Function

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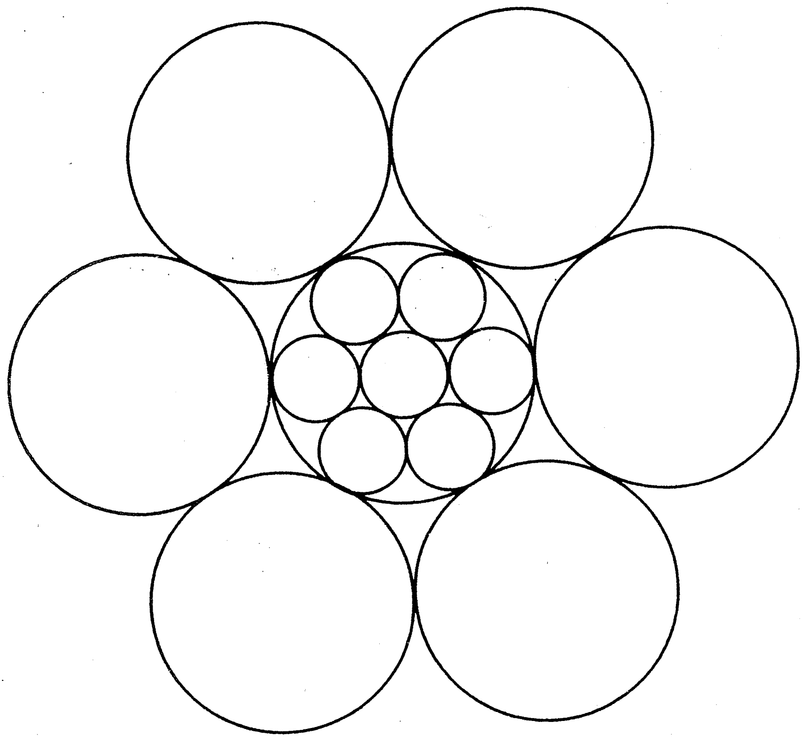

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FUNCTION is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. FUNCTION is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

FUNCTION deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of FUNCTION include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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*$8.50 for *bona fide* secondary or tertiary students.

EDITORIAL

Welcome to our last issue of *Function* for 1994.

This issue has a strong geometric flavour. In a longer than usual front cover article, "Chains of Circles", Bert Bolton and Bill Boundy examine the geometry arising out of the patterns of farms around towns. This leads to a concept known as inversion, which turns out to be a very useful technique for investigating certain geometric problems. The well-known theorem of Pythagoras is examined in a new light in K Sastry's article on n-gonal numbers, "The Pythagorean Theorem: A Pythagorean Generalization". In our regular History column, we take a look at a graph arising from a mathematical model of the transmission of malaria which turns out to be helpful in developing malaria control programs. In our column on Computers and Computing, we explain how you can generate stunning fractal images on a computer screen with the aid of random numbers and some simple transformations.

Also in this issue are the results of this year's Asian Pacific Mathematics Olympiad and International Mathematical Olympiad in which the Australian teams distinguished themselves, and of course there is our regular Problems section.

This year we have introduced a number of changes to *Function*. We hope that by doing this we have made *Function* more interesting and appealing. However, the only way we can be sure is if our readers respond. We are always pleased to hear what readers think of *Function*, so don't hesitate to drop us a line!

To those readers who are nearing exams, especially those who are approaching the end of their final year of school, we offer our best wishes for your future work or studies, and we hope to renew your acquaintance in 1995.

* * * * *

THE FRONT COVER
Chains of Circles

Bert Bolton, University of Melbourne,
and Bill Boundy, SA

The front cover shows a central circle surrounded by a ring of six circles all with a common radius. Each circle in the ring touches its two neighbours and the inner circle. A further outside ring of six touching circles is also shown. The origin of this and other similar diagrams lies in an analysis of the geography of farms which surrounded a walled, fortified city. The mathematical analysis involves ideas of scaling, inversion and symmetry which are straightforward to apply in this case but are powerful enough to be used in more complicated situations.

In the early nineteenth century before railways were started, a German geographer noted that the farms in his native countryside of the North German plain were arranged in a pattern of rings and circles surrounding a town or city. His name was Johann Heinrich von Thünen (1806-1850); the nearest pronunciation of his name is "Toonen". He wrote a book called *The Isolated State* in which he said: "If one assumed that in a province of about 60 km diameter, a big town lay at the centre and that the farms of this province could only send their products to this town, and that the agriculture in the district had attained the highest level of cultivation, then one could assume that four types of farming systems would exist around this town. ... Fairly sharply differentiated concentric rings or belts will form around the town, each with its own particular staple product". Thünen's book is in German and translations have been made, but they are not well known. The same pattern of rings of farms is known round the city states on the North Italian Lombardy plain. We will not follow Thünen's development of his pattern of rings which dealt with the economics of the town, but look only at the geometry of patterns of rings of circles surrounding the town.

We will reduce the real geographical situation into an ideal problem. The central town is a circle; the farms are also circles and for simplicity at first, we take the radii of the town and the farms to be the same. We then have the central part of the pattern on the front cover. It will be noticed that there is some spare land round the town that is not used for farming. If we were "Greenies" we might allow this spare land to be left as a relic of the original forest, but that is not part of the geometry. First we must recognise that the geometrical problem is one of packing. The farms need to be as big as possible and we expect each to touch its two neighbours. We know a similar pattern to this and we pause to look at it in Figure 1. Honey bees store the honey for their winter feed in beeswax cells, which they create and build into a pattern of hexagonal tunnels called honeycombs. Figure 1 can be taken as a cross-section through these tunnels. The start of the pattern is the hexagon marked 0; the first ring is placed round it and there are just 6 hexagons marked 1. At this stage we want to encourage readers to draw their own patterns. Even a sketch helps, but it needs only a pair of compasses and the fact that the side of the inscribed regular hexagon in a circle has the same length as its radius. A good radius to choose for a diagram is 1 centimetre. Another way of looking at a hexagon is to note that it is made up of six equilateral triangles which fit into a circle such that its radius is equal to the side of the hexagon.

