Function

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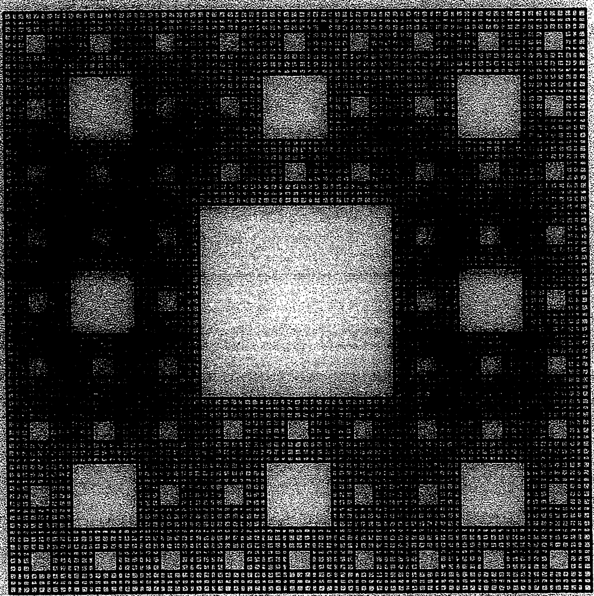

Mathematics Department - Monash University

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FUNCTION is a mathematics magazine produced by the Department of Mathematics at Monash University; it was founded in 1977 by Professor G.B. Preston. FUNCTION is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

FUNCTION deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of FUNCTION include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

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*\$8.50 for *bona fide* secondary or tertiary students.

EDITORIAL

We welcome readers to this second issue of the new-look *Function.* This time we have three feature articles. Michael Deakin looks at the geometry of the Tetra® Pak and its construction, while M. J. Englefield presents a useful but not well known relationship between logarithms. K. R. S. Sastry looks at whether or not the length of a triangle median equals the "mean" of the lengths of certain sides of the triangle, using three different definitions of "mean". In addition, John Stillwell gives, in our *History of Mathematics* section, a fascinating account of the history of Fermat's Last Theorem, following up his previous note about Andrew Wiles's proof. He describes how famous mathematicians such as Euler, Lamé, Kummer and Fermat himself struggled over the years to prove (or disprove) Fermat's famous conjecture.

Our front cover illustrates the famous fractal image known as the Sierpinski Carpet. This is produced by an iterative process, and has some intriguing properties, including a *fractal dimension* of 1·8929.

This issue also includes our other regular sections – *Problem Corner* and *Computers and Computing*, where in the latter you can learn how to construct magic squares. We report on the recent Telecom 1994 Australian Mathematical Olympiad, listing the problems posed and the winners of Gold Certificates. Finally, we include, as a follow-up to our cover story in August 1993 on sundials, a reader's letter describing a most unusual sundial, which uses your own shadow to tell the time!

We continue to welcome readers' contributions, whether they are letters, articles, new problems or solutions to earlier problems. Send them to the Editors of *Function* at the address on the inside back cover. We look forward to hearing from you.

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THE FRONT COVER

THE SIERPINSKI CARPET

Cristina Varsavsky, Monash University

The front cover illustrates the famous fractal image known as the Sierpinski Carpet. This computer-drawn image is the result of 5 iterations of the geometric transformation described in Exercise 2 in the article "Construct your own Fractal" in the previous issue of *Function*. This exercise required you to write a computer program which would iteratively carry out the following basic step.

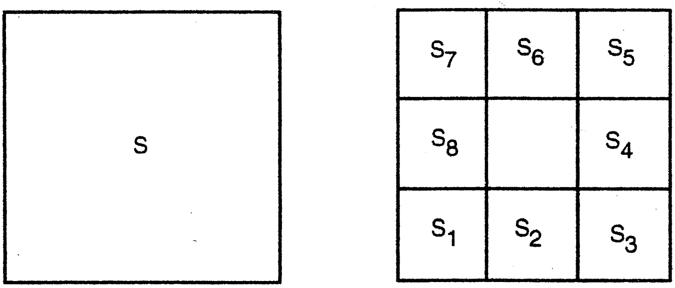

Figure 1

In other words, divide each square into nine identical squares, and omit the middle one.

If we carry out this iterative process infinitely many times, the resulting set of points is known as the Sierpinski Carpet. This structure is one of several fractal images created by Waclaw Sierpinski (1882–1969), a prominent Polish mathematician[1] of his time.

A *QuickBasic* program for producing up to 10 iterations of this iterative process is given on pages 59 and 60, in the *Computers and Computing* Section of this issue.

A fractal is more than a beautiful picture. The theory of fractals involves many sophisticated concepts; one of them is that of *fractal dimension*. We

[1]There is also a moon crater named after W. Sierspinski

usually regard a line as having dimension one because it does not contain area or volume elements. Similarly, a square has dimension two and a cube has dimension three. Applying this definition, we might intuitively argue that both the Sierpinski Triangle and the Sierpinski Carpet have dimension one, since in the limit they cannot contain any area.

Mathematicians at the beginning of this century came up with another way of defining dimension. Let us start with a square, and apply the transformation (shown in Figure 1) that reduces the square by a factor of $\frac{1}{3}$, but this time we do not omit the middle square. If N is the number of reduced squares (in this case 9) and r is the reduction factor (in this case $\frac{1}{3}$), we have the following relation:

$$N = \frac{1}{r^D}$$

where $D = 2$, the dimension of the square. This relation holds for any N and r, and it is also true for a line $(D = 1)$ and a cube $(D = 3)$.

In the Sierpinski Carpet the number of squares left is 8, so $N = 8$ and $r = \frac{1}{3}$. Substituting these values in the above formula, we get

$$8 = \frac{1}{\left(\frac{1}{3}\right)^D}.$$

Taking logarithms of both sides of this equation gives

$$\log(8) = D \log(3)$$

hence

$$D = \frac{\log(8)}{\log(3)} = 1.8928.$$

This number, 1.8928, is called the *fractal dimension* of the Sierpinski Carpet.

Similarly, for the Sierpinski Triangle, the reduction factor is $\frac{1}{2}$ and the number of parts left in at each iteration is 3. Therefore its fractal dimension is $D = \frac{\log 3}{\log 2} = 1.5850$. In both cases the fractal dimension lies between 1, the Euclidean dimension of the line, and 2, the Euclidean dimension of the plane.

