Function

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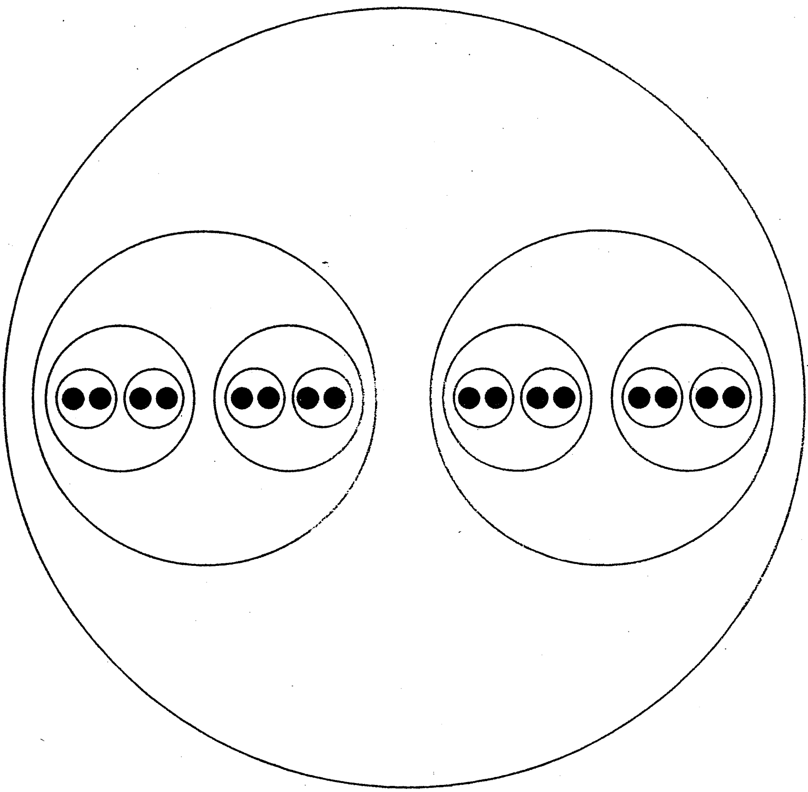

Mathematics Department - Monash University

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Function is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors, *Function*
Department of Mathematics, Monash University
900 Dandenong Rd
Caulfield East VIC 3145, Australia
Fax: +61 (03) 9903 2227
e-mail: function@maths.monash.edu.au

Alternatively, correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

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*$8.50 for *bona fide* secondary or tertiary students.

EDITORIAL

We hope you will find in this issue of *Function* a good variety of interesting articles.

The diagram on the front cover depicts an iterative construction based on an object known as the Cantor set, which could be considered as the oldest fractal.

This issue of *Function* includes two feature articles. The first is by Aidan Sudbury, who derives a formula for calculating the length of a day. You need a calculator, the latitude of your location and the date. (The newspaper may still be useful to check the accuracy of your estimate.) In a lighter tone, Michael Deakin introduces, in his usual entertaining style, some rather amusing names and symbols for the numbers used in the hexadecimal system. They are based on a proposal made by J Bowden early this century to introduce base sixteen into general use.

Our regular *History* column is devoted to the distinguished English mathematician G H Hardy. He was a pure mathematician, concerned with the depth and beauty of mathematics as a creative art.

You will find in the *Computers and Computing* column a short explanation of the representation of a two-variable function as a surface in three dimensions. A simple program to produce these plots on a computer screen is included, a program which could be a starting point for you to do more sophisticated plots.

The *Problem Corner* includes, as usual, some challenging problems, solutions to the ones published in the February issue, and also a solution to an earlier one. Hans Lausch, our correspondent on competitions and olympiads, has provided the questions of the Seventh Asian Pacific Mathematics Olympiad and the names of the six highly gifted year twelve students who will represent Australia at the XXXVI International Mathematical Olympiad.

Happy reading!

* * * * *

THE FRONT COVER

The Cantor Cheese

Cristina Varsavsky

The German mathematician Georg Cantor [1845-1918] was one of the founders of set theory, which is of great significance for the foundation of mathematics. We also owe him what could be called the oldest fractal, known as the *Cantor set*. The set is obtained iteratively: start with the closed unit interval $[0, 1]$. Remove the middle third but not the endpoints, as shown in Figure 1, stage 1; two closed intervals of length one third remain. At the second stage, remove the middle third of each of these two intervals; four closed intervals of length one ninth remain. The process of removing the middle third of every interval formed at each stage continues *ad infinitum*. No intervals remain at the end, only a dust of points. This set of disconnected points is the Cantor set.