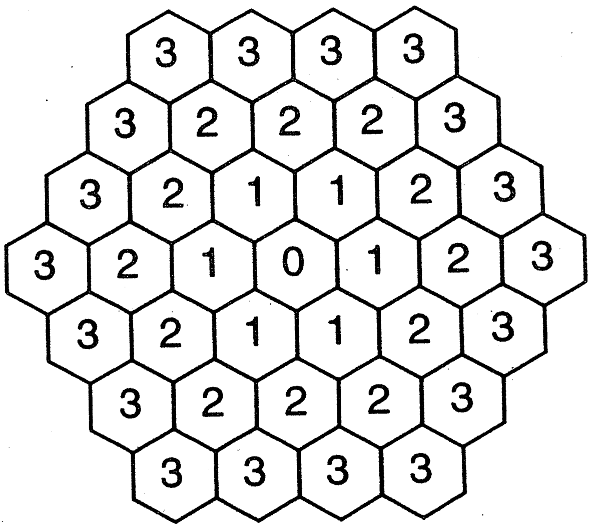

Figure 1

The growth of the beeswax hexagonal cells continues with the ring marked 2. There are 12 such hexagons. The pattern continues. It is tempting to guess that the ring of hexagons marked n has $6n$ hexagons. We can see that this is so by noting that the pattern at every ring maintains the 6-fold symmetry, and we need only discuss what happens inside one-sixth of the whole pattern. To help with this we have outlined in Figure 2 a sixty-degree sector in dotted lines. If the rings are marked $1, 2, \ldots, n, n+1, \ldots$ then the $(n+1)$th ring receives one more hexagon per sector than was found on the nth ring. Note in passing how efficient the bees' pattern is; there are no spare gaps. All the space is used either for the wax walls or for containing honey. Is it a fair question to ask if the bees "know" geometry?

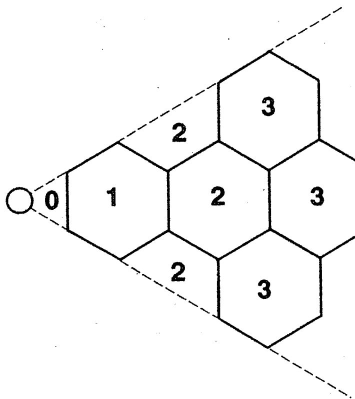

Figure 2

No, the bees don't know geometry. The hexagonal shape of the wax cells is achieved by the bees working to make cells for the honey and while each bee is working by and for itself, it is in competition with its neighbours for space and the hexagonal cells are the outcome. A discussion of this is given in the book *Growth and Form* by D'Arcy Wentworth Thompson (1917); this book applies mathematics to many biological problems.

Let us return to the front cover with this experience of the hexagons in mind. The first ring of 6 circles (farms) defines the 6-fold symmetry. On the front cover we have drawn an outer circle touching all the circles in the first ring; it is called the *circumscribing* circle. We will take the common radius of the town and the first farms as unity and then the radius of the circumscribing circle is 3. We can now see one way at least to make progress to the second ring; treat the circumscribing circle of radius 3 as a new "town" and then the second ring has 6 "farms" of radius 3. We are now letting the geometry take over from the geography. There may have been more than six farms in each ring and we are strictly exploring the pattern when we have restricted the number of farms to six and allowed them to touch three neighbours.

The move from ring 1 to ring 2 is just the same as the move from the centre farm to ring 1. The second ring has farms each of radius 3. Strictly speaking, the farms of the second ring can be anywhere round the circumscribing circle, but we have selected the pattern so that the farms touch on a common radial line and share a common tangent with the circumscribing circle. By doing this we have preserved the 6-fold symmetry. Another way of describing the construction of the second ring is to say that it is a version of the first ring scaled by a factor 3.

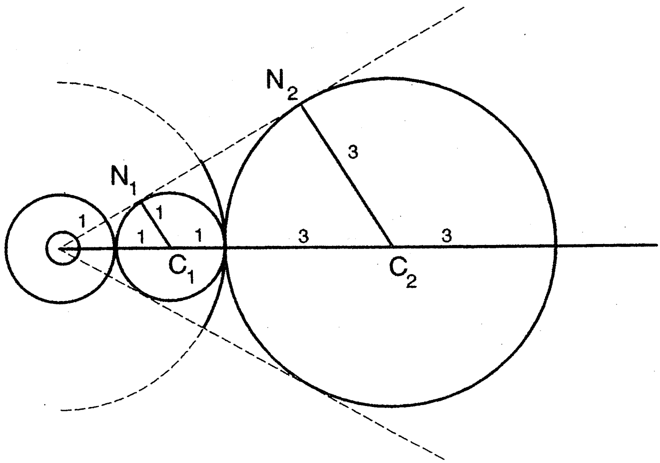

Figure 3

We examine the geometry more fully by choosing a sector of 60° as in Figure 3, just as we did for the hexagons. The radii are marked. C_1, C_2 are the centres of circles in the first and second rings. The upper part of the sector is bounded by the dotted line touching the two circles at N_1 and N_2. An arc of the circumscribing circle is shown; its radius is 3. The triangle ΔOC_1N_1 is a $(30°, 60°, 90°)$ triangle and by Pythagoras's Theorem the sidelength ON_1 is

$$ON_1 = \sqrt{OC_1^2 - C_1N_1^2} = \sqrt{4-1} = \sqrt{3}.$$

Similarly, $ON_2 = \sqrt{OC_2^2 - C_2N_2^2} = \sqrt{36-9} = 3\sqrt{3}$. Notice that the scale factor of 3 has appeared between the first and second ring. Note that

$$ON_1 \times ON_2 = \sqrt{3} \times \sqrt{3} \times 3 = 3^2$$

the square of the scale factor.

There is another relationship between the two rings expressed through the phrase that one can be inverted into the other. The process of inversion is defined as follows. Given a point O in a plane, the so-called *centre of inversion*, and a constant k, the so-called *inversion constant*, then any point P of the plane can have associated with it a point Q on the ray $\overrightarrow{OP}$ such that

$$OP \times OQ = k$$

The points P and Q are said to be *inverse* to each other. To see this more clearly, we repeat Figure 3 with another general radius intersecting the two circles in P_1, Q_1 and P_2, Q_2; this is Figure 4. Consider the line $\overrightarrow{OQ_2}$ moving towards $\overrightarrow{OC_2}$. The point Q_2 moves towards $\overrightarrow{OC_2}$ and the length OQ_2 increases. At the same time P_1 moves towards $\overrightarrow{OC_2}$ and the length of $\overline{OP_1}$ decreases. This way of looking at points "moving" along the curves is one of the ways that geometry was examined after Isaac Newton taught us about the dynamics of representing a body by a point with mass, and is very useful in a problem such as we face here. OQ_2 increases as OP_1 decreases; let us look at their product $OP_1 \times OQ_2$.

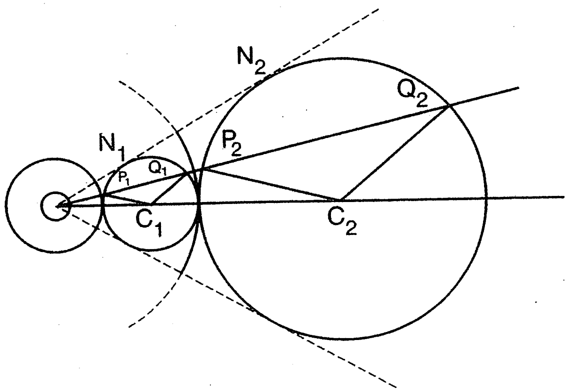

Figure 4

The fact that the second circle is scaled up from the first suggests that the triangles ΔOC_1P_1 and ΔOC_2P_2 may be similar. This is true; the proof is as follows.