* * * * *

THE TETRA® PAK

Michael A.B. Deakin, Monash University

I don't use much milk in my home, but I like always to have some on hand for visitors who take it in their tea or coffee. So what I buy are small portions of long-life (UHT) milk. These used to take the form of small bucket-shaped plastic containers, each holding 18 ml. But recently I've found these replaced by a differently packaged product. The milk now comes in a cardboard tetrahedron with seals on two of its six edges. This new shape is marketed under the brand-name *Tetra® Pak*.

Figure 1 shows a perspective view of a tetrahedron, with four triangular faces, six edges and four vertices. The four triangles are congruent isosceles triangles; in the special case of a regular tetrahedron, all faces are in fact equilateral triangles.

The two thicker edges DA and CB represent the sealed edges of the Tetra® Pak, and are taken to have length a. The other four edges will each be taken to have length b.

Of course, if the tetrahedron were a regular one then we would have

$$a = b$$

but for the Tetra® Pak, as we shall see, this is not the case.

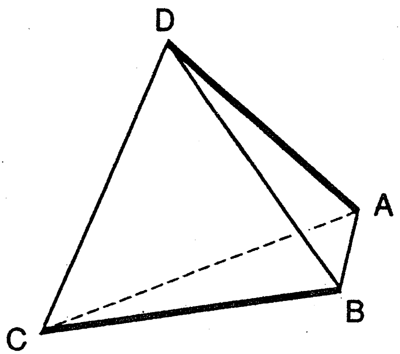

Figure 1

Because of the perspective drawing in Figure 1, DA and CB may appear to have different lengths. However, you may check from a real Tetra® Pak that they really do have the same length!

We now analyse the tetrahedron, starting with the base as shown in Figure 2.

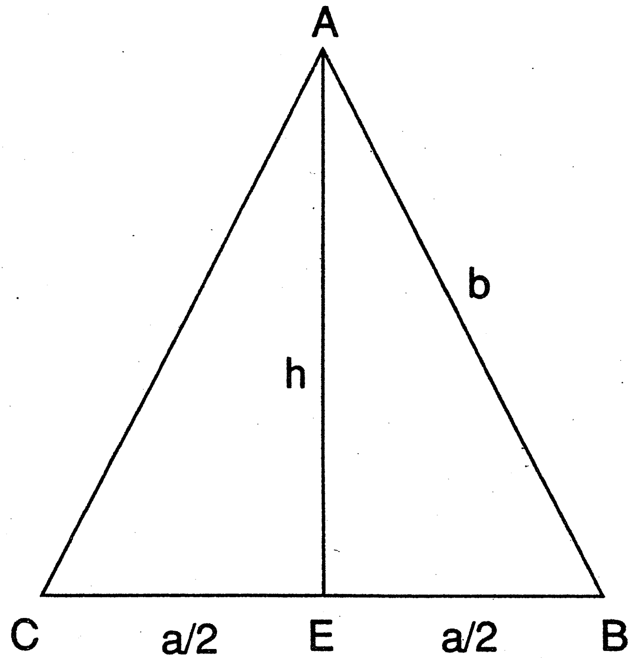

Figure 2

This is an isosceles triangle ABC, and we will place the point E at the midpoint of its base BC. Let the length of the base be a, let the height of the triangle be h, and let the sides AB and AC each have length b. Then, applying Pythagoras' Theorem to the triangle ABE :

$$b^2 = h^2 + (a/2)^2.$$

Now, by symmetry, the fourth vertex of the tetrahedron, D, is directly above a point on the line EA. Call this point F, and let y and z be the lengths of EF and FD respectively. Arguing by symmetry, we can see that DE has the same length, h, as AE, and AD has the same length, a, as BC.

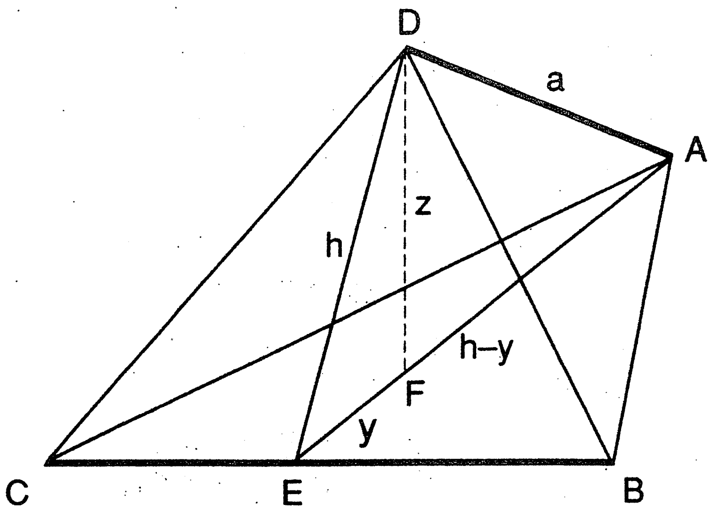

Figure 3

From triangle EDF :

$$y^2 + z^2 = h^2.$$

From triangle AFD :

$$(h-y)^2 + z^2 = a^2.$$

Solving these two equations for y and z is tedious but not particularly difficult. I will spare you the details. The results are

$$y = h - a^2/(2h) \text{ and } z^2 = a^2(4h^2 - a^2)/(4h^2).$$

Let us now find out some lengths, areas and volumes. The area of the base ABC is $ah/2$, as is readily seen. The height of the tetrahedron is z and it is known that the volume of a tetrahedron is

$$(1/3) \times \text{Area of Base} \times \text{Perpendicular Height.}$$

Thus we may use this formula with the value of z given above to find the volume to be $(1/12)a^2\sqrt{4h^2 - a^2}$.

To measure the Tetra® Pak, open it along one of the sealed edges (say DA) and then flatten it carefully out from the other seal. This produces a rectangle of sides a and h. (See Figure 4.) Measurement of the rectangle gives

$$a = h = 5 \text{ cm},$$

so that the rectangle is a square. This may well be deliberate. In any case, we find $b = 5.59$ cm, so the tetrahedron is not regular.

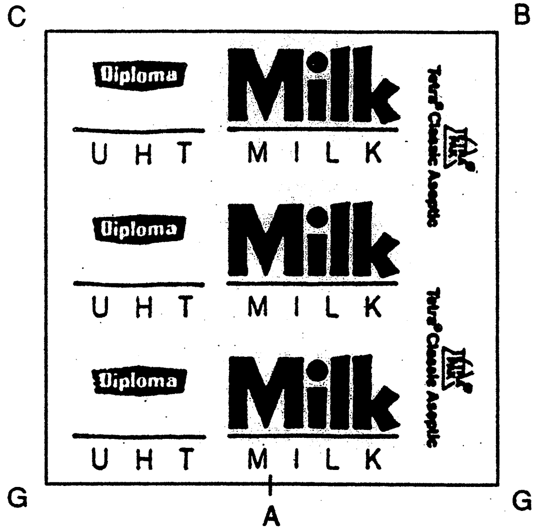

Figure 4

Substitution of the measured values into the formula for the volume yields a volume of 18 ml, which is a little under the claimed 20 ml. However, I measured the amount of milk in a Tetra® Pak and found it was indeed 20 ml. I attribute the extra to a slight rounding out of the faces of the underlying tetrahedron.