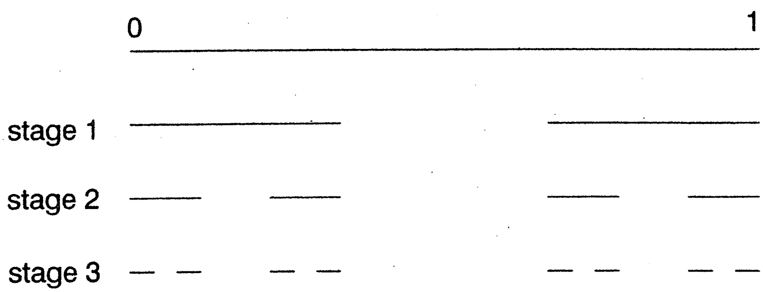

Figure 1

It is interesting to see which points from the original interval $[0, 1]$ actually belong to the Cantor set. Certainly, the endpoints of the intervals formed at each stage are not removed in the construction process, so $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, etc. all belong to the set. As division by three is a fundamental part of the process, it is useful here to express the numbers in the ternary system. That is, we write every number x between 0 and 1 as $x = 0.a_1a_2a_3a_4\ldots$ where a_i can be 0, 1, or 2, and

$$x = \frac{a_1}{3} + \frac{a_2}{9} + \frac{a_3}{27} + \frac{a_4}{81} + \dots$$

This last formula tells us how to convert from the ternary to the decimal representation. To convert from decimal to ternary follow the example to obtain the ternary expansion for $0.3_{10} = 0.\overline{0220}_3$:

$$\begin{array}{ll} 0.3 \times 3 = 0.9 & \mathbf{0} \\ 0.9 \times 3 = 2.7 & \mathbf{2} \\ 0.7 \times 3 = 2.1 & \mathbf{2} \\ 0.1 \times 3 = 0.3 & \mathbf{0} \\ 0.3 \times 3 = 0.9 & \mathbf{0} \\ 0.9 \times 3 = 2.7 & \mathbf{2} \\ \vdots & \vdots \end{array}$$

We start by multiplying the fraction to be converted by 3, and record the integer part of the result, 0, in the second column. Next we multiply by 3 only the fractional part of the result of the previous multiplication. Again, we store the integer part of this result, 2, in the second column. We repeat this process four more times; a repeating pattern appears. The first six digits after the "ternary point" in the ternary expansion of the decimal appear, top to bottom, in bold in the second column. We indicate the repeating pattern with a line over the sequence of repeating digits.

In the process of generating the Cantor set the first interval to be removed is $(\frac{1}{3}, \frac{2}{3})$. Each number in this interval has in its ternary representation a 1 immediately following the "ternary point". (Can you prove this?) Similarly, the numbers removed at the second stage are those with a 1 in the second position to the right of the point in their ternary representation. The pattern repeats again and again; at the end, only numbers that are represented with 0's and 2's in the ternary system remain. For example, $0.7_{10} = 0.\overline{2002}_3$ belongs to the Cantor set, but $0.8_{10} = 0.\overline{2101}_3$ is removed at the second stage. Observe that some numbers have two ternary representations; for example, fractions of the form

$$\frac{1}{3^n} \quad \text{for } n = 1, 2, 3, \dots$$

can be represented by a terminating ternary number containing a 1, but they can also be represented by a non-terminating sequence of 0's and 2's:

$$\begin{aligned} (1/3)_{10} &= 0.1_3 &&= 0.0\overline{2}_3 \\ (1/9)_{10} &= 0.01_3 &&= 0.00\overline{2}_3 \\ (1/27)_{10} &= 0.001_3 &&= 0.000\overline{2}_3 \\ &\vdots \end{aligned}$$

Therefore 1/3, 1/9, 1/27, ..., all belong to the Cantor set.