The angle P_1OC_1 is common to both triangles. The ratio C_2P_2/C_1P_1 is equal to 3 (the scale factor). The ratio $OC_2/OC_1 = 6/2$ which equals 3. These three conditions are enough to make ΔOC_1P_1 similar to ΔOC_2P_2. The triangles ΔOC_1Q_1 and ΔOC_2Q_2 are similar for the same reasons. Thus $OP_2/OP_1 = OQ_2/OQ_1 = 3$ and, cross-multiplying,

$$OP_1 \times OQ_2 = OP_2 \times OQ_1$$

This is enough to show that P_1 inverts into Q_2 and similarly that Q_1 inverts into P_2. The constant value of the product can be seen from the position where P_1 and Q_1 merge into N_1 and, simultaneously, where P_2 and Q_2 merge into N_2. We have already found that

$$ON_1 \times ON_2 = 3^2$$

so that the full statement of inversion is

$$OP_1 \times OQ_2 = OP_2 \times OQ_1 = 3^2$$

We can say that the first ring inverts into the second ring through the circumscribing circle.

Inversion is a powerful general technique. It can be used across a straight line or across curved lines. A good introduction, especially for problems of curves, is given by E H Lockwood, *A Book of Curves* (1961), Cambridge University Press, Chapter 23, p. 172.

But we still have not completed our analysis of the Thünen problem. The front cover showed the pattern when the radius of the first farm was chosen to be the same as that of the town. We relax this condition and choose a smaller radius for the farm; for instance, we pack 12 touching farms round the town in the first ring, 12 in the second ring, and so on. The proof that inversion solves this problem also is as follows. We do not need to draw the whole pattern; we use the symmetry of the pattern which is now 12-fold, and draw a sector of 30° as in Figure 5. The points marked in Figure 5 have the same meanings as in Figure 4. The radius of the town can be taken as 1 and the radii of the first set of circles as r_1, and of the second set of circles as r_2. We start by treating the problem as one of scaling. We need the value of $\sin 15°$ which we get by using

$$\cos 2\theta = 1 - 2\sin^2\theta;$$

thus,

$$\sin 15^\circ = \frac{\sqrt{2-\sqrt{3}}}{2}$$

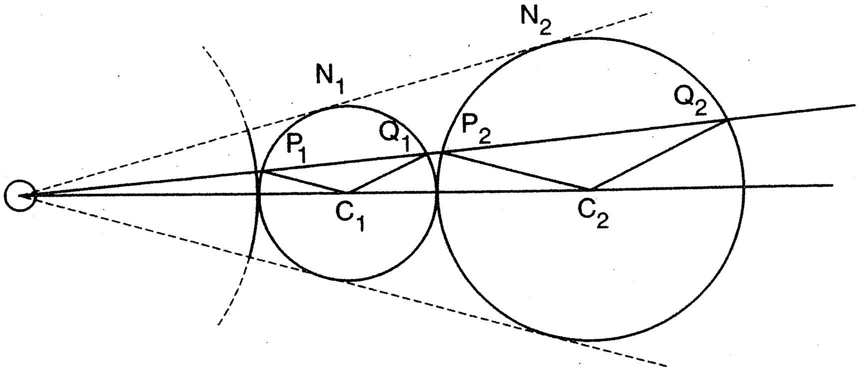

Figure 5

In the triangle ΔOC_1N_1, $\sin 15^\circ = \dfrac{r_1}{1+r_1}$ and

$$r_1 = \frac{\sqrt{2-\sqrt{3}}}{2-\sqrt{2-\sqrt{3}}} = 0.349198\ldots$$

The scaling of the pattern is represented by the scaling ratio of the circumscribing circles. Thus

$$\frac{r_2}{r_1} = \frac{1+2r_1}{1}$$

and $r_2 = 0.593076\ldots$ Exactly as we found for the ring of 6 circles, the triangles ΔOC_1P_1 and ΔOC_2P_2 are similar and the triangles ΔOC_1Q_1 and ΔOC_2Q_2 are similar.

Thus

$$\frac{OP_2}{OP_1} = \frac{OQ_2}{OQ_1}$$

and

$$OP_1 \times OQ_2 = OP_2 \times OQ_1$$

which is the condition for inversion. We again use $ON_1 \times ON_2$ and Pythagoras's Theorem to obtain the inversion constant as

$$(1+2r_1)^2$$

which is the square of the radius of the first circumscribing circle.

There are further ring patterns to find. One in which the town and all the farms have the same radius occurs by looking at Figure 1 in the following way. A circle can be inscribed in each hexagon and the pattern of these circles has each circle touching its 6 neighbours. If the side of the hexagon is given the value unity, then the radius of the inscribed circle can be proved by Pythagoras's Theorem to be $\frac{\sqrt{3}}{2}$. This pattern of circles is another realization of Thünen's pattern of rings.

We conclude this discussion of rings of circles (or farms round a town) by stating that the problems can be generalized not only to any number of equal touching circles, but also to unequal circles. An account of the chains of circles was first given by the Swiss mathematician Jacob Steiner (1796-1863), a contemporary of Thünen. Steiner explored these problems in great generality. His work is described in the *Dictionary of Scientific Biography* (a good source for the descriptions of mathematicians' work) and a short discussion is presented in the book of C Stanley Ogilvy, *Excursions in Geometry* (1969), Oxford University Press, in the chapter on the applications of inverse geometry. Ogilvy's book is a stimulating account of advanced geometry.

* * * * *

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Bill Boundy has taught Mathematics and Physics in various South Australian institutions, including the University of Adelaide. He is currently undertaking some collaborative research with Bert Bolton. His personal interests include radio, photography and recreational mathematics.

THE PYTHAGOREAN THEOREM: A PYTHAGOREAN GENERALIZATION

K R S Sastry, Addis Ababa, Ethiopia

The Pythagorean Theorem, that the sum of the squares of the legs of a right-angled triangle equals the square of the hypotenuse, is very well known. Can we look at it differently in a way that will lead us to find a new generalization? The answer is yes. To do this, we need to go back in time to the ancient Pythagoreans – the pupils of Pythagoras – who evolved the concept of *polygonal* or *n-gonal* numbers.

The ancient Pythagoreans used the sequence

$$1, 3, 6, 10, 15, \ldots$$

of numbers of objects (say, sea shells) to depict the sequence of triangles shown in Figure 1.

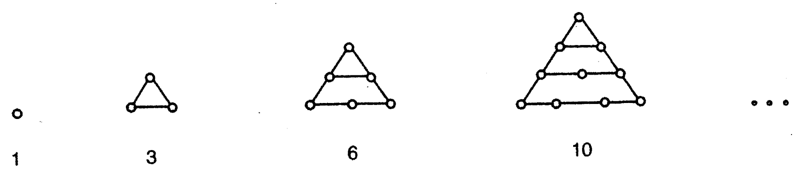

Figure 1

Hence they called the numbers in this sequence *triangular* numbers. Notice that each triangle is obtained geometrically from the previous one by adding objects in a row adjacent to one of its sides. It is a simple exercise to obtain a formula for the rth triangular number in terms of r.

