientific Biography*) finds that "although he was not ignorant of the new trends [in mathematical logic], their importance either escaped him or was discountenanced".

Both books (and I include the second part of *Symbolic Logic*, reconstructed by Bartley) are original, quirky, and, to my mind, ultimately sterile. They both post-date Boole's *Laws of Thought*, which Dodgson is known to have possessed, but neither shows the slightest acquaintance with that work.

The predominant concern of these books is not really modern symbolic logic, but a rather baroque outgrowth from the puzzle world – the *sorites* (pronounced sore-*eye*-teas).

I will discuss but one very simple example. Three premisses are given:

(1) No potatoes of mine, that are new, have been boiled;

(2) All my potatoes in this dish are fit to eat;

(3) No unboiled potatoes of mine are fit to eat.

These all concern "my potatoes", which may be: a (boiled), b (fit to eat), c (in this dish), d (new). The object is to construct a valid conclusion from the premisses. The method is perfectly mechanical and one of several equivalent techniques employed by Carroll proceeds as follows.

Write $\Rightarrow$ for "implies" and $\sim$ for "not". The premisses now translate as:

$$\begin{array}{ll}(1) & d \Rightarrow \sim a \\ (2) & c \Rightarrow b \\ (3) & \sim a \Rightarrow \sim b.\end{array}$$

Four letters are involved, of which two (a, b) occur twice and the others (c, d) only once. (More generally, if n premisses are involved, and some of Carroll's soriteses involve over 50, there are $(n+1)$ letters, $(n-1)$ of them occurring twice and two occurring once.) The problem is to find a logical connection between the two which are listed only once.

In the above example, write (1) and (3) in the alternative (and equivalent) forms:

$$\begin{array}{ll}(1) & a \Rightarrow \sim d \\ (3) & b \Rightarrow a.\end{array}$$

We thus find

$$c \Rightarrow b \Rightarrow a \Rightarrow \sim d \quad \text{whence} \quad c \Rightarrow \sim d$$

which translates as "My potatoes in this dish are not new".

For more on these topics, see *Function, Vol.* 1, *Part* 5.

There are some nice things in *Symbolic Logic* and *The Game of Logic.* Venn diagrams are used in an elegant way with coloured counters and some subjects are raised which still occupy some (usually less mathematically inclined) logicians today.

Of Carroll's other mathematical recreations, the best known is his proof that all triangles are equilateral. This featured in the April Fools' Day column of *Function, Vol.* 7, *Part* 2. Although Carroll did not realise it, this is a significant result, for the proof is not technically incorrect. Where it goes wrong is in its translation into reality. Points G, H are there constructed, one necessarily lying *inside* and the other *outside* the triangle and this causes the proof to fail (in almost its last line). Euclid's axioms do not, however, refer to the "inside" or "outside" of a triangle, and thus what Carroll had done was to prove the inadequacy of Euclid's system of axioms, a conclusion he would not have liked at all!

This example is found in a collection called *Curiosa Mathematica*, as is this next problem (the relevant part has also been published as *Pillow Problems*). The kindest thing that can be said about Carroll's error here is that it shows how easy it is to make mistakes in elementary probability theory.

Problem

> "A bag contains 2 counters as to which nothing is known except that each is either white or black. Ascertain their colours without taking them out of the bag."

Now this is nonsense, but Carroll confidently gives the answer "one white, one black" and, moreover, argues for it by means of a specious probability argument. We leave it to the reader to find the error, noting merely with one commentator (Eperson, *Mathematical Gazette*, *Vol.* 17 (1933), p. 99), that a similar argument, applied to the case of 3 counters, shows that there were not 3 after all. Here is Carroll's "solution".

"We know that if a bag contains 3 counters, 2 being black and one white, the chance of drawing a black one is $\frac{2}{3}$, and that any other state of things would not give this chance.
Now the chances that the given bag contains, (1) BB, (2) BW, (3) WW, are respectively $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.
Add a black counter.
Then the chances that it contains (1) BBB, (2) BBW, (3) BWW, are as before, $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.
Hence the chance of now drawing a black one

$$= \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{2}{3}.$$

Hence the bag now contains BBW (since any other state of things would not give this chance).
Hence before the black counter was added, it contained BW, i.e. one black and one white counter."

Regrettably, one encounters other such lapses in Dodgson's writing. He clearly enjoyed mathematics as a recreation and kept a journal of mathematical thoughts.