The six edges, AB, BC, CA, DA, DB, DC, of a tetrahedron come in three pairs of "opposites". Opposite the side BC is DA and these two sides, although they do not intersect, are nonetheless at right angles, because if at any point on BC we place a line parallel to DA, this line will be perpendicular to BC. This is quite obvious geometrically, but it may also be proved formally, by vector algebra, for example.

Now take G to be the mid-point of DA. The line EG is perpendicular to both BC and DA. (Again this is rather obvious, but a formal proof could be given.) This means that of all the lines that connect points on BC to points on DA, EG is the shortest. It is sometimes referred to as the "distance" between BC and DA.

We may now calculate this distance by applying Pythagoras' Theorem to triangle DGE. The result is $\sqrt{h^2 - a^2/4}$. This works out to be 4.33 cm, which seems about right.

The point G lies at the mid-point of the base of triangle ADC *and* of triangle ADB. This is why in Fig. 4 the two bottom corners of the flattened tetrahedron both correspond to the point G.

To make Tetra® Paks, we take a continuous tube of cardboard and make seals along its length by flattening the tube at equally spaced intervals, making adjacent seals at right angles to each other. These will have to be about 4.3 cm apart and each will have a length of 5 cm. The circumference of the tube will have to make up both sides of the seal and thus we need a tube of circumference 10 cm. More generally, we would have a circumference of $2a$ and a separation of $\sqrt{h^2 - a^2/4}$. Then cut along the seals, and hey presto!

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MATHEMATICAL NURSERY RHYME

Simple Simon met a π man
Going to the fair.
Said Simple Simon to the π man
"You have unusual ware.
The π's I've seen before were round
But, gosh, your π's r^2."

From: *The Surprise Attack in Mathematical Problems*
by L.A. Graham (Dover, 1968).

A LAW OF LOGARITHMS

M. J. Englefield, Monash University

Recently, while working on something quite different, I stumbled across the result

$$a^{\log b} = b^{\log a}. \tag{1}$$

The logarithms may be taken to any base, say c.

To prove Equation (1), we may write

$$\begin{aligned} a^{\log_c b} &= \left(c^{\log_c a}\right)^{\log_c b} \\ &= c^{\log_c a \, \log_c b} \\ &= \left(c^{\log_c b}\right)^{\log_c a} \\ &= b^{\log_c a}. \end{aligned} \tag{2}$$

Here we have used the property that for any positive numbers c and u,

$$c^{\log_c u} = u,$$

and the index law for powers, namely

$$(c^x)^y = (c^y)^x = c^{xy},$$

with $x = \log_c a$, $y = \log_c b$.

Take the logarithm to base a of both sides of Equation (2). This gives

$$\log_c b = \log_c a \, \log_a b \tag{3}$$

since $\log_a a = 1$.

Rearranging Equation (3) produces the change-of-base law.

We may apply Equation (1) to certain problems of antidifferentiation. Consider the problem of antidifferentiating

$$f(x) = 2^{\ln x} \tag{4}$$

where $\ln x = \log_e x$, the natural logarithm of x. Using Equation (1),

$$f(x) = x^{\ln 2}$$

so that the required antiderivative is

$$\begin{aligned} F(x) &= \frac{x^{1+\ln 2}}{1+\ln 2} + c \\ &= \frac{x.x^{\ln 2}}{1+\ln 2} + c \\ &= \frac{x\ 2^{\ln x}}{1+\ln 2} + c. \end{aligned}$$

You may find other applications of Equation (1). I find it strange that this law is not more widely known.

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M. J. Englefield has been in the Department of Mathematics at Monash University since 1965. His publications include the text Mathematical Methods for Engineering and Science Students *and his research interests are in methods of solving differential equations, with applications to theoretical physics.*

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MEANS AND TRIANGLE MEDIANS

K. R. S. Sastry, Addis Ababa, Ethiopia

Given two positive numbers u and v, the three well-known *means* of these numbers are (i) the arithmetic mean $\frac{1}{2}(u+v)$, (ii) the geometric mean $\sqrt{uv}$ and (iii) the harmonic mean $\frac{2uv}{u+v}$. (see article by Ken Evans in *Function*, *Vol.* 15, *Part* 4, pp. 98–106).

A triangle *median* is a line segment between a vertex and the midpoint of the opposite side. For example, one such triangle median is the line segment AD, where D is the mid-point of BC in triangle ABC (see Figure 1).

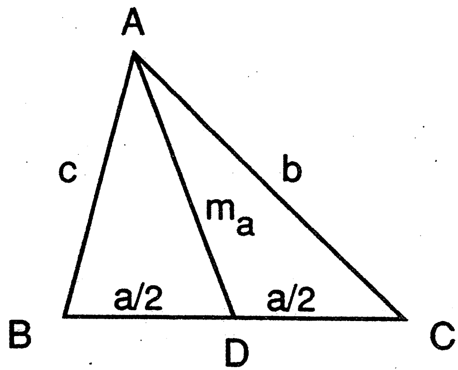

Figure 1

Using standard notation, we let a, b, c respectively denote the lengths of the sides BC, CA and AB. We denote by m_a the length of the triangle median from vertex A, i.e. the length of the line. AD. Analogously we define m_b and m_c as the lengths of the triangle medians from vertices B and C respectively.

In this article, we ask:

Are there triangles in which m_a is the (i) arithmetic mean, or (ii) geometric mean, or (iii) harmonic mean of b and c?

We can derive an expression for m_a in terms of a, b, c by applying the cosine rule. From triangles ADC and ADB we can write

$$b^2 = m_a^2 + \tfrac{a^2}{4} - 2m_a(\tfrac{a}{2})\cos\angle ADC$$

and

$$c^2 = m_a^2 + \tfrac{a^2}{4} - 2m_a(\tfrac{a}{2})\cos\angle ADB.$$

Since $\angle ADC$ and $\angle ADB$ are supplementary angles,

$$\cos\angle ADC + \cos\angle ADB = 0.$$

We add the above equations and solve for $4m_a^2$ to avoid fractions. This yields

$$4m_a^2 = 2b^2 + 2c^2 - a^2. \qquad (1)$$

Likewise, $4m_b^2 = 2c^2 + 2a^2 - b^2$ and $4m_c^2 = 2a^2 + 2b^2 - c^2$.