In a similar way, the Pythagoreans depicted the *square* numbers

$$1, 4, 9, 16, 25, \ldots$$

as shown in Figure 2.

Each square is obtained from the previous one by adding objects in rows alongside *two* adjacent sides. The rth square number is of course r^2.

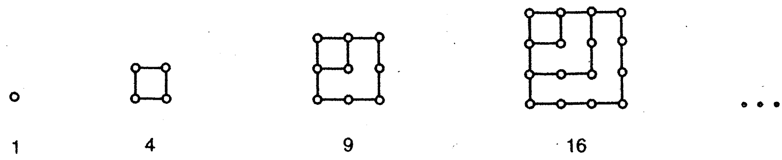

Figure 2

For a similar reason, the numbers

$$1, 5, 12, 22, 35, \ldots$$

are called *pentagonal* numbers. See Figure 3.

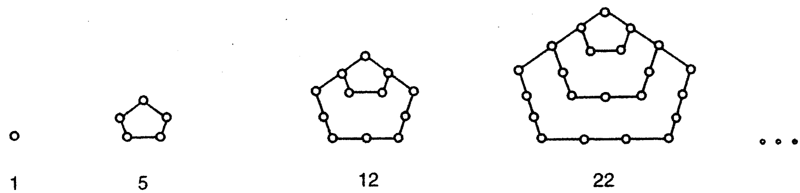

Figure 3

Each pentagon is obtained from the previous one by adding objects in rows alongside *three* adjacent sides. As another simple exercise, you could find a formula for the rth pentagonal number; we will give the answer shortly.

This idea can be extended to obtain a sequence of polygonal numbers of side n (or n-gonal numbers) of rank r, where $n \geq 3$ and $r = 1, 2, 3, \ldots$ The rank r denotes the number of objects on each side. If we use the symbol P_n^r for the polygonal number of side n and rank r, then the formula for P_n^r is

$$P_n^r = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2}, \quad n \geq 3, \quad r = 1, 2, 3, \ldots \tag{1}$$

It is left as an exercise for you to derive this formula. Observe that $n = 3$ gives the triangular numbers $P_3^r = \frac{1}{2}r(r+1)$, $n = 4$ gives the square numbers $P_4^r = r^2$, $n = 5$ gives the pentagonal numbers $P_5^r = \frac{1}{2}r(3r-1)$, $n = 6$ the hexagonal numbers, and so on. It is this polygonal number idea that we would like to use to find a Pythagorean generalization of the Pythagorean theorem.

Suppose ABC is a triangle, right angled at C. Let a and b denote respectively the lengths of the legs $\overline{BC}$ and $\overline{CA}$, and let c denote the length of the hypotenuse $\overline{AB}$ (Figure 4).

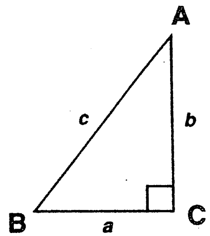

Figure 4

Then the Pythagorean theorem translates into the equation

$$a^2 + b^2 = c^2 \tag{2}$$

The traditional geometrical interpretation of Equation 2 is that the sum of the areas of the squares described on the legs of a right-angled triangle equals the area of the square described on the hypotenuse.

However, if we simultaneously look at the sequence of square numbers and Equation 2, we get the following polygonal number idea:

$$a\text{th square number} + b\text{th square number} = c\text{th square number} \tag{3}$$

where a, b and c are now restricted to taking natural number values.

That is, if we can depict squares with a objects, b objects, and c objects per side on the legs with lengths a and b and the hypotenuse with length c respectively of a right-angled triangle, then

$$a^2 \text{ objects} + b^2 \text{ objects} = c^2 \text{ objects}$$

Therefore we may look upon the Pythagorean theorem as giving rise to a Pythagorean relation among certain triples of the square numbers. Figures 5 and 6 respectively show the traditional geometric view and the present polygonal number view of the Pythagorean relation, for the case of the Pythagorean triple $(a, b, c) = (3, 4, 5)$.

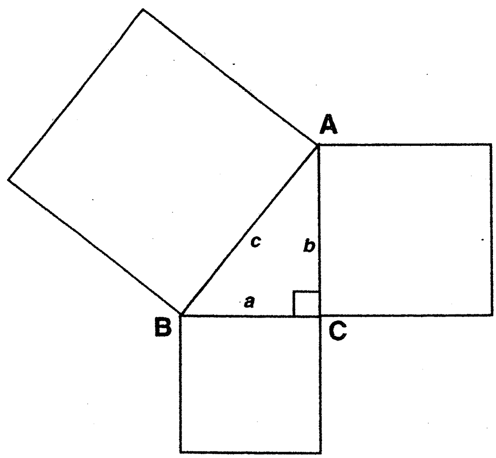

Figure 5
Traditional geometric view: $a^2 + b^2 = c^2$

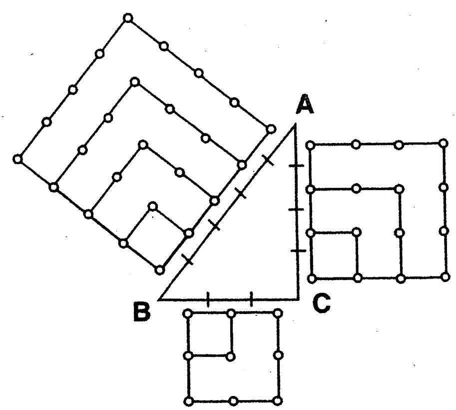

Figure 6
Polygonal number view: a^2 objects + b^2 objects = c^2 objects

Now let us define a Pythagorean relation for the triangular numbers. Looking again at (3), the following definition suggests itself.

A triple (a, b, c) of natural numbers is said to satisfy the *Pythagorean relation for the triangular numbers* if

ath triangular number + bth triangular number = cth triangular number

i.e., if $\frac{1}{2}a(a+1) + \frac{1}{2}b(b+1) = \frac{1}{2}c(c+1)$, or

$$a^2 + a + b^2 + b = c^2 + c \tag{4}$$

For example, the triple $(a, b, c) = (2, 2, 3)$ satisfies (4), the Pythagorean relation for the triangular numbers, as you can verify. Some other examples of triples that satisfy (4) are (3,5,6), (5,6,8), (14,14,20) and (34,35,49). By analogy with the usual Pythagorean triples such as (3,4,5), let us call a triple (a, b, c) a *Pythagorean triple of triangular numbers*, or briefly, a *triangular Pythagorean triple*, if a, b, c satisfy Equation (4).

As a more difficult exercise, you could show that any triangular Pythagorean triple (a, b, c) can form the lengths of three sides of a triangle. In order to do this, you need to prove that $a + b > c$ for any natural numbers a, b, c that satisfy (4).