"31st October, 1890. This morning, thinking over the problem of finding two squares whose sum is a square, I chanced on the theorem (which seems true, though I cannot prove it) that if x^2+y^2 is even, its half is the sum of two squares. A kindred theorem that $2(x^2 + y^2)$ is always the sum of two squares also seems true but inproveable."

"True but unproveable" would seem to presage Gödel's Theorem, but this is not what Dodgson had in mind. He found a proof five days later. It is one line long and we leave you to discover it for yourself. On the 5th of November, he proved also the related "theorem": "Any number whose square is the sum of two squares is itself the sum of two squares".

The result, as stated, is in fact false, e.g., $15^2 = 12^2 + 9^2$, but 15 is not the sum of two squares. However, it is true that if

$$z^2 = x^2 + y^2$$

and if z, x, y have no common factor, then there exist integers u, v, such that $z = u^2 + v^2$. This result was known to tbe Babylonians[5] and proofs

[5]See *Function, Vol.* 15, *Part* 3, pp. 85-91.

had been available for hundreds of years before Dodgson. See Problem 7.3.5. See also for more background *Function, Vol.* 4, *Part* 1, p. 27 and the article "Pythagorean Triples" by F. Schweiger in *Vol.* 6, *Part* 3.

Some commentators, like Bartley, rate Dodgson's logical works more highly than I have done. But Gridgeman's assessment of his mathematical work takes the least-known of it (the work on tournaments and elections) to be the best.

This work is contained in a number of pamphlets, letters and broadsheets, all very rare, and one so rare that it survives merely as a single copy. The best account of this work is in the book *The Theory of Committees and Elections* by Duncan Black and I draw on this.

There is an extensive mathematical theory concerned with the organisation of tournaments, elections and fair decision-making procedures. Black's book is a good introduction to an area we can only touch on here. Some of this theory predates Dodgson, but Black shows quite conclusively that Dodgson did not know of this.

Dodgson's sources were his practical experience in the organisation of tennis tournaments and his work on committees at Christ Church. In this latter capacity he used his work to further his quarrel with Dean Liddell (Alice's father).

Again, I would hardly consider Dodgson's work in the area earth-shattering, but he does consider a number of unusual and imaginative voting schemes, methods for multiple decision-making and allowance for the expression of degrees of preference in a ballot. There are many numerical examples, which at least serve to show the limitations of methods in vogue.

To those new to this field, consider the differences between:

(a) the "first-past-the-post" system as used in Britain,

(b) the Australian Federal lower house system,

(c) the Australian Senate system,

(d) Tasmania's Hare-Clark system,

(e) the system used to decide *The Age* footballer of the year.

Gridgeman, rather generously, remarks that Dodgson was the first to use matrices in multiple decision-making, and, if the tabulation of results in a rectangular array is to be called a use of matrices, so be it. No use is made of *matrix algebra*, of which, apart from the relatively elementary theory of determinants, Dodgson seems to have been ignorant.

The picture that emerges of Dodgson the mathematician is one of a pedant (his *Notes on Euclid* includes definitions of "problem" and "theorem", and *Symbolic Logic* has a definition of "definition"), original enough, but out of touch with the mathematics of his day. He was a mediocre mathematician, in other words.

Of course, his talents in both literature and photography were much greater and for these he is justly famous. He is deservedly best remembered for the things he did best.

An update:

The above article appeared in *Function* in 1983. In the following year, E. Seneta of the University of Sydney published an article entitled "Lewis Carroll as a Probabilist and Mathematician" in the journal *The Mathematical Scientist*. This covers much the same ground as my own article, but provides a rather different assessment. There is a lengthy discussion of Dodgson's work on probabilities and a somewhat shorter one on determinants. A final section considers his other mathematical interests. In particular, Seneta makes a case for regarding Dodgson's work on determinants more highly than I have done. He quotes several other authors in support of his view. However, I still retain my own conviction, even after reading Seneta, that not only is it the almost universal belief, but it is also correct, that Dodgson's contributions to mathematics itself were extremely slight.

* * * * *

CONSTRUCT YOUR OWN FRACTAL

Cristina Varsavsky, Monash University

Fractals are without doubt fascinating and beautiful. You have certainly seen and appreciated fractal images like beautiful ferns, Mandelbrot sets, Julia sets, and many others. You might have also come across some computer programs, usually public domain ones, with which you can put your imagination to work and generate your own fractal. In this article we will explore a powerful yet simple mathematical concept behind fractals: *iteration*. Iteration has already been covered in various *Function* issues: Fibonacci sequences, solution of algebraic equations, etc. In this article we will use iteration in a geometric construction of a famous fractal image and we will design a computer program that would generate it on the screen.