The Median and the Arithmetic Mean of the Sides

Can the length of the median AD of triangle ABC equal the arithmetic mean of the lengths of sides AB and AC? To find out the answer let us substitute $m_a = \frac{1}{2}(b+c)$ in (1). Then we find that

$$(b+c)^2 = 2b^2 + 2c^2 - a^2.$$

After simplification the equation yields $a = b - c$ if $b > c$ or $a = c - b$ if $c > b$. In either case the triangle degenerates to a line segment. Hence the answer is

Theorem 1. *In a non-degenerate triangle ABC the length of the median AD cannot equal the arithmetic mean of the lengths of the sides AB and AC.*

Of course this also means that $m_b \neq \frac{1}{2}(c+a)$ and $m_c \neq \frac{1}{2}(a+b)$.

The Median and the Geometric Mean of the Two Sides

Is it possible for the median AD to be the geometric mean of the sides AB and AC of triangle ABC? The answer is yes.

Theorem 2. *The median AD will be the geometric mean of the sides AB and AC of triangle ABC if and only if $a = \sqrt{2}(b-c)$ if $b > c$ or $a = \sqrt{2}(c-b)$ if $c > b$.*

Proof. Put $m_a = \sqrt{bc}$ in $4m_a^2 = 2b^2 + 2c^2 - a^2$. Then we get

$$4bc = 2b^2 + 2c^2 - a^2$$

and therefore

$$a^2 = 2(b - c)^2.$$

So

$$a = \sqrt{2}(b - c) \text{ if } b > c;$$

otherwise

$$a = \sqrt{2}(c - b).$$

Conversely, from

$$a^2 = 2(b - c)^2$$

we get

$$m_a = \sqrt{bc}.$$

Hence the theorem follows.

Remark: With the constraint $a = \sqrt{2}(b - c)$ it is no longer certain that such lengths a, b, c form a triangle. Note that $b > c$ implies $a + b > c$, which is necessary if the lengths a, b and c are to form a triangle (See Figure 1). We must also have $b+c > a$. That is, $b+c > \sqrt{2}(b-c)$. Hence b and c should be such that $c < b < (3 + 2\sqrt{2})c$. For example, if $c = 1$ then $1 < b < 3 + 2\sqrt{2}$. If we choose $b = 2$ then $a = \sqrt{2}(b - c) = \sqrt{2}$. This yields $4m_a^2 = 2b^2 + 2c^2 - a^2 = 8$, $m_a = \sqrt{2}$ which equals $\sqrt{bc}$ for $b = 2$, $c = 1$.

The Median and the Harmonic Mean of Two Sides

Theorem 3. *The median AD of a triangle ABC will be the harmonic mean of the sides AB and AC if and only if $a = \frac{b-c}{b+c}\sqrt{2(b^2 + 4bc + c^2)}$ if $b > c$ or $a = \frac{c-b}{b+c}\sqrt{2(b^2 + 4bc + c^2)}$ if $c > b$.*

Proof. Put $m_a = \frac{2bc}{b+c}$ in (1) and solve for a.

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K. R. S. Sastry, an expatriate mathematics teacher with the Ethiopian Education Ministry, earned his mathematics degrees from Mysore University, India. A contributor to several mathematics journals, his hobby is photography.

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HISTORY OF MATHEMATICS

FERMAT'S LAST THEOREM

John Stillwell, Monash University

Fermat's last theorem is the most famous theorem in mathematics and was, until recently, its most famous open problem. In the 350 years since the theorem was first stated, some of the greatest mathematicians have attempted to prove it, all without success until the proof of Andrew Wiles in June 1993. At the time of writing, this proof still needs some work, but seems essentially correct. While we wait for the full story to emerge, it may be of interest to survey some of the classic attempts to prove the theorem – correct proofs of special cases, and others simply incorrect – together with some of the recent research on which Wiles based his proof.

Pierre de Fermat (1601-1665) conjectured the theorem around 1637, while reflecting on problems about sums of squares. Since ancient times, people have known examples of whole number squares that are sums of two whole number squares, for instance

$$\begin{aligned} 5^2 &= 3^2+4^2 \\ 13^2 &= 5^2+12^2 \\ 17^2 &= 8^2+15^2. \end{aligned}$$

Around 200 AD, the Greek mathematician Diophantus showed, in his *Arithmetica*, that any whole number square could be split into two *fractional* squares. For example,

$$4^2 = \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2$$

Fermat was a keen student of the *Arithmetica*, and jotted many of his thoughts on number theory in the margin of his copy. Next to Diophantus' explanation of how to split 4^2 into two squares, Fermat wrote:

> It is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or in general any power higher than second into powers of like degree; I have discovered a truly marvellous proof of this which however this margin is too small to contain.

In modern notation, Fermat's statement is that the equation

$$a^n + b^n = c^n$$

has no solution in positive integers a, b, c when n is an integer greater than 2.

The statement became known as Fermat's *last* theorem after other theorems claimed by Fermat were eventually proved (or, in a few cases, disproved). What makes the last theorem so difficult is its enormous scope: it has to be proved for the infinitely many values 3, 4, 5, 6, ... of n, and even proofs for particular values of n are difficult. Until recently, in fact, almost all progress has been by hand-to-hand fighting against individual values of n. Fermat himself left a proof for $n = 4$. In 1770 Euler gave a proof for $n = 3$ (not quite complete, but capable of being patched up). Legendre and Dirichlet disposed of $n = 5$ between them in the 1820s. Dirichlet got $n = 14$ from an unsuccessful attempt to do $n = 7$, and Lamé finally settled $n = 7$ in 1840.

Then in 1847 Lamé announced a proof for all n. This created a sensation at the Paris Academy, though not quite the one that Lamé expected. Almost immediately, a mistake was found in his proof! Meanwhile, in Germany, Kummer had encountered the same difficulty, and was developing a way round it. The "ideal numbers" he invented for this purpose were a great success – elsewhere in number theory – but they turned out to be only partially successful with Fermat's last theorem. Kummer was able to deal with many values of n, but not all. In fact, it is still unknown whether his method can cope with infinitely many values of n.