What polygonal number interpretation can be given to a triangular Pythagorean triple (a, b, c)? We can depict appropriate triangles – in the manner squares were depicted in Figure 6 – on the sides of a triangle with side lengths a, b, c. This is done in Figure 7 for the triangular Pythagorean triple (3,5,6).

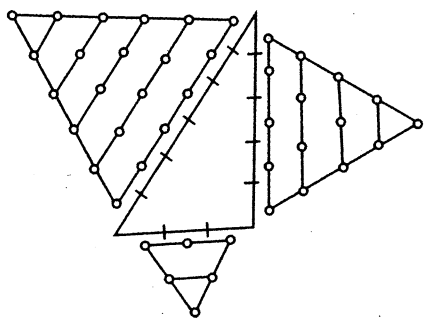

Figure 7

It is now a simple matter to give the extended definition of the Pythagorean relation for the polygonal numbers:

A triple (a, b, c) of natural numbers is said to satisfy the *Pythagorean relation for the n-gonal numbers* (or to be an *n-gonal Pythagorean triple*) if

$$a\text{th } n\text{-gonal number} + b\text{th } n\text{-gonal number} = c\text{th } n\text{-gonal number}$$

Using the notation of (1), we may write the equation above more compactly as:

$$P_n^a + P_n^b = P_n^c \tag{5}$$

For example, (5,5,7) is a pentagonal Pythagorean triple, as you can easily check.

As a further exercise, you may like to prove that, for any value of n, any triple (a, b, c) of natural numbers that satisfies (5) can form the lengths of the sides of a triangle. Let us call such a triangle an *n-gonal Pythagorean triangle.* These triangles have some interesting properties. Here is a partial list. If $n = 3$ they are all obtuse-angled triangles. If $n = 4$, of course, they are right-angled triangles. If $n > 4$ they are all acute-angled triangles. Curiously, all these obtuse and acute angles are nearly right angles. If $n \neq 4$, the triangles can be isosceles (for example, the triangles corresponding to the triangular Pythagorean triple (2,2,3) and the pentagonal Pythagorean triple (5,5,7)), but no two distinct triangles can be similar for a given value of n if $n \neq 4$. A large variety of problems – several of them very deep – is suggested when we look up A H Beiler's *Recreations in the Theory of Numbers*, Dover (1964), pages 104-133, 185-199 and 248-268, and L E Dickson's *History of the Theory of Numbers*, Volume II, Chelsea (1971), pages 1-39, 165-190, 341-400 and 615-627.

The reader can find a discussion of the properties mentioned above and more in the article "Pythagorean triangles of the polygonal numbers", which appeared in Volume 27, Number 2 (Spring 1993) of the journal *Mathematics and Computer Education* on pages 135-142. Another interesting property of the polygonal numbers may be found in the article "Cubes of natural numbers in arithmetic progression", which appeared in the June 1992 issue of *Crux Mathematicorum* on pages 161-164.

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HISTORY OF MATHEMATICS

Mathematics and Malaria[1]

Michael A B Deakin

The British epidemiologist and pioneer of tropical medicine, Sir Ronald Ross (1857-1932), is best remembered today for his life's work on the transmission of malaria, for which he was awarded the Nobel Prize in Physiology and Medicine in 1902. It was Ross who showed conclusively that malaria is transmitted by mosquitoes and who analysed that transmission with a view to control or eradication of this virulent disease.

Ross was a considerable mathematician as well as being a medical researcher (he also wrote quite meritorious poetry). Among the studies he made was the development of a mathematical model of malaria and its transmission. This led to what are now called the *Ross Malaria Equations.* Figure 1 shows the so-called *trajectories* of these equations.

Let a population of mosquitoes live in the same locality as a population of humans, and let x be the proportion of humans infected with malaria and y the proportion of mosquitoes so infected. In a given time, a proportion R of the infected humans recovers and a proportion M of the mosquitoes dies of malaria. If each human suffers B bites in this given time and each mosquito delivers b bites in this same time, then it can be shown that in the long run

$$\begin{aligned} x = \overline{x} &= \frac{Bb - RM}{B(M+b)} \\ y = \overline{y} &= \frac{Bb - RM}{b(B+R)} \end{aligned} \qquad (1)$$

provided these numbers are positive.

Before this "long-run" or "equilibrium" situation is achieved, however, x and y will vary with time. While it is possible to graph the values of x versus the time, or of y versus the time, it is also possible to construct a graph of the values of y versus the values of x. Such a graph is referred to as a "phase-plane graph" and the technique is a most useful and important one in mathematical modelling.

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[1]Dr Deakin has recently returned from overseas. We reprint here, with a few modifications, his cover story for *Function, Vol.* 11, *Part* 2.

Figure 1 shows such a graph. Every point in the square defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$ other than $(0,0)$ and $(\overline{x}, \overline{y})$ lies on exactly one curve or *trajectory*; as time goes by, this curve traces the subsequent values of x, y which continue to travel along it, in the direction of the arrows toward the long-run solution $x = \overline{x}$, $y = \overline{y}$.

The diagram here shows the case for which $\overline{x} = 0.35$, $\overline{y} = 0.30$.

If $\overline{x} < 0$, or $\overline{y} < 0$, then the situation $x = \overline{x}$, $y = \overline{y}$ cannot be achieved and the arrows would then show that in the long run $x = 0$, $y = 0$. This corresponds to there being no malaria present – a much wished-for situation.

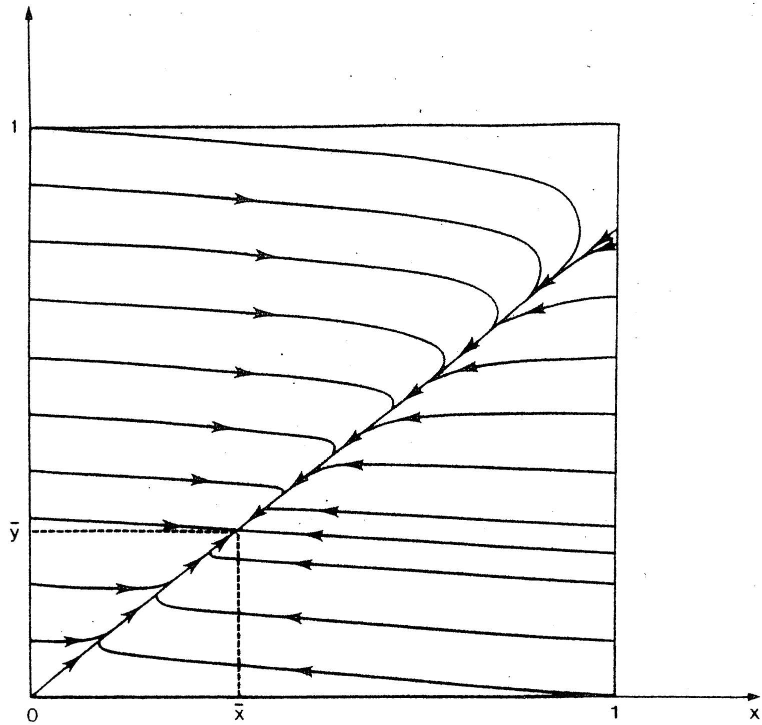

Figure 1

Equations (1) show us that if $Bb > RM$, then both $\overline{x}$, $\overline{y}$ are positive as in Figure 1, and malaria remains endemic. If, on the other hand, $Bb < RM$, then both $\overline{x}$, $\overline{y}$ are negative and so the malaria is eradicated.