The construction on the front cover is related to that of the Cantor set. The basic repetitive step is represented in Figure 2. Start with a disc and draw two smaller discs inside; then remove everything but the two smaller discs. We treat each of these two remaining discs as the original one: in each of them we draw two smaller discs and remove everything but those discs. Repeat this iterative process *ad infinitum*. The front cover shows the first four stages of the construction. In his book *Does God Play Dice?*, Ian Stewart calls this construction, for obvious reasons, the *Cantor cheese.*

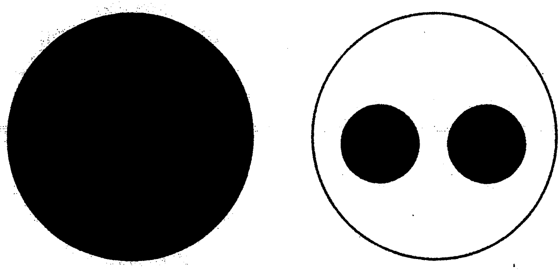

Figure 2

Although the visual representations of the two processes explained here seem to have very little in common, the two sets are what mathematicians call "topologically equivalent", but the proof of this is well beyond the level of this magazine!

* * * * *

HOW TO WORK OUT THE LENGTH OF A DAY WITHOUT LOOKING AT THE NEWSPAPER

Aidan Sudbury, Monash University

Everyone knows that the days are long in summer when the sun is high, and short in winter when the sun is low. What we'll do in this article is find a way to calculate how long the day is, given the date and the latitude. Although the result will not be 100% accurate, the fun is that I'm not going to use any detailed astronomical information; only that the earth is almost spherical, its orbit very nearly circular and that the tropics of Cancer and Capricorn are about 23.5° away from the equator. As I suspect you already know, the tropics are the latitudes at which the sun passes directly overhead on midsummer day.

In the northern summer the north pole is in permanent sunlight and the south pole in permanent darkness. In the southern summer the situation is reversed but, contrary to what you might think, this is not because the earth's axis wobbles, but rather because it does not.[1] A simple diagram (Figure 1) illustrates this.

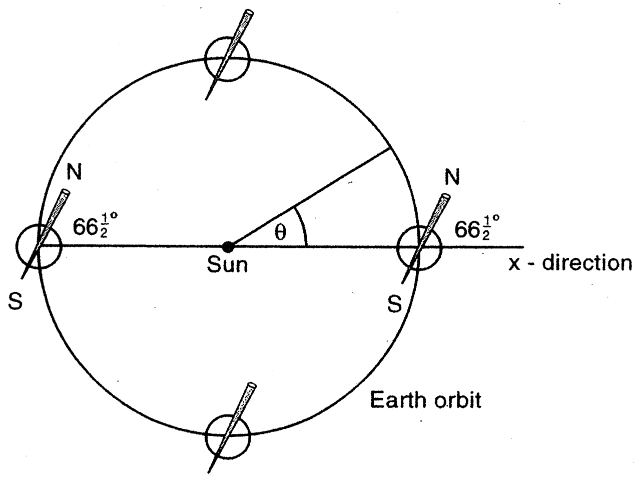

Figure 1

The needle shapes represent different positions of the earth and in particular its axis as it orbits the sun along an elliptical, but almost circular,

[1]Or rather, the small "wobble" that *does* exist may be ignored.

path. Begin with the earth in the position at the far right and proceed to follow the diagram in the anticlockwise sense. The respective positions shown depict the situation of the southern summer solstice (December 22nd), the southern autumn equinox (March 22nd), the southern winter (northern summer) solstice (June 22nd), and the southern spring equinox (September 22nd).

Note that the axis of the earth preserves its constant orientation in space with one pole towards the sun in summer, and away from it in winter.[2]

The earth's axis makes an angle of approximately 66.5° to the plane of its orbit, so that at the solstices the sun is overhead at midday at the latitudes 23.5°N (in June) and 23.5°S (in December), since

$$23.5^\circ + 66.5^\circ = 90^\circ.$$

We will use three-dimensional cartesian axes to represent our problem. The unit vectors **i**, **j**, and **k** give the directions of the x-, y-, and z-axes respectively. Any vector can be expressed as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ as shown in Figure 2.