Let us start by drawing a square. The basic construction step goes as follows: replace the square with three squares with side lengths halved, two of them sitting side to side at the bottom of the original one, and the third square on top of them centred horizontally with respect to the old square. Figure 1 shows the basic step, in which the square S is replaced by the squares S_1, S_2 and S_3.

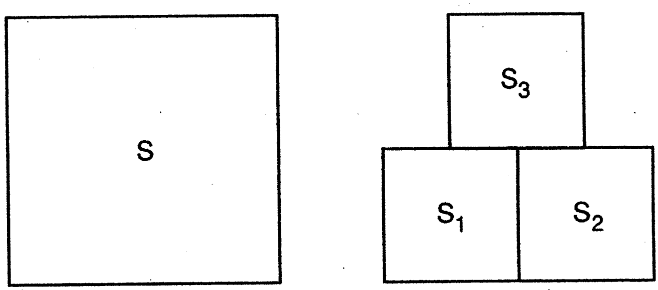


Figure 1

This basic constructive step can be repeated again; each of the three squares, namely S_1, S_2 and S_3, is replaced with three new squares following the same rule as above. The nine squares are displayed in Figure 2.

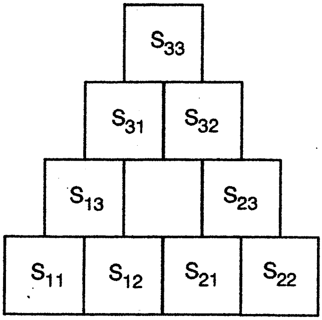


Figure 2

We can repeat this process as often as desired. Figure 3 displays the image generated by this iterative process using seven iterations. The image generated by applying this basic constructive step infinitely many times is called the Sierpinski triangle which was first produced by the great Polish mathematician Waclaw Sierpinski in 1916.

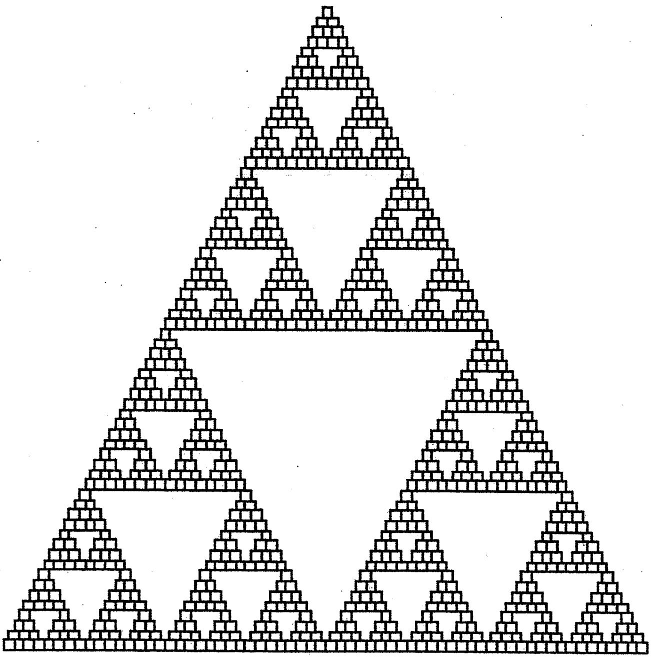


Figure 3

Since constructing the Sierpinski triangle involves repeating the same process several times, a computer is an ideal tool to carry out that task. Let us design a computer program to produce this fractal image on the computer screen. We will use QuickBasic as this is the computer language most widely available to secondary students today. If you want to implement it in a different language, it should not be too difficult to translate it as the subroutines are pretty simple and clear.

The first thing we need to specify is the screen resolution. Choose SCREEN 9 (640x350 graphics). The basic square we start with, which is sometimes also called *seed*, will take 300x300 pixels. We label the vertices of the square as shown in Figure 4.