In order to eradicate malaria therefore we need to achieve a situation in which

$$RM > Bb. \tag{2}$$

That is to say, we need to combine several factors:

Increase the recovery rate of the humans,
Increase the mortality of the mosquitoes,
Decrease the rate at which mosquitoes bite humans.

Inequality (2) shows an important point. It is not necessary to eliminate *all* mosquitoes, nor to cure *every* human case of malaria, nor need all biting be prevented. As long as Inequality (2) continues to apply, malaria cannot spread in the population. This led to considerable hopes that malaria might be eradicated, as smallpox has been.

Unfortunately, matters not taken into account in the analysis complicate matters. Malaria *plasmodia* are becoming resistant to the drugs (derivatives of quinine) used to keep the value of R high, and mosquitoes have become resistant to the pesticides used to keep the value of M up. It has also been discovered that malaria infects other animals besides humans and this corresponds (in essence) to higher values of b and lower values of R than we might at first suppose.

Nonetheless, the analysis has been very helpful, and continues to be helpful, in malaria control programs.

In 1923, the *American Journal of Hygiene* devoted a special supplement to the Ross Malaria Equations. Most of this was written by Alfred J Lotka, a mathematician at the Johns Hopkins University in Baltimore. (We tend to forget that malaria was endemic in Baltimore and in many other parts of the United States until well into this century.) Lotka produced a very full study of the equations in the course of this work, and Figure 1 is based on one of his diagrams.

* * * * *

COMPUTERS AND COMPUTING

Chance and Beauty

Cristina Varsavsky

In our February issue we looked at the geometrical construction of fractals, more precisely the Sierpinski triangle. This was generated by repeatedly applying a basic constructive procedure which consisted of replacing a square with three new squares scaled down by a factor of 0.5. We also observed that the final fractal, that is, the set of points obtained after applying the basic procedure infinitely many times, does not depend on the figure we start from but only on the three transformations involved.

There are other ways of constructing such a fractal image. We will introduce here a method based on chance (any method based on chance is usually called a *stochastic* method). A fractal constructed using the method described above consists of an infinite number of points. Since we can only plot a finite number of them on the screen, we need to distribute them in such a way as to create the illusion of the fractal. It is here that chance plays an important rôle: we generate a sequence of points by means of a random process.

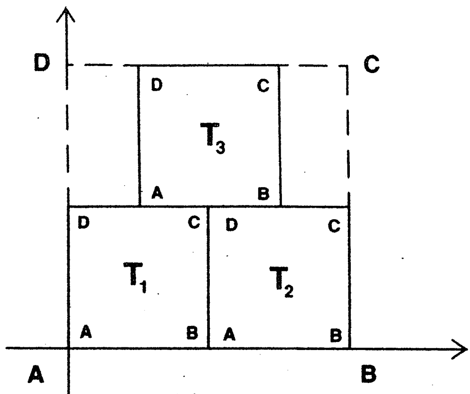

Figure 1

Let us illustrate this random process with the Sierpinski triangle. There are three transformations involved here – we name them T_1, T_2 and T_3. They are represented in Figure 1, which shows the effect of each of them on the square drawn with dotted lines. The three transformations consist of a contraction by a scale factor of 0.5; in addition, T_2 involves a horizontal shift and T_3 involves a horizontal and a vertical shift. We represent these transformations through the following equations,

$$T_1 \begin{cases} x' = 0.5x \\ y' = 0.5y \end{cases} \qquad T_2 \begin{cases} x' = 0.5x + 0.5 \\ y' = 0.5y \end{cases} \qquad T_3 \begin{cases} x' = 0.5x + 0.25 \\ y' = 0.5y + 0.5 \end{cases}$$

which transform the point (x, y) into the point (x', y').

We start with a point P_0 – say $(0, 0)$. Then we randomly select one of the three transformations which we apply to P_0 to get a new point P_1. We repeat this process, that is, we generate P_2 by applying a randomly selected transformation to P_1. This is repeated again and again until enough points are generated to produce the fractal image.

Now, since we intend to produce this fractal with a computer program, we need to design a procedure for choosing randomly one of the three transformations. Nowadays computers come with a generator of random numbers equally distributed between 0 and 1[1]. We can use this facility as follows: we generate a random number and then we choose T_1 if it is less than 1/3, T_2 if it lies between 1/3 and 2/3, and T_3 if it is greater than 2/3.

The following computer program implements this algorithm in Quick-Basic (in this language, RND is the instruction for the random number generation):

```
SCREEN 9
WINDOW (-.5, -.5) - (1.5, 1)
iterations = 10000
x = 0 : y = 0

FOR i = 1 TO iterations
        choice = RND
        IF choice < .33 THEN
           xn = .5 * x
           yn = .5 * y
           ELSEIF choice < .66 THEN
                 xn = .5 * x + .5
                 yn = .5 * y
```

[1]More precisely, computers generate *pseudo-random* numbers, but this in itself could be the subject of another article.

```
                ELSE
                        xn = .5 * x + .25
                        yn = .5 * y + .5
        END IF
        PSET (xn, yn)
        x = xn : y = yn
NEXT i
END
```

The output of the program, shown in Figure 2, is the Sierpinski triangle, which is already familiar to us.

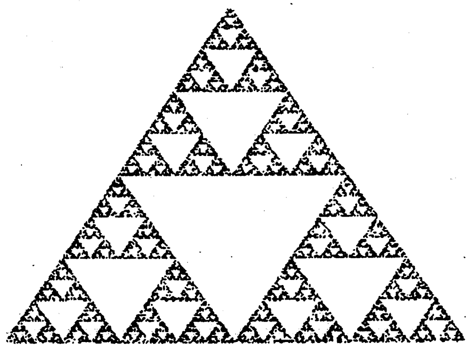

Figure 2

Take as a second example the fractal generated with the five transformations illustrated in Figure 3. Each transformation is a contraction by a factor of 1/3, followed by a translation. The FOR ... TO ... loop now should be written as:

```
FOR i = 1 TO iterations
        choice = RND
        IF choice < .2 THEN
            xn = .33 * x
            yn = .33 * y
            ELSEIF choice < .4 THEN
```

```
                xn = .33 * x + .66
                yn = .33 * y
                ELSEIF choice < .6 THEN
                        xn = .33 * x + .33
                        yn = .33 * y + .33
                        ELSEIF choice < .8 THEN
                                xn = .33 * x + .66
                                yn = .33 * y + .66
                                ELSE
                                        xn = .33 * x
                                        yn = .33 * y + .66
        END IF
        PSET (xn, yn)
        x = xn : y = yn
NEXT i
```