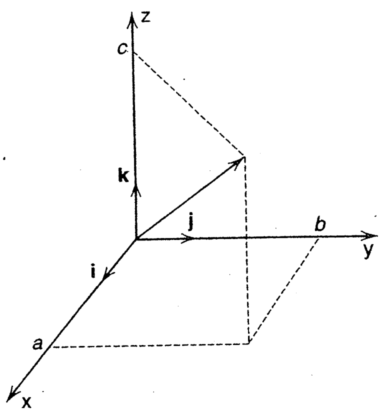

Figure 2

We take an origin at the sun, an x-axis as shown in Figure 1 (i.e. joining the sun to the earth's position at the December solstice) and a y-axis joining

[2]The fact that the orbit is not exactly circular results in the southern summer's being a few days shorter than the northern, but this is a minor complication which is here ignored.

the sun to the earth's position at the March equinox (i.e. up the page in Figure 1). Perpendicular to both of these axes (i.e. coming out of the page) is the z-axis.

The permanently fixed direction of the earth's axis in space is now given by the unit vector[3] $\mathbf{i}\cos 66.5° + \mathbf{k}\sin 66.5°$. At the southern summer solstice the direction from the sun to the earth is given by the unit vector $\mathbf{i}$. Over the next twelve months (365.25 days), the earth orbits 360° about the sun at a rate of about 30° a month or, more accurately, $360°/365.25 = 0.9856°$ per day.

Now we shall need the only bit of mathematics you may not know. Suppose $a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two unit vectors. Then the angle φ between them is given by

$$\cos\varphi = a_1a_2 + b_1b_2 + c_1c_2. \tag{1}$$

Now suppose the earth to be in a general position, in which the line joining the sun to the earth makes an angle θ with the x-axis, as shown in Figure 1. Then the direction from the sun to the earth is that of the unit vector

$$\mathbf{i}\cos\theta + \mathbf{j}\sin\theta.$$

We may now use equation (1) to find φ, the angle the earth's axis makes with the direction from the sun to the earth. We find

$$\cos\varphi = \cos 66.5° \cos\theta. \tag{2}$$

When the earth's axis makes an angle φ with the vector joining the sun to the earth, the sun will be overhead at midday at latitude α, where

$$\varphi + \alpha = 90°.$$

Because of this last equation, $\cos\varphi = \sin\alpha$. Furthermore, as the earth advances in its orbit, θ increases by about 30° per month or more accurately by 0.9856° per day. Thus if m months, or d days, have elapsed since December 22nd, then $\theta = (30m)° = (0.9856d)°$ and thus

$$\sin\alpha = \cos 66.5° \cos(30m)° = \cos 66.5° \cos(0.9856d)°. \tag{3}$$

As an example, take $m = 3$ (i.e. March 22nd). We now find $\alpha = 0$, which is to say that the sun is directly over the equator, as things should be at the equinox.

[3]A unit vector is a vector with length one.

Now consider a more general situation. I am writing this on November 22nd, one month (30 days) before the December solstice. So $m = -1$, $d = -30$. This gives us $\alpha = 20.3°$ (if we use the more accurate measure involving days).[4]

We may now use this figure to determine the length of the day. For this we need a new point of view and so a new set of axes. This time put the origin at the centre of the earth and let the x-axis connect this to that point of the equator for which it is noon. The y-axis also lies in the plane of the equator and makes an angle of 90° with the x-axis, while the z-axis is the earth's axis itself. See Figure 3. Set up unit vectors **i**, **j**, **k** along the x-, y-, z-axes respectively.