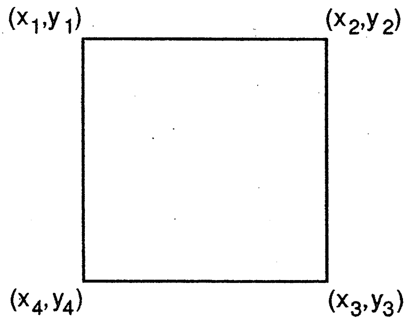


Figure 4

To implement the basic step, we need to define three transformations, one for each of the smaller squares. The first transformation, that is the one that transforms S into S_1, is just a scaling down by a factor of 0.5 in both directions followed by a vertical shift of 150 pixels. To transform S into S_2, again we scale by a factor of 0.5 but we also need to move borizontally 150 pixels (half the size of S) to the right and vertically 150 pixels down. Finally the third transformation involves scaling by a factor of 0.5 and a shift of 75 pixels to the right. We will call **Transform** the subroutine performing this task. This subroutine has to be repeated again and again until the last level is reached; only then are the squares drawn.

So, what are the variables we need? There must be a variable to control the number of iterations; we call it **number**. Then there are the vertices of the squares which are modified by the subroutine **Transform**. We will use arrays following the same notation as in Figure 4; as 10 iterations are usually more than enough to generate an image with good resolution, we dimension them to 10. (It may take a while to run the program with more than 10 iterations.)

Here is the code:

```
REM Geometric construction of the Sierpinski triangle

SCREEN 9
DIM x1(10), x2(10), x3(10), x4(10), y1(10), y1(10), y2(10), y3(10), y4(10)

INPUT "Enter the number of iterations [1-10] : ", number
REM initialisation of variables
   horz = 150: vert = 30
   x1(number) = 0: y1(number) = 0
   x2(number) = 300: y2(number) = 0
   x3(number) = 300: y3(number) = 300
   x4(number) = 0: y4(number) = 300
   xdisp(1) = 0: xdisp(2) = 150: xdisp(3) = 75
   ydisp(1) = 150: ydisp(2) = 150: ydisp(3) = 0

GOSUB Drawing
END

REM Next iteration

Iterate:
   number = number - 1
   trans = 1
   GOSUB Transform
   trans = 2
   GOSUB Transform
   trans = 3
   GOSUB Transform
   number = number + 1
   RETURN

REM Basic step: transformation of square in terms of the previous one

Transform:
   x1(number) = .5 * x1(number + 1) + xdisp(trans)
```

```
    y1(number) = .5 * y1(number + 1) + ydisp(trans)
    x2(number) = .5 * x2(number + 1) + xdisp(trans)
    y2(number) = .5 * y2(number + 1) + ydisp(trans)
    x3(number) = .5 * x3(number + 1) + xdisp(trans)
    y3(number) = .5 * y3(number + 1) + ydisp(trans)
    x4(number) = .5 * x4(number + 1) + xdisp(trans)
    y4(number) = .5 * y4(number + 1) + ydisp(trans)


REM Drawing of square at the last iteration

Drawing:
    IF number > 1 GOTO Iterate
    LINE (horz + x1(1), vert + y1(1)) - (horz + x2(1), vert + y2(1))
    LINE - (horz + x3(1), vert + y3(1))
    LINE - (horz + x4(1), vert + y4(1))
    LINE - (horz + x1(1), vert + y1(1))
    RETURN
```

In the initialisation section, the constants **horz** and **vert** are set so that the fractal is displayed on the centre of the screen. The vertices of the seed are set to (0,0), (300,0), (300,300), (0,300) (the origin (0,0) is at the top left corner of the screen). The two arrays, **xdisp** and **ydisp** are the horizontal and vertical displacements for each of the three transformations. (Can you see why **xdisp** and **ydisp** are necessary, and are unchanged from iteration to iteration?) As you can see, the subroutine **Iterate** controls the number of iterations for each transformation. The subroutine **Transform** is already described above, and finally **Drawing** draws the squares corresponding to the final iteration using the LINE statement. Type in the program and run it with **number** set to 7 to see the Sierpinski triangle shown in Figure 3. I recommend you use the Procedure step from the Debug menu to see step by step how the squares are drawn at the last iteration.

Why is it that, although we start with a square, we end up with a triangular shape? What if we apply the same process starting with a circle? The basic step would be the same, but replacing the circle C with three new circles, C_1, C_2 and C_3, with halved radii and positioned in the same way as before (see Figure 5).