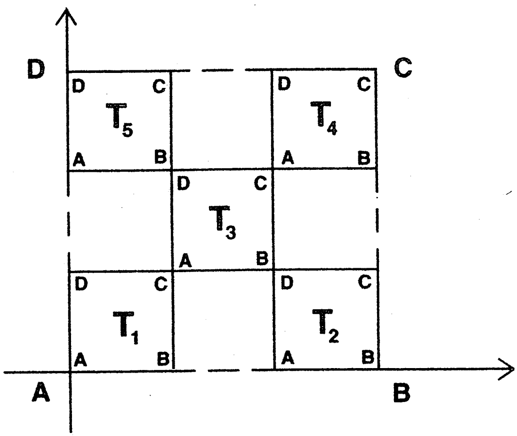

Figure 3

These five transformations are the clue to the fractal depicted in Figure 4.

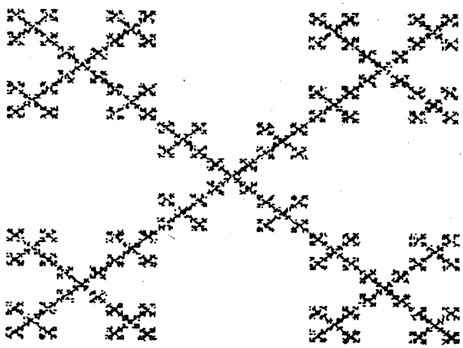

Figure 4

So far we have only worked with transformations involving contractions (scaling), but there are other transformations we could use to produce more complex fractal images. Let us do this more systematically. We could think of a point (x, y) being transformed to the point (x', y') by performing the following matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad (1)$$

For example, the contraction involved in the three transformations for the Sierpinski triangle could be expressed as in (1) with $a = d = 0.5$ and $c = b = 0$, as you can check by carrying out the matrix multiplication. In general, a matrix product of the form

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

will result in scaling by a factor of a in the horizontal direction, and by a factor of d in the vertical direction.

Similarly, an anticlockwise rotation through an angle θ and centred at the origin is expressed as:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Reflections in the x- and y-axes are achieved through the matrix equations

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{respectively.}$$

If more than one transformation is applied to a point, the overall transformation consists of the product of the corresponding matrices from right to left in the order they are applied (you can check that the order in which transformations are applied usually matters).

We will illustrate these ideas with the set of transformations shown in Figure 5. What are the matrices for T_1, T_2 and T_3? T_1 consists of an anticlockwise rotation through an angle of 45° followed by a contraction by the factor $\frac{\sqrt{2}}{2}$. This results in the matrix product

$$\begin{bmatrix} \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

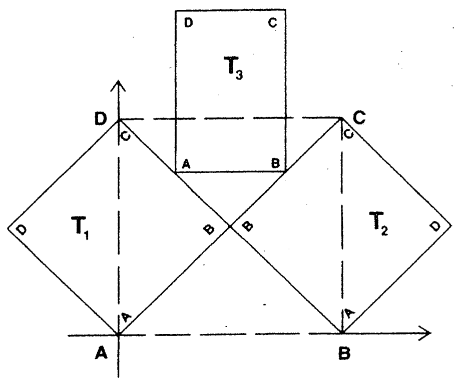

Figure 5

Similarly, T_2 involves an anticlockwise rotation through an angle of 45° followed by a contraction by a factor of $\frac{\sqrt{2}}{2}$, and then a reflection in the y-axis, which is represented by the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

This is followed by a shift of one unit in the horizontal direction.

Finally, T_3 is a contraction by the scale factor 0.5 in the horizontal direction and the scale factor 0.75 in the vertical direction, followed by a horizontal and a vertical shift.

The corresponding section in the program follows:

```
FOR i = 1 TO iterations
   choice = RND
   IF choice < .33 THEN
         xn = .5 * x - .5 * y
         yn = .5 * x + .5 * y
         ELSEIF choice < .66 THEN
            xn = -.5 * x + .5 * y + 1
            yn = .5 * x + .5 * y
            ELSE
               xn = .5 * x + .25
               yn = .75 * y + .75
   END IF
```

The program, run with window set to (-2, -2) - (3,3) and 20000 iterations, produces the maple leaf displayed in Figure 6.

This article gives you enough insight into the theory of transformations for you to experiment with them and design your own aesthetic figures generated by a stochastic process. An alternative you may wish to consider is to add colour to make them look even more beautiful.

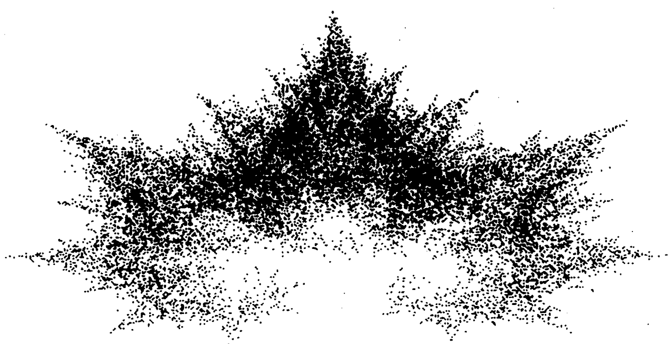

Figure 6

OLYMPIAD NEWS

Hans Lausch, Special Correspondent on Competitions and Olympiads

1. The Sixth Asian Pacific Mathematics Olympiad

The Asian Pacific Mathematics Olympiad (APMO), an annual competition, was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the number of participating Pacific Rim countries has grown to fourteen. Besides students from the founding countries, participants in the 1994 APMO were from Chile, Colombia, Indonesia, Malaysia, Mexico, New Zealand, the Philippines, the Republic of China, the Republic of Korea and Thailand. Here are the questions from this four-hour examination:

Time allowed: 4 hours.
No calculators to be used.
Each question is worth seven points.

Question 1.

Let $f : R \rightarrow R$ a function such that

i) for all $x, y \in R$

$$f(x) + f(y) + 1 \geq f(x+y) \geq f(x) + f(y)$$

ii) for all $x \in [0,1), f(0) \geq f(x)$.

iii) $-f(-1) = f(1) = 1$.

Find all such functions.

Question 2.

Given a nondegenerate triangle ABC, with circumcentre O, orthocentre H, and circumradius R, prove that $|OH| < 3R$.

Question 3.

Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab. Determine all such n.

Question 4.

Is there an infinite set of points of the plane such that no three elements of it are collinear, and the distance between any two of them is rational?

Question 5.