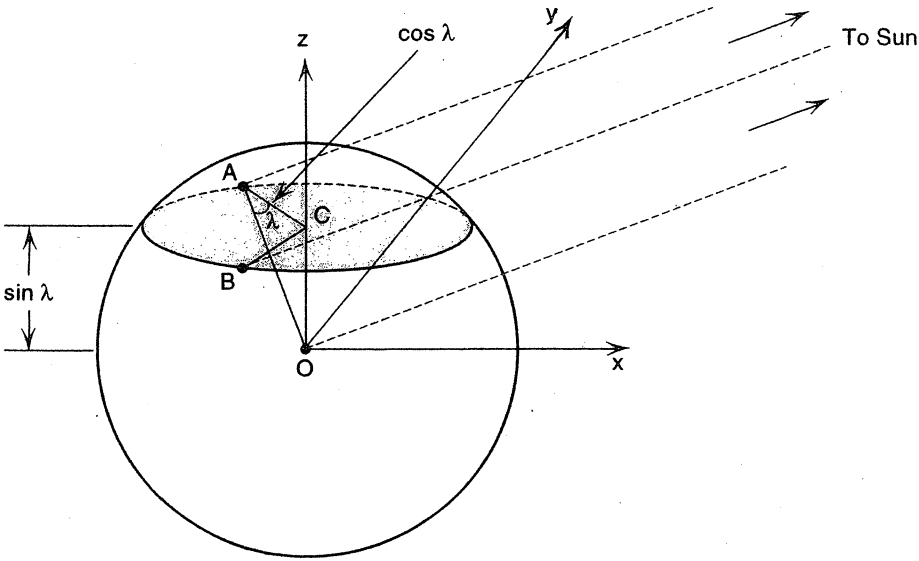

Figure 3

Suppose now that the sun is directly overhead at latitude α. The direction from the centre of the earth out to the sun is given by the unit vector

$$\mathbf{i}\cos\alpha + \mathbf{k}\sin\alpha.$$

[4]The latitude of Melbourne is 37.75° and so now we can determine the maximum height of the sun above the horizon for Melbourne on November 22nd. The result, which I leave you to justify for yourselves, is [90 − (37.75 − 20.30)]°, i.e. about 72.5°.

We now set out to calculate what proportion of the circle at latitude λ is illuminated at this moment.

Figure 3 displays a perspective rendering of the earth as the sun's rays shine on it. The latitude-circle joining all points of latitude λ is shown shaded. Its centre is on the z-axis at a point C, where $OC = \sin\lambda$ (where, for convenience, we have used the radius of the earth as our unit of length); the radius of the latitude-circle is $\cos\lambda$.

Figure 4 depicts the latitude-circle, shown "full-on". In both Figure 3 and Figure 4, A, B are the positions from which the sun is just visible; for one of them it is sunset, for the other, sunrise. The sun's rays are tangent to the earth at A and at B, so therefore the direction of the sun is perpendicular to the radii from the centre of the earth to A and to B. We can now calculate the position of A in terms of our new unit vectors.

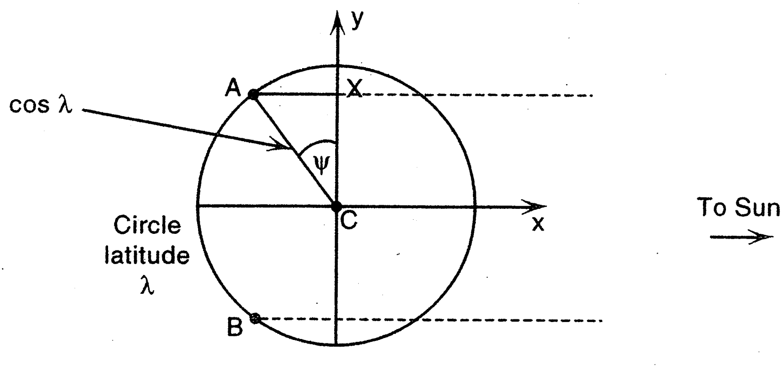

Figure 4

The y-coordinate of the point A (i.e. the distance CX) may now be seen to be $\cos\lambda\cos\psi$, where ψ is as shown. Similarly, we discover that the x-coordinate is $-\cos\lambda\sin\psi$, and we already have the z-coordinate as $\sin\lambda$. Thus the unit vector joining the centre of the earth to the point A is

$$-\mathbf{i}\cos\lambda\sin\psi + \mathbf{j}\cos\lambda\cos\psi + \mathbf{k}\sin\lambda.$$

This vector is to be perpendicular to the direction from the earth to the sun, which we previously found to be $\mathbf{i}\cos\alpha + \mathbf{k}\sin\alpha$.

Thus the angle φ between these two vectors is 90° and so the left-hand side of equation (1) is zero in this case[5]. Thus we find

$$-\cos\lambda\sin\psi\cos\alpha+\sin\lambda\sin\alpha=0. \tag{4}$$

The portion of the latitude circle that is illuminated is depicted in Figure 4 as that arc AB that is facing the sun. This occupies an angle of $180°+2\psi$ and so the *proportion* illuminated is $(180°+2\psi)/360°$.