All the same, Kummer was far more successful than any other mathematician who had attacked the problem, and his work put Fermat's last theorem on the map. In 1908 the hunt for a proof intensified with the offer of a prize of 100,000 marks, left in the will of mathematician Paul Wolfskehl. The value of the Wolfskehl prize was wiped out by the German hyperinflation in the 1920s, but in its early years it attracted thousands of entries – all of them wrong. Most of the entrants were amateur mathematicians, encouraged by the fact that Fermat himself was an amateur, and hoping to discover a lost proof by Fermat's own methods.

Alas, it is wrong for amateurs (and even professionals) to think Fermat's last theorem is that easy. Fermat was an amateur, yes, but a master of all the mathematics then known, and able to compete with the top mathematicians of his day (most of whom were also amateurs, e.g. Descartes and Pascal). And anyway, did Fermat have a proof? Probably not. His marginal note was written when he was a beginner in number theory, and was most likely based on a mistake. He never repeated the claim, except for the cases $n = 3$ and $n = 4$. We know his proof for $n = 4$ is correct, and perhaps he could have done $n = 3$ the way Euler did, but that was probably his limit.

Be that as it may, there was virtually no progress on the theorem between Kummer and the 1980s, when a wild idea by the German mathematician Gerhardt Frey reduced Fermat's last theorem from a question about n^{th} powers to a question about squares and cubes. Frey said: suppose (contrary to Fermat's last theorem) that there *are* positive integers a, b, c such that

$$a^n + b^n = c^n,$$

and consider the curve with equation

$$y^2 = x(x - a^n)(x + c^n).$$

Frey guessed that the unlikely numbers a^n and c^n would give this curve an unlikely property, called "nonmodularity" in the trade. His guess was proved by the American Ken Ribet in 1987. Then to prove Fermat's last theorem it only remained to prove that nonmodularity is in fact impossible. This is what Andrew Wiles did; it took him seven years and about 250 pages, using some of the most sophisticated methods of modern mathematics. Even if the proof is simplified in the future, there will probably never be a margin large enough to contain it!

No doubt it will also be beyond the scope of *Function*, but it is possible to give an idea of the classical results of Fermat, Euler and Kummer.

Pierre de Fermat (1601 - 1665)

As mentioned above, Fermat proved the case $n = 4$, that the sum of two fourth powers cannot be a fourth power. In fact, he proved that the sum of two fourth powers cannot even be a square. His method was one of his own invention called "infinite descent".

Supposing there are positive integers a_1, b_1, c_1 such that

$$a_1{}^4 + b_1{}^4 = c_1{}^2,$$

Fermat finds *smaller* positive integers a_2, b_2, c_2 such that

$$a_2{}^4 + b_2{}^4 = c_2{}^2.$$

this is the first step of the "descent", which can be continued indefinitely. applying the same argument to a_2, b_2, c_2 gives smaller positive integers a_3, b_3, c_3 with the same property, and so on. However, it is *impossible* to find smaller positive integers indefinitely, so we have a contradiction. No such a_1, b_1, c_1 exist.

This is the logic of Fermat's argument. The mathematical part is to find a way to "descend" from a_1, b_1, c_1 to a_2, b_2, c_2. Fermat used an ancient result about sums of squares (apparently known to the Babylonians around 2000 BC) which goes as follows.

If x, y, z are integers without common divisor, and

$$x^2 + y^2 = z^2$$

with x even, then there exist integers u, v such that

$$x = 2uv, \quad y = u^2 - v^2, \quad z = u^2 + v^2.$$

For example, if $x = 4$, $y = 3$ and $z = 5$, take $u = 2$ and $v = 1$.

Applying this result to

$$({a_1}^2)^2 + ({b_1}^2)^2 = {c_1}^2$$

after removing any common divisors, we can conclude that

$$\begin{aligned} {a_1}^2 &= 2uv, \\ {b_1}^2 &= u^2 - v^2, \\ c_1 &= u^2 + v^2. \end{aligned}$$

But then $v^2 + {b_1}^2 = u^2$, and so we also have integers u_1, v_1 with

$$\begin{aligned} v &= 2u_1v_1, \\ b_1 &= {u_1}^2 - {v_1}^2, \\ u &= {u_1}^2 + {v_1}^2 \end{aligned}$$

It follows in particular that

$${a_1}^2 = 2uv = 4({u_1}^2 + {v_1}^2)u_1v_1.$$

It can be checked the factors ${u_1}^2 + {v_1}^2$, u_1 and v_1 have no common divisor. Since their product is the square ${a_1}^2/4$, they must themselves be squares, say

$$u_1 = {a_2}^2, \quad v_1 = {b_2}^2, \quad {u_1}^2 + {v_1}^2 = {c_2}^2.$$

This gives

$$a_2^4 + b_2^4 = u_1^2 + v_1^2 = c_2^2.$$

Finally, it can be checked that a_2, b_2, c_2 are smaller than a_1, b_1, c_1. As already mentioned, this leads to an infinite descent, which is impossible. Hence there is no triple a_1, b_1, c_1 with the fourth powers of the first two adding up to the square of the third. In particular, there are no positive integer fourth powers whose sum is a fourth power.

Leonhard Euler (1707 - 1783)

If you thought that Fermat's proof for $n = 4$ was tricky, I have bad news for you. The $n = 4$ case is by far the easiest. No other case has yet been proved by reasoning of the kind used by Fermat, using only simple properties of the positive integers. All known proofs for $n = 3$, $n = 5$, etc. use more sophisticated concepts, in particular they use *irrational numbers.* The idea of using irrational numbers to prove results about integers appeared around 1770; Euler used it to give the first proof of Fermat's last theorem for $n = 3$. His proof is too complicated to summarise here, but I can tell you the key idea.

If a, b, c are positive integers such that

$$a^3 + b^3 = c^3,$$

then

$$a^3 = c^3 - b^3 = (c-b)(c^2 + cb + b^2),$$

and the right hand side can be factorised completely to

$$a^3 = (c-b)(c-\zeta b)(c-\zeta^2 b)$$

using the irrational (and imaginary) number $\zeta = (1+\sqrt{-3})/2$.

But so what? The numbers $c - \zeta b$ and $c - \zeta^2 b$ are not integers, so how can they help? Euler's bold and successful idea was to treat them *as if* they were integers, and to look for a contradiction in the wider world of "integers" of the form $r + \zeta s + \zeta^2 t$, where r, s, t are ordinary integers. In particular, he claimed that since $c-b$, $c-\zeta b$, $c-\zeta^2 b$ have product equal to a cube, they must themselves be cubes. Euler's justification for this claim was rather weak, but amazingly, he was right! Later mathematicians were able to explain clearly why Euler's "integers" behave like ordinary integers, and hence put the rest of his argument (some complicated but conventional algebra) on a solid foundation.