You are given three lists, A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

A	B	C
10	1010	20
100	1100100	400
1000	1111101000	13000

Prove that for every integer $n > 1$, there is exactly *one* number in exactly *one* of the sets B or C that has exactly n digits.

Australian certificate winners (school year in parenthesis) were:

Gold:	William Hawkins (12), ACT, Canberra Grammar School
Silver:	James Lefevre (12), Tas, Launceston College
	Akshay Venkatesh (12), WA, Scotch College
Bronze:	Anthony Wirth (12), Vic, Melbourne Church of England Grammar School
	Ren Hou (12), NSW, North Sydney Boys' High School
	Andrew Rogers (12), Vic, Scotch College
	Nigel Tao (12), SA, Westminster School

2. The XXXV International Mathematical Olympiad (IMO)

In April, the ten-day Team Selection School of the Australian Mathematical Olympiad Committee took place at Sydney Church of England Grammar School ("Shore School") in North Sydney. Candidates for the Australian team at this year's IMO and other highly-gifted students who look forward to at least one more year of secondary education were there to undergo a day and evening programme consisting of tests and examinations, problem sessions and lectures by mathematicians. Finally, the 1994 Australian IMO Team was selected.

The following students were selected as team members: William Hawkins, James Lefevre, Akshay Venkatesh, Andrew Rogers, Nigel Tao and Chaitanya Rao (Melbourne Church of England Grammar School); reserve: Anthony Wirth.

Hong Kong was this year's venue for the IMO. There the Australian team had to contend with six problems during nine hours spread equally over two days in succession. The Australian team finished in twelfth place, out of 68 participating countries (behind USA, China, Russia, Bulgaria, Hungary, Vietnam, United Kingdom, Iran, Romania, Japan and Germany). This matches some of our best performances to date. Members of the team received awards as follows:

James Lefevre, Silver medal

William Hawkins, Silver medal

Andrew Rogers, Bronze medal

Akshay Venkatesh, Bronze medal

Nigel Tao, Bronze medal

Chaitanya Rao, Honourable Mention.

Well done!

*****

MATHEMATICAL NURSERY RHYME

Hey diddle diddle, the cat and the fiddle,
The cow jumped over the moon;
Which requires computation of its orbit's equation
To avoid jumping late or too soon.
If Earth's mass be m, gravitation be g,
The product must equal indeed
Moon's distance from Earth (cancel out the moon's mass)
Multiplied by the square of its speed.

From: *The Surprise Attack in Mathematical Problems*
by L A Graham (Dover, 1968)

*****

PROBLEM CORNER

SOLUTIONS

PROBLEM 18.3.1

A man's shirt is normally buttoned up with the left side overlapping the right side. If a man puts his shirt on inside-out and buttons it up, which side will be outermost?

SOLUTION

With the shirt on inside-out, the buttons will be on the left side instead of the right. However, they will be facing inward, so the left side will still be outermost.

PROBLEM 18.3.2

Find the unique 5-digit number which, when multiplied by 4, yields the number formed by writing the digits of the original number in the reverse order.

SOLUTION

The problem can be written as shown below, where each letter corresponds to a digit:

$$\begin{array}{ccccc} A & B & C & D & E \\ & & & & 4 \\ \hline \\ E & D & C & B & A \end{array}$$

Since 4 times the first digit, A, plus the carry from the adjacent column, equals the single-digit number E, A must be 1 or 2. By looking at the rightmost column we can see that $4E$ has A as its last digit, so A is even. Therefore $A = 2$.

We now know that $4E$ ends in 2, so E must be either 3 or 8. We also know that E is at least $4A$, so we conclude that $E = 8$.

It now follows that there is no carry when B is multiplied by 4, so B must be equal to 0, 1 or 2. By examining the second column from the right, we can see that $4D + 3$ ends in B, and so B is odd. Hence $B = 1$.

Now that we know that $4D+3$ ends in 1, we can deduce that D is either 2 or 7. If $D = 2$ then there is a carry of 1 into the middle column, and so

$4C + 1$ would have to end in C. This would give $C = 3$, and it is easy to check that this doesn't work. Therefore $D = 7$, and we readily deduce that $C = 9$. Hence the number is 21978.

More on one of our earlier problems

PROBLEM 15.4.7

Prove that the polynomial

$$f_n(x) = x^n \sin a - x \sin(na) + \sin[(n-1)a]$$

is exactly divisible by

$$h(x) = x^2 - 2x \cos a + 1$$

where a is a real number, and n is an integer greater than 1.

This problem appeared in *Function, Vol.* 15, *Part* 4 (1991), and a solution using complex numbers was given in *Vol.* 17, *Part* 5 (1993). Keith Anker of Monash University has provided an alternative proof, which does not use complex numbers, and which proceeds by induction on n. (He actually proves the slightly stronger statement that the claim is true for any integer $n \geq 0$.) For reasons of space, and because a solution to this problem has already been published, we will not reproduce the proof here, but some readers might like to try to discover it for themselves. (A proof by induction proceeds by showing, firstly, that the claim is true when $n = 0$, and, secondly, that *if* the claim is true for any particular value of n, say $n = k$, *then* it must also be true for the next value of n, namely $n = k + 1$.)

PROBLEMS

PROBLEM 18.5.1 (D F Charles, Oak Park, Vic.)

In Figure 1, the point P is free to move along $\overrightarrow{OL}$.

(a) What is the maximum value of θ ($\theta_{\max}$)?

(b) With AB remaining at 1 unit, what is the length of $\overline{OA}$ so that $\theta_{\max} = 30°$?

(The problem can be solved without calculus.)

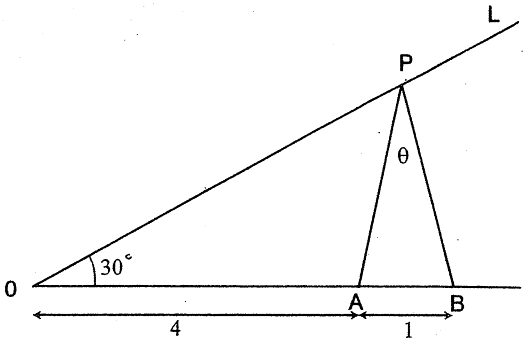

Figure 1

PROBLEM 18.5.2 (Peter Oliphant, student, Monash University)

The numbers in the equilateral triangle shown in Figure 2 represent the areas of their respective regions. Find the area of the central triangle.

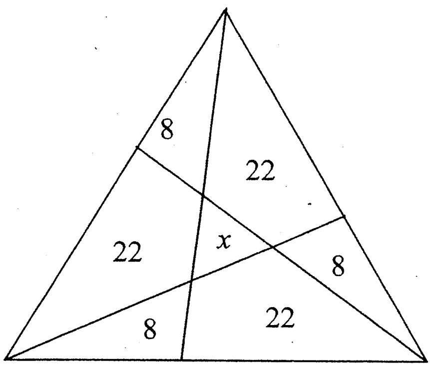

Figure 2

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(*C*) – Computers and Computing Section

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