We now express this as a number of hours, remembering that the earth rotates 360° every 24 hours. We thus find for the length of the day the expression $(12+2\psi/15)$ hours, if ψ is measured in degrees. All we need now is the value of this measure.

From equation (4),

$$\sin\psi=\tan\alpha\tan\lambda. \tag{5}$$

Equations (3) and (5) thus tell us that at latitude λ, d days after the summer solstice, the length of the day is approximately $12+2\psi/15$ *hours where*

$$\sin\psi=\tan\alpha\tan\lambda,\ \textit{with}\ \sin\alpha=\cos 66.5°\cos(0.9856d)°.$$

We may note several features of equation (5). First, if the sun is overhead in the opposite hemisphere (i.e. during wintertime), $\tan\alpha$ and $\tan\lambda$ will have opposite signs and thus $\tan\alpha\tan\lambda$ will be negative. Therefore ψ too will be negative and the day less than 12 hours long.

Secondly, if $|\tan\alpha\tan\lambda|>1$, we will be unable to solve for ψ. In fact when $\tan\alpha\tan\lambda>1$, latitude λ will be in permanent daylight, while if on the other hand $\tan\alpha\tan\lambda<-1$, latitude λ will be in permanent darkness.[6]

Finally note that at the equator $\lambda=0$, so that $\psi=0$ and the day is always exactly 12 hours long.

Now let's see how the theory works for Melbourne, where $\lambda=37.75°$S, and on November 22nd as before. We previously calculated that $\alpha=20.3°$, and so we have

$$\sin\psi=\tan 20.3°\tan 37.75°,$$

[5]It is left to the reader to prove that the vectors involved are unit vectors.

[6]If the sun is at α south latitude, then there is permanent daylight for latitudes further south than $(90-\alpha)°$S and permanent darkness for latitudes further north than $(90-\alpha)°$N. Conversely when the sun is in the north.

and so $\psi = 16.64°$ and the total length of the day is 14.22 hours or 14 hours and 13 minutes. In fact, on that day sunrise was at 5:56 a.m. and sunset at 8:16 p.m., giving a length of day of 14 hours and 20 minutes.

The slight discrepancy is in part due to the simplifying assumptions made. We have already alluded to the fact that the earth is not exactly spherical (and does "wobble" somewhat), nor is its orbit perfectly circular. The main source of error, however, is that the sun is not a point but has a definite size. Half the sun has already risen when its centre clears the horizon, and when the centre passes below the horizon, half the sun has yet to set. Thus the calculations given here are systematic underestimates. There are also complications arising from refraction in the earth's atmosphere.

At Melbourne, we have a difference of some five and a quarter hours between the length of daylight on the longest and the shortest days.[7] For higher latitudes this difference is even greater. In southern England (about 50°N latitude), for example, this difference is over 9 hours, and the higher the latitude the more pronounced the effect, until on the arctic and antarctic circles the difference is exactly 24 hours.

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Aidan Sudbury studied mathematics at Cambridge University, but decided he preferred philosophy. However, two years of philosophy convinced him that mathematics wasn't so bad after all and he came to Monash University to do a PhD in Astrophysics. Unable to find a job in astrophysics, he became a lecturer in statistics at Bristol even though he had never attended a course in the subject. He learnt some, however, and is now a Senior Lecturer at Monash.

* * * * *

[7]This is left as an exercise for the reader to calculate.

THRUN, FRON, FEEN, WUNTY

Michael A B Deakin

We, and indeed all contemporary cultures of advanced numeracy, use ten as the base of our counting system. Almost certainly this is because we have ten fingers, for there is nothing sacrosanct about the number ten itself. The Mayans (*Function*, *Vol* 12, *Part* 4) actually used a base of twenty and the Babylonians (*Function*, *Vol* 15, *Part* 3) a system usually described as being based on sixty. Otherwise, ten has reigned supreme.

However, from time to time there are calls for a "reform" of our number system by going over to some base other than ten. In *Function*, *Vol* 9, *Part* 1, I considered the competing claims of various bases[1] and concluded that ten is actually a very good base. The "reformers" mainly press the claims of base twelve on the grounds that twelve is rich in divisors, and so a duodecimal expansion of a fraction is more likely to terminate than is a decimal one. Indeed, there is a Duodecimal Society of America who lobby for the adoption of base twelve.