Ernst Eduard Kummer (1810 - 1893)

The numbers $r+\zeta s+\zeta^2 t$ are examples of what are now called *algebraic integers.* The algebraic integers relevant to Fermat's last theorem are of the form $a_0+\zeta_n a_1+\cdots+\zeta_n{}^{n-1}a_{n-1}$ where $\zeta_n=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$, where i denotes the imaginary number $\sqrt{-1}$. (Euler's ζ is ζ_3.) They enable us to factorise the expression c^n-b^n arising from a hypothetical counterexample to Fermat's last theorem, as follows:

$$a^n=c^n-b^n=(c-b)(c-\zeta_n b)(c-\zeta_n{}^2 b)\cdots(c-\zeta_n{}^{n-1}b).$$

It is then tempting to generalise Euler's argument and claim that each factor on the right hand side must be an n^{th} power. Unfortunately, what works for $n=3$ does not work for all values of n. Euler got away with his argument for $n=3$ only because there is a decent concept of *prime factorisation* in the algebraic integers $r+\zeta_3 s+\zeta_3{}^2 t$. Each such algebraic integer has a *unique* prime factorisation, and this is the real reason that factors of a cube are also cubes (provided they have no common factor).

Lamé's "proof" of Fermat's last theorem attempted to generalise Euler's idea, assuming that the above factors of the n^{th} power a^n are also n^{th} powers. This assumes unique prime factorisation of the ζ_n integers for all values of n, which alas is not the case. Kummer showed that it first breaks down for $n=23$. Not deterred by this, he created an even more general concept of "integer", and "prime", for which unique prime factorisation is valid. Kummer's integers can be used to settle many case of Fermat's last theorem, but not all.

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* * * * *

AN ALGORITHM FOR MAGIC SQUARES

Cristina Varsavsky, Monash University

A magic square is a square matrix whose columns, rows and diagonals all have the same sum. Magic squares have fascinated the great minds of the world and many mathematicians have amused themselves with them. They have been found in paintings, on vases, fortune bowls and seals of the ancient world, and they were also used by Arabian astrologers to predict the future. One of the world's famous engravings, Albrecht Dürer's *Melancholia* (1514), depicts, among other mathematical concepts, the magic square shown in Figure 1.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 1

Today magic squares are still an interesting diversion to exercise our minds. How can we create one? How should the different numbers relate to each other? Let us explore this with a 3×3 square with generic entries a, b, c, d, e, f, g, h and i (see Figure 2).

a	b	c
d	e	f
g	h	i

Figure 2

The sum of all rows, columns and diagonals should give the same number, say k. Therefore the following equations must be satisfied.

$$\begin{aligned}
a+b+c &= k\\
d+e+f &= k\\
g+h+i &= k\\
a+d+g &= k\\
b+e+h &= k\\
c+f+i &= k\\
a+e+i &= k\\
c+e+g &= k
\end{aligned}$$

So any solution to this system of eight equations and 10 unknowns would result in a magic square. How many solutions are there? I asked the computer algebra system *MAPLE* to solve it for me, and received the answer shown in Figure 3, where solutions are given in terms of b, i and k (I asked *MAPLE* to do this), meaning that for each choice of the triplet b, i and k, we get a magic square by plugging those values into each expression.

$-i+\frac{2}{3}k$	b	$-b+i+\frac{1}{3}k$
$-b+2i$	$\frac{1}{3}k$	$b-2i+\frac{2}{3}k$
$b+\frac{1}{3}k-i$	$-b+\frac{2}{3}k$	i

Figure 3

Let us inspect the solutions shown in Figure 3. Firstly, the middle square is always occupied by a third of the total sum, also called the *magic constant*. Since all entries must be integers, it follows that the magic constant must be a multiple of three. Secondly, although any integer choices of b and i will produce integer entries, not all of them will be positive. We need b and i such that

$$-i+\tfrac{2}{3}k>0 \quad \text{or} \quad i<\tfrac{2}{3}k \tag{1}$$

$$-b+2i>0 \quad \text{or} \quad b<2i \tag{2}$$

$$b+\tfrac{1}{3}k-i>0 \quad \text{or} \quad i-b<\tfrac{1}{3}k \tag{3}$$

$$-b+\tfrac{2}{3}k>0 \quad \text{or} \quad b<\tfrac{2}{3}k \tag{4}$$

$$-b+i+\tfrac{1}{3}k>0 \quad \text{or} \quad b-i<\tfrac{1}{3}k \tag{5}$$

$$b-2i+\tfrac{2}{3}k>0 \quad \text{or} \quad 2i-b<\tfrac{2}{3}k \tag{6}$$

Let us find a square with a magic constant $k=30$ (which of course is a multiple of three). According to (1) and (4), b and i must be less than 20. Conditions (3) and (5) indicate that the difference between b and i must be less than 10. Now, to satisfy (2), one possible choice is $b=3$ and $i=7$, which also satisfies the remaining condition (6). Replacing these values in the expressions, we get the magic square depicted in Figure 4.

13	3	14
11	10	9
6	17	7

Figure 4

Usually, magic squares are more interesting than this. For example, the magic square of the *Melancholia* engraving has some other special properties: not only do the columns, rows and diagonals add to the magic constant 34, but also the four corners of the square, and opposite pairs (5, 9, 8, 12 and 3, 2, 15, 14), to name a few. Also, the middle numbers in the bottom row represent the year, 1514, in which the engraving was made.

A well-known method, called the *De La Louvère Procedure*[1], produces magic squares of an odd number of columns, n, with the special feature that a set of n^2 consecutive numbers is used to fill up the square. Here is the algorithm:

Algorithm De la Louvère

1. Start with any number x and place it in the middle position of the first row.
2. Increment x by one unit. This becomes the next entry.
3. To find the position for this entry, move one position up and one to the right. If you fall outside the square on its top, move down the column till you reach the bottom of the square. If you fall outside the square on its right side, move to the left till you find the left side of the square.
4. Repeat steps 2 and 3 until you complete n entries.
5. Increment x by one and place it in the position immediately below the current position.
6. Repeat steps 2, 3, 4 and 5 until you fill up the square.

[1]Antoine de la Louvère was a Jesuit, born in Languedoc, France, in 1600.

Figure 5 shows the first few steps for a 5×5 square starting with $x = 11$, and Figure 6 displays the completed square.