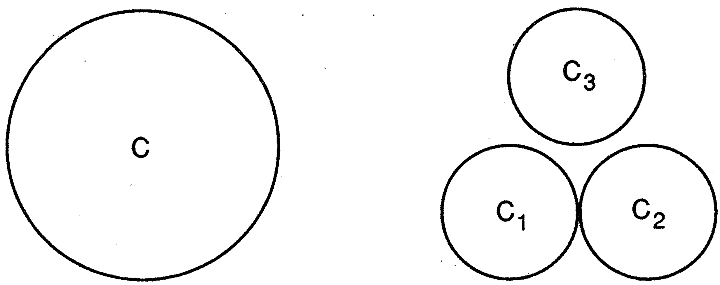


Figure 5

A circle is fully determined by the radius and the coordinates of the centre. We will use the variables **radius, xcentre** and **ycentre** instead of the set of coordinates for the vertices of the circle. In the initialisation section those settings for the vertices must be replaced with

```
radius(number) = 150
xcentre(number) = 150
ycentre(number) = 150
```

The subroutine **Iterate** remains unchanged. The new subroutine **Transform** should be

```
Transform:
    xcentre(number) = 0.5 * xcentre(number+1) + xdisp(trans)
    ycentre(number) = 0.5 * ycentre(number+1) + ydisp(trans)
    radius(number) = 0.5 * radius(number+1)
```

The new **Drawing** subroutine is much shorter, as only one statement is needed to draw a circle:

```
Drawing:
    If number > 1 GOTO Iterate
    CIRCLE(horz + xcentre(1), vert + ycentre(1)), radius(1)
```

If you run the program with these modifications, you will see that the final figure you obtain is very much the same as the one produced with squares; the more times you iterate, the greater the similarity. Compare

Figures 3 and 6: both images are produced with seven iterations, but the seeds are a square and a circle. You could experiment with different types of seeds, and you will soon discover that it does not matter what the shape of the seed is, the fractal produced in this way will be the same. So the secret of the final image does not reside in the seed but rather in the basic constructive principle, in how the three new squares (or circles, or any other seed) relate to the previous one.

This basic principle of iteration applied to different sets of transformations will generate an infinite variety of images. You need to decide how many transformed squares (or any other seed) are to replace the previous one, and how the new "smaller" copies are transformed and positioned with respect to the previous ones. The example we used scales down by a factor of 0.5 in both directions, but it does not have to be necessarily so. We may reduce by different factors in the horizontal and vertical directions; we could even use rotation, shifting and reflection. Beautiful images like ferns and trees could be produced in this way, but it takes a bit of practice to design the right set of transformations to be used in the iterative process.

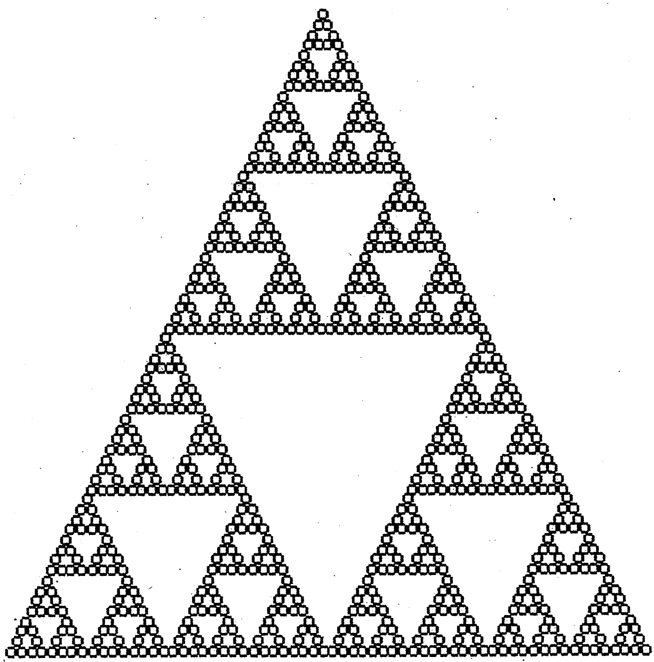


Figure 6

Here you have some suggestions you can easily try out. You can also put your imagination to work to produce your own images. Have fun!

Exercise 1 : Replace the square with three new squares as shown in Figure 7.