Less widely known are attempts to have us use base sixteen. This is known as the hexadecimal system.[2] In May 1964, Martin Gardner devoted his regular *Scientific American* column to the properties of base three, but mentioned two attempts to introduce base sixteen. The first was a highly eccentric 1862 attempt that I shall ignore here, but the second was a rather more sensible proposal.

In 1936, Joseph Bowden, a mathematician at Adelphi College (in New York State) published his book *Special Topics in Theoretical Arithmetic* and devoted Chapter 2 of this to his advocacy of base sixteen. The book is extremely rare and I have not managed to see a copy, although I almost succeeded some years ago. The University of Chicago, where once I studied, has one, and on a recent trip there I tried to get a look at it. However, they could not make it available to me within the very limited time I had and so, reluctantly, I had to forego the chance.

[1]This article was also reprinted in *Composite Function*, a collection of articles culled from *Function*'s first ten volumes and published by the Mathematical Association of Victoria.

[2]Strictly it should be "sexadecimal" as the main root is Latin, not Greek, but this risks confusion with "sexagesimal", referring to base sixty, and perhaps prudery has also played a part. The word "television" mixes Greek and Latin roots; Logie-Baird, who invented the gadget, wanted to call it "teleopsis" to avoid this complication, but wiser counsel prevailed.

What follows, therefore, is based on the limited information in Gardner's account, on deductions from that, on plausible extensions of those deductions and on some suggestions of my own. So what I describe is not Bowden's system but a reasonably close approximation to it.

It seemed that Bowden kept the names of the numbers more or less (see below) intact up to and including the word "twelve". "Thirteen", however, he must have seen as making reference to base ten[3] and so he rechristened it *thrun*. Similarly he substituted *fron* for "fourteen" and *feen* for "fifteen". "Sixteen" he called *wunty*.

From this last name I deduce that Bowden was one of those who also wished to reform at least the most obvious spelling anomalies and so wrote *wun* for "one". We may imagine him also writing *too* for "two", *ait* or perhaps *eit* for "eight", and possibly *fore* for "four". The other spellings he would probably have left intact.

Gardner shows his symbol for *feen* as a reversed numeral 5 (as shown in Table 1) and this device would also produce usable symbols for *twelve*, *thrun* and *fron*. However, this approach won't work for *ten* and is chancy at best with *eleven*. My own suggestion, though I doubt that Bowden would have used it, is to press into service two letters from the Cyrillic (Russian) alphabet and write *ten* as Ю ("yu") and *eleven* as И (the Cyrillic *i*) as these symbols remind us of the familiar base ten symbols.

My CASIO *fx*-570 will do some hexadecimal arithmetic, though it does not use any of this notation and it only works in integers. For the numbers ten, ..., fifteen (*feen*) it uses the symbols A, b, C, d, E, F respectively. (Can you see why the letters b, d are lower case?) Of course it has no need of names for these integers, but it is very good at converting integers from decimal (base ten) form to hexadecimal and *vice versa* and also for going between these bases and octal (base eight).

Because of the difficulty in reproducing Cyrillic characters and Bowden's symbols, this article will use the CASIO notation in what follows. The various names and symbols are, however, summarised below in Table 1.

[3]As a matter of linguistic record, the words "eleven" and "twelve" also relate to base ten (see *Function*, *Vol* 15, *Part* 5, p.153) but the connection is not an obvious one and Bowden may well have been unaware of it.

Number (base ten)	Bowden	CASIO
1	1	1
2	2	2
3	3	3
4	4	4
5	5	5
6	6	6
7	7	7
8	8	8
9	9	9
10	Ю	A
11	И	b
12	ƨ	C
13	Ɛ	d
14	ᔭ	E
15	ƽ	F
16	10	10
17	11	11
18	12	12

Table 1. The first eighteen natural numbers in Bowden's notation with my suggestions for ten and eleven. The final column gives the symbols from the CASIO *fx*-570 as used in the article.

It is not difficult to convert base ten numbers to base sixteen. E.g.