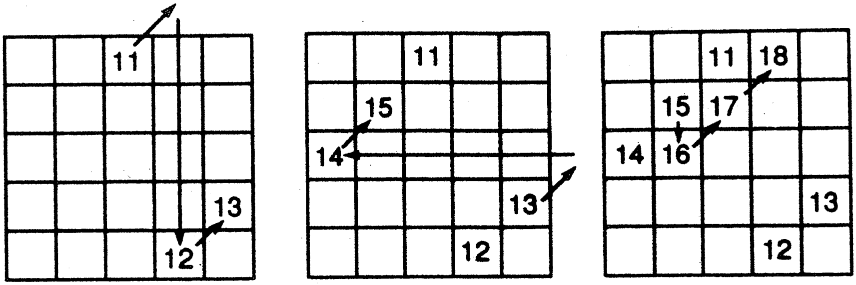

Figure 5

27	34	11	18	25
33	15	17	24	26
14	16	23	30	32
20	22	29	31	13
21	28	35	12	19

Figure 6

It is possible to prove that the square produced through this method will have a magic constant calculated according to the formula

$$\text{magic constant} = \frac{n^3 - n}{2} + nx$$

Here is a program in *QuickBasic* which implements this algorithm:

```
REM De la Louvere Algorithm for odd magic squares

REM you may need to change the following statement
DIM square(25, 25)

INPUT " Enter an odd number, the size of the magic square: ", size
INPUT " Enter the starting number: ", start
```

```
    number = start: i = 1: j = (size + 1)/2

 WHILE number  <=  size * size + start - 1
        square(i, j) = number
        number = number + 1

    FOR k = 2 TO size

        i = i - 1:  j = j + 1
        IF i = 0 THEN
             i = size
        ELSEIF j > size THEN
             j = 1
        ENDIF
        square(i, j) = number
        number = number + 1

    NEXT

        i = i + 1

    WEND

    REM Printing the Magic Square

    PRINT : PRINT
    FOR i = 1 TO size
        FOR j = 1 TO size
             PRINT square(i, j): TAB(5 * j);
    NEXT j
    PRINT
    PRINT
 NEXT i

 PRINT "Columns, rows and diagonals add up to"; (size^3 - size) /
 2 + size * start
 END
```

Running this program for a 9×9 square and initial value 11 produces the following output:

57	68	79	90	11	22	33	44	55
67	78	89	19	21	32	43	54	56
77	88	18	20	31	42	53	64	66
87	17	28	30	41	52	63	65	76
16	27	29	40	51	62	73	75	86
26	37	39	50	61	72	74	85	15
36	38	49	60	71	82	84	14	25
46	48	59	70	81	83	13	24	35
47	58	69	80	91	12	23	34	45

```
Columns, rows and diagonals add up to 459
```

Exercise: Here is an algorithm to produce magic squares with an even number of columns. Write a computer program that implements this algorithm.

Algorithm for Magic Squares of Size 4

1. Mark the two main diagonals, say with the letter A (Figure 7).
2. Select a starting number x ($x = 4$ in Figure 8), and move to the upper left corner of the square.
3. Move to the next position to the right, and increment x by 1. If that position is not already marked, enter the current value of x.
4. Repeat step 3 until the row is completed, then move to the left of the next row.
5. Repeat steps 3 and 4 until all rows are completed.
6. Fill the marked positions using essentially the same rule as in 3 and 4. Start at the top left corner, but now fill in the marked positions and skip the ones with numbers in it. The value of x is now *decreased* by 1 each time you move one position to the right; in the bottom right square you should have the original value of x (see Figure 8).

A			A
	A	A	
	A	A	
A			A

Figure 7

A	5	6	A
8	A	A	11
12	A	A	15
A	17	18	A

19	5	6	16
8	14	13	11
12	10	9	15
7	17	18	4

Figure 8

A similar procedure could be followed to form an 8×8 magic square. Divide the square into four squares of size 4, mark the diagonals of the four 4×4 squares, and fill it up proceeding through steps 2 to 6.

Have fun!

* * * * *

THE SIERSPINSKI CARPET

Here is the *QuickBasic* program used to produce the Sierspinski Carpet illustration on the front cover. The program has the same structure as that given on pages 22–23 of the previous issue of *Function.*

```
REM SIERPINSKI CARPET
SCREEN 9
COLOR 11
DIM x1(10), x2(10), x3(10), x4(10), y1(10),
y2(10), y3(10), y4(10)

INPUT "Enter the number of iterations [1-10] : ", number

REM initialisation of variables
    horz = 150: vert = 30
    x1(number) = 0:      y1(number) = 0
    x2(number) = 300:    y2(number) = 0
    x3(number) = 300:    y3(number) = 300
    x4(number) = 0:      y4(number) = 300
    xdisp(1) = 0: xdisp(2) = 100: xdisp(3) = 200:
    xdisp(4) = 200:
    xdisp(5) = 200: xdisp(6) = 100: xdisp(7) = 0:
    xdisp(8) = 0
    ydisp(1) = 200: ydisp(2) = 200: ydisp(3) = 200:
    ydisp(4) = 100
    ydisp(5) = 0: ydisp(6) = 0: ydisp(7) = 0:
    ydisp(8) = 100

GOSUB Drawing
END

REM Next iteration
Iterate:
    number = number - 1
    trans = 1
    GOSUB Transform
    trans = 2
    GOSUB Transform
```

```
    trans = 3
    GOSUB Transform
    trans = 4
    GOSUB Transform
    trans = 5
    GOSUB Transform
    trans = 6
    GOSUB Transform
    trans = 7
    GOSUB Transform
    trans = 8
    GOSUB Transform
    number = number + 1
    RETURN

REM Basic step: transformation of square in terms
of the previous one

Transform:
    x1(number) = x1(number + 1)/3 + xdisp(trans)
    y1(number) = y1(number + 1)/3 + ydisp(trans)
    x2(number) = x2(number + 1)/3 + xdisp(trans)
    y2(number) = y2(number + 1)/3 + ydisp(trans)
    x3(number) = x3(number + 1)/3 + xdisp(trans)
    y3(number) = y3(number + 1)/3 + ydisp(trans)
    x4(number) = x4(number + 1)/3 + xdisp(trans)
    y4(number) = y4(number + 1)/3 + ydisp(trans)

REM Drawing of square at the last iteration

Drawing:
    IF number > 1 GOTO Iterate
    LINE (horz + x1(1), vert + y1(1)) - (horz + x2(1),
    vert + y2(1))
    LINE - (horz + x3(1), vert + y3(1))
    LINE - (horz + x4(1), vert + y4(1))
    LINE - (horz + x1(1), vert + y1(1))
    RETURN
```

* * * * *

PROBLEM CORNER

SOLUTIONS

We are holding over solutions of the problems set last time until our next issue. We invite readers to provide us with solutions to those problems or with alternative solutions or corrections to solutions published in previous issues of *Function*.