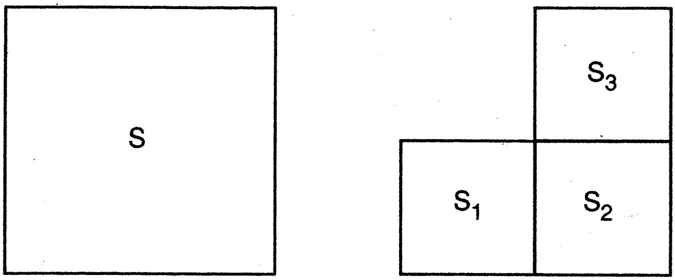


Figure 7

Exercise 2 : Replace the square with nine squares as in Figure 8. (The fractal obtained with this set of 8 transformations is called Sierpinski Carpet.)

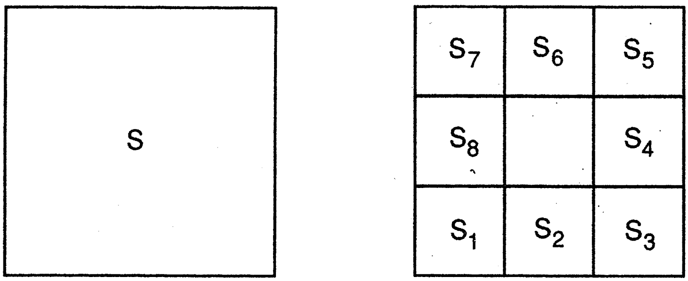


Figure 8

* * * * *

Cristina Varsavsky received her Ph.D. from Buenos Aires University and is currently Assistant Lecturer at Monash University. Her interests include computer algebra and the use of computers to make the learning of Mathematics more attractive and meaningful for students.

* * * * *

PROBLEM CORNER

SOLUTIONS

We always welcome letters from readers who provide us with alternative solutions or corrections to solutions published in previous issues of *Function*. Two such letters have reached us in the past few months. David Shaw, of Newtown, Geelong, Vic., sent us alternative solutions to Problems 14.1.3 (February 1990) and 14.2.10 (April 1990), for which we had published solutions in the April 1993 issue. His solution to Problem 14.2.10 (which asked for the solution in positive integers of the equation $x + y + xy = 1990$) is more elegant than ours; in essence, it involves rewriting the equation as $(1 + x)(1 + y) = 1991$ and looking for factors of 1991. David Shaw also provided solutions to some of the Australian Mathematical Olympiad which appeared in the April 1993 issue.

Andy Liu, of the University of Alberta, Canada, pointed out a mistake in our solution to Problem 14.3.2 in the June 1993 issue. He writes: "I like to solve geometry problems geometrically. In general, they turn out to be simpler, and provide more insight into the configuration". We think he has made a good point. Here is a modified version of his solution to Problem 14.3.2.

PROBLEM 14.3.2

The problem read: ABC is a triangle right-angled at A, and D is the foot of the altitude from A. Let X and Y be the incentres of triangles ABD and ADC respectively. Determine the angles of triangle AXY in terms of triangle ABC.

SOLUTION by Andy Liu

The triangle DAC is similar to ABC since both contain the angle $\angle ACD = \gamma$ and a right angle. Analogously, DBA has angle $\angle ABD = \beta$ in common and is similar to ABC and thus to DAC. Both smaller triangles contain angles β and γ at corresponding vertices.

The line from any vertex of a triangle to the incentre bisects the angle at that vertex. Therefore:

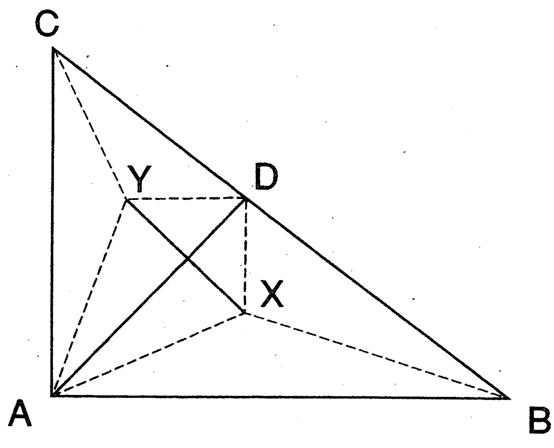
$$\angle DAX = \gamma/2 \quad (1)$$
$$\angle DAY = \beta/2 \quad (2)$$
$$\angle ADY = \angle ADX = \pi/4. \quad (3)$$

We now return to the similar triangles DAC and DBA. Since in similar figures the ratio of the lengths of any pair of corresponding lines is constant, DX/DY (the ratio of distances from apex to incentre) equals c/b (the ratio of the hypotenuses). The angle $\angle YDX$ being $\pi/2$ as shown from (3) above, the triangle DXY must be similar to the other three mentioned and the angles $\angle DYX$ and $\angle DXY$ are respectively γ and β. We now derive the angles of triangle AXY.

$\angle XAY$ is seen from equations (1) and (2) to be $(\beta + \gamma)/2$ which is $\pi/4$.

Two of the angles in triangle AXD are known, namely $\pi/4$ and $\gamma/2$, and, bearing in mind that $\gamma = \pi/2 - \beta$, the remaining angle $\angle AXD$ is $\pi/2 + \beta/2$. Hence $\angle AXY = \pi/2 - \beta/2$.

Similarly $\angle AYX = \pi/2 - \gamma/2$, or, in terms of angle β, $\angle AYX = \pi/4 + \beta/2$.



Andy Liu also provided a solution to Problem 14.4.5 that is more "geometric" than ours.

SOME MORE SOLUTIONS

PROBLEM 16.1.2

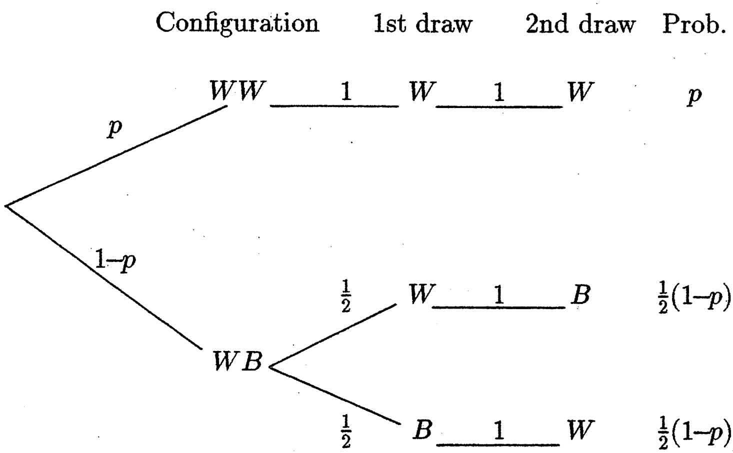
A bag contains a counter which is known to be either black or white. A white counter is added, the bag shaken, and a counter drawn out which proves to be white. What is now the chance (probability) of drawing a white counter?

SOLUTION by Malcolm Clark, Monash University

Let p denote the probability that the initial counter is white. Then after the second counter is added, there are only two possible configurations of the counters in the bag:

either	Both counters white – probability p
or	One white, one black – probability $1-p$.

The following tree-diagram summarises what happens when two counters are drawn, one at a time, from the bag.



Let W_1 denote the event that a white counter is drawn on the first draw, and W_2 the event that a white counter is drawn on the second draw. We require the conditional probability

$$\begin{aligned} P(W_2|W_1) &= \frac{P(W_2 \cap W_1)}{P(W_1)} = \frac{P(WW)}{P(WW)+P(WB)} \\ &= \frac{p}{p+\frac{1}{2}(1-p)} = \frac{2p}{p+1}. \end{aligned}$$

It is not possible to give a specific numerical value for this probability without making some further assumption about the mechanism for inserting the initial counter. One possibility is that the initial counter is selected at random from another bag containing a proportion p of white counters. At the other extreme, the initial counter could always be white, in which case $P(W_2|W_1) = 1$. A third possibility is that the colour of the initial counter is decided by flipping a coin. In this case, $p = 0.5$ but $P(W_2|W_1) = \frac{2}{3}$.

PROBLEM 16.5.1 Determine all positive integers m and n such that

$$\frac{1}{m} + \frac{1}{n} - \frac{1}{mn} = \frac{2}{5}.$$

SOLUTION by Keith Anker, Monash University

The equation given in the problem is equivalent to the following one:

$$\frac{3}{m-1} + \frac{5}{n} = 2. \qquad (1)$$

(This can be readily checked by multiplying through to clear the fractions in both equations, and simplifying.)