PROBLEMS

Our first three problems are motivated by the discussion in K. R. S. Sastry's article on *Means and Triangle Medians* on pp. 43–45 of this issue, and use the same standard notation.

PROBLEM 18.2.1

Is there an isosceles triangle ABC with $a = c$ in which $m_a = a$? (Note that m_a would then trivially be equal to the arithmetic, geometric and harmonic means of a and c.)

PROBLEM 18.2.2

Is there an isosceles triangle ABC for which $m_a = \frac{1}{3}(a + b + c)$, the arithmetic mean of the lengths of all three sides?

PROBLEM 18.2.3

Under what circumstances (if any) is it possible to have a triangle ABC in which $m_a = \sqrt{bc}$ and $m_b = \sqrt{ca}$?

PROBLEM 18.2.4 (R. D. Coote, Katoomba High School, N.S.W.)

Express each integer from 1 to 100 in terms of an equation involving all four digits 1, 9, 9, 4 (in *that* order) and any other mathematical symbols. Note: 100 equations are needed!

Example: $1 = 1 + (9 - 9) \times 4$

* * * * *

THE TELECOM 1994
AUSTRALIAN MATHEMATICAL OLYMPIAD

The contest was held in Australian schools on February 8 and 9. On either day students had to sit a paper consisting of four problems, for which they were given four hours.

Paper 1

1. Let ABC be a triangle and M and N points on BC such that $BM = MN = NC$. A line parallel to AC meets lines AB, AM and AN in points D, E and F respectively. Show that $EF = 3DE$.

2. Prove that for every integer x, the number
$$\tfrac{1}{5}x^5 + \tfrac{1}{3}x^3 + \tfrac{7}{15}x$$
is an integer.

3. Let ABC be a triangle with side lengths being integers and AB and AC being relatively prime. Let the tangent at A to the circumcircle of ABC meet BC produced at D. Prove that both AD and CD are rational, but that neither is an integer.

4. Determine all functions f, defined for all rational numbers and having real values, such that
$$f(x+y) = f(x) + f(y) + 2xy.$$

Paper 2

5. Let q be an arbitrary positive real number and a_n, $n = 1, 2, \ldots$, be real numbers such that $a_0 = 1$, $a_1 = 1 + q$, and for all positive integers k the following equations are satisfied:

   (i) $\frac{a_{2k-1}}{a_{2k-2}} = \frac{a_{2k}}{a_{2k-1}}$,

   (ii) $a_{2k} - a_{2k-1} = a_{2k+1} - a_{2k}$.

   Show that for each q as given one can find a positive integer N such that $a_n > 1994$ for all $n > N$.

6. Let n be a positive integer. Prove that both $2n+1$ and $3n+1$ are perfect squares if and only if $n+1$ is both the sum of two successive perfect squares and the sum of a perfect square and twice the succeeding perfect square.

7. Students from 13 different countries participated in the Fifth Asian Pacific Mathematics Olympiad (1993). These students belonged to 5 different age groups, namely 14, 15, 16, 17 and 18. Prove that there were at least 9 participants in that competition, each of whom had more fellow participants in his or her age group than fellow participants from his or her own country.

8. Let $ABCD$ be a parallelogram, E a point on AB and F a point on CD. Let AF intersect ED in G and EC intersect FB in H. Further, let GH produced intersect AD in L and BC in M.
Prove that $DL = BM$.

There were 98 entrants. Gold Certificates were received by:

William Hawkins (year 12), Canberra Grammar School, ACT

Akshay Venkatesh (12), Scotch College, Western Australia

Ren Hou (12), North Sydney Boys' High School, NSW

Nigel Tao (11), Westminster School, South Australia

Andrew Rogers (12), Scotch College, Victoria

Chaitanya Rao (12), Melbourne Church of England Grammar School, Victoria

Anthony Wirth (12), Melbourne Church of England Grammar School, Victoria

John Ho (12), James Ruse Agricultural High School, NSW

James Lefevre (12), Launceston College, Tasmania.

Congratulations to all! Twenty-eight students, including all Gold Certificate winners, were invited to represent Australia at the Sixth Asian Pacific Mathematics Olympiad. Students from thirteen countries of the Asia-Pacific Region take part in this competition, which started in 1989.

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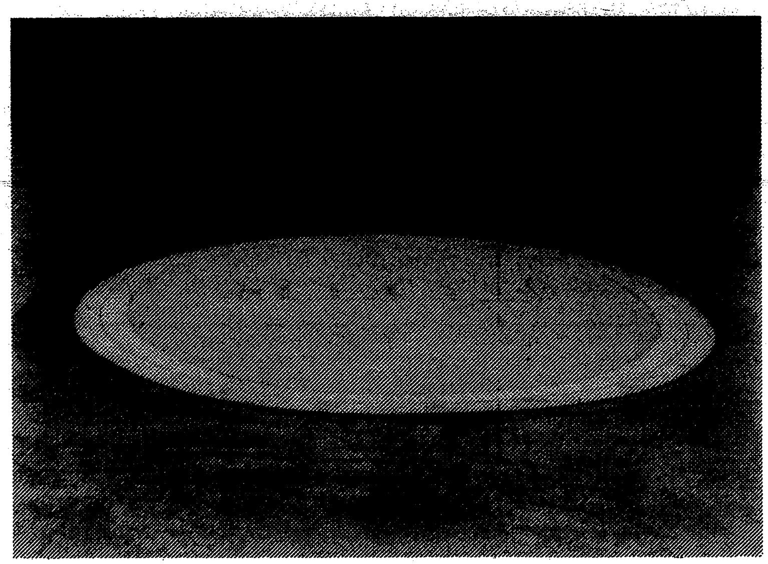
LETTER TO THE EDITOR

Another Unusual Sundial

I was interested to read your account of the Monash Sundial (*Function, Vol.* 17, *Part* 4). Recently I was in Montpellier (France) and came across the sundial pictured below. The French describe it as an analemmic dial. It is in the shape of an ellipse (major axis 536 cm, minor axis 388 cm) and the distances from the "June line" to each of the other lines in turn are 12 cm, 44 cm, 81 cm, 119 cm, 150 cm and 167 cm. One stands at the appropriate position for the time of year and one's shadow indicates solar time.

Peter Ransom
12 Annaside Mews, Leadgate
Consett, Co. Durham, England.

[*The writer is President of the British Sundial Society. Eds.*]



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