It follows from Equation (1) that $\frac{3}{m-1} < 2$. Therefore $m > \frac{5}{2}$ and hence $m \geq 3$, since m is an integer. Thus $\frac{3}{m-1} \leq \frac{3}{2}$, and therefore $\frac{5}{n} \leq \frac{1}{2}$, which implies that $n \leq 10$. But m and n appear symmetrically in the original question, so both inequalities must apply to both m and n:

$$3 \leq m \leq 10, \quad 3 \leq n \leq 10$$

Checking each possible value of m in turn yields the following solutions:

$$\begin{aligned} m &= 3, & n &= 10 \\ m &= 4, & n &= 5 \\ m &= 5, & n &= 4 \\ m &= 10, & n &= 3 \end{aligned}$$

PROBLEMS

PROBLEM 18.1.1. (K.R.S. Sastry, Addis Ababa, Ethiopia). Through a fixed point K a variable line is drawn to cut the parabola $y = x^2$ in the points P and Q. Let R be the midpoint of the chord PQ of the parabola. Find the locus of R.

PROBLEM 18.1.2 The polynomial factorisation shown below was written down so hastily that most of the digits are illegible.

$$x^2 + {*}x - {*}1 = (x + {*}{*})(x - {*})$$

(Each asterisk denotes an illegible digit.) How should it read?

* * * * *

FERMAT'S LAST THEOREM – STOP PRESS

John Stillwell, Monash University

The most famous problem in mathematics – Fermat's Last Theorem – appears to have been solved, or very nearly. First stated by the French mathematician Pierre de Fermat around 1637, the theorem is that there are no positive integers a, b, c such that

$$a^n + b^n = c^n,$$

where n is an integer greater than 2.

Fermat wrote the statement in the margin of a book with the claim that he had "a marvellous proof, but the margin is too small to contain it". Over the centuries, the mystery of the theorem has deepened, as other mathematicians failed to prove it, despite the development of more and more powerful methods.

Finally, on 23 June 1993, a proof was announced by Andrew Wiles, an English mathematician now at Princeton University in the U.S. Wiles's proof is far too long to fit in any margin – it is likely to be a book in itself – but experts are confident it is basically correct.

But not *entirely* correct, unfortunately. As the months went by after the first announcement, and Wiles's proof remained in the hands of a few expert referees, rumours about gaps and errors began to surface. Most of them were unfounded, but in early December Wiles conceded that some details still needed fixing.

The most recent news is from Ken Ribet, one of the referees, who spoke about the proof at a conference on January 13. Ribet, whose own work is crucial to Wiles's proof, believes that only a small gap remains to be filled. The next few months should show whether he (and Wiles, and Fermat) are right.

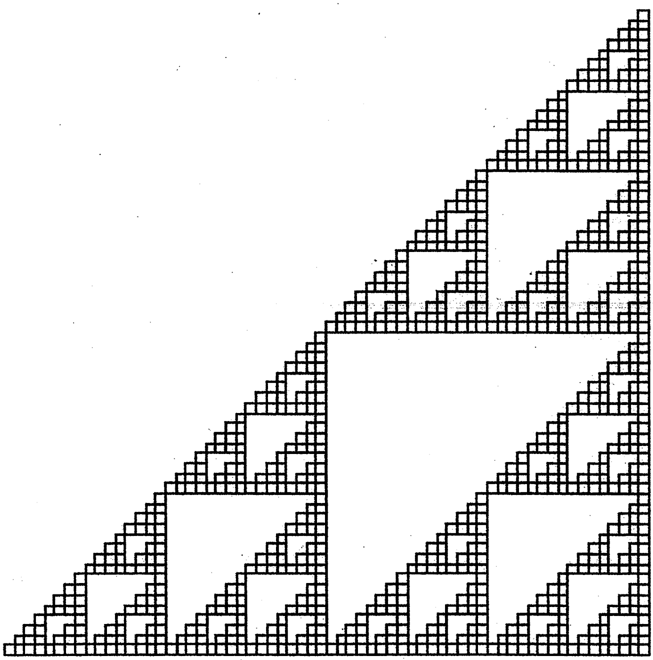
* * * * *

Correction

On p. 154 of *Function, Vol.* 17 the third last paragraph (after the exercise) should be replaced with:

This correcting procedure will not work when more than one error occurs in the transmission. This can be easily seen with the following example: if errors occur at the third and fourth positions during the transmission of 0010011, the receiver would read 0001011. Multiplying the received word by G gives us 100, indicating that the error occurred at the fifth position, which is not true. This Hamming code is capable of properly correcting a word only when one error occurs. If two errors occur they will be detected because multiplication by the matrix would not lead to [000], but the correction would not produce the original word.

* * * * *



This is the fractal produced by seven iterations of the transformation illustrated in Figure 7, page 26.

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