255 (base ten) $= 15 \times 16 + 15$ (base ten) $= F0 + F$ (base sixteen) $= FF$,

which is "feenty-feen". Conversely:

$$\begin{aligned} 6FF \text{ (base sixteen)} &= 6 \times 16^2 + 15 \times 16 + 15 \text{ (base ten)} \\ &= 1536 + 240 + 15 \text{ (base ten)} \\ &= 1791. \end{aligned}$$

We may also readily enough teach ourselves to count in base sixteen, and indeed my two daughters learned to do this while they were in primary school. They used to amuse themselves on long car journeys by counting up to large (or at any rate larg*ish*) numbers using this system. Table 2 shows how it goes.

wun	wunty-wun	tooty-wun	...	feenty-wun
too	wunty-too	tooty-too	...	feenty-too
three	wunty-three	tooty-three	...	feenty-three
fore	wunty-fore	tooty-fore	...	feenty-fore
five	wunty-five	tooty-five	...	feenty-five
six	wunty-six	tooty-six	...	feenty-six
seven	wunty-seven	tooty-seven	...	feenty-seven
ait	wunty-ait	tooty-ait	...	feenty-ait
nine	wunty-nine	tooty-nine	...	feenty-nine
ten	wunty-ten	tooty-ten	...	feenty-ten
eleven	wunty-eleven	tooty-eleven	...	feenty-eleven
twelve	wunty-twelve	tooty-twelve	...	feenty-twelve
thrun	wunty-thrun	tooty-thrun	...	feenty-thrun
fron	wunty-fron	tooty-fron	...	feenty-fron
feen	wunty-feen	tooty-feen	...	feenty-feen
wunty	tooty	threety	...	wundred

Table 2. Counting in base sixteen

The secret thrill of saying (e.g.) "fivety-eleven" and knowing it was all right really to do so and was not an admission of innumeracy gave them much pleasure.

Of course, once we get to *feenty-feen*, we need a new name for the next number (256 in base ten). I decided to call that *wundred*, and similarly named *toodred, threedred, ..., feendred.* This enables us to count to *feendred and feenty-feen* (4095 in base ten) before we need a new name. My choice for this was *wunsand*, which allows us similarly to count to *feensand, feendred and feenty-feen* (our 65535). The next name was never actually needed but I thought perhaps *wunllion*, though it is actually well short of a million (*wunty* times *wunllion* is just over a million). Beyond this no-one ventured.

Computers use at the very fundamental level binary (base two) arithmetic, for which the only symbols are 0 and 1. A single binary digit conveys the information entailed in one choice between exactly two possibilities. This unit of information is called a *bit*, which is short for **b**inary dig**it**. Eight bits of information make up a larger and more practical unit called a *byte*. One byte of information is that amount equivalent to pointing to one of 2^8, i.e. 256, numbers. That is to say, a byte is a single choice from the numbers wun to wundred (or zero to feenty-feen). A single hexadecimal digit contains half a byte of information, as wunty is the square root of wundred. However, there seems not to be a special word for this unit.

The connection between hexadecimal arithmetic and computing could clearly not have been known to Bowden writing in 1936. Were we to reform our number system (which of course we won't), this would now be the strongest argument in favour. However, Bowden must have relied on other points.

The strongest of Bowden's points concerns the hexadecimal expansions of reciprocals. These are shown in Table 3.

1/2	0.8	1/9	0.1C71C7...
1/3	0.555...	1/A	0.1999...
1/4	0.4	1/b	0.1745d1745d...
1/5	0.333...	1/C	0.1555...
1/6	0.2AAA...	1/d	0.13b13b...
1/7	0.249249...	1/E	0.1249249...
1/8	0.2	1/F	0.111...

Table 3. Reciprocals of the numbers two up to fifteen expressed as hexadecimal expansions

To see how these entries are generated, consider the second last. Writing this out in base ten, we have

$$\frac{1}{14} = \frac{1}{16} + \frac{2}{16^2} + \frac{4}{16^3} + \frac{9}{16^4} + \frac{2}{16^5} + \frac{4}{16^6} + \frac{9}{16^7} + \dots.$$

This equation may readily be checked by multiplying successively by 16, and this is in fact how it is derived also[4].

The worst expansion in this table is the five-digit *repetend* (i.e. the set of digits that recur) in 1/b. In fact there is nothing very dreadful after this until we encounter a nine-digit repetend in 1/13 (i.e. 1/19 in base ten).

This compares *very* favourably with base ten which has:

a six-digit repetend in the expansion of 1/7,
another six-digit repetend in the expansion of 1/13,
yet another six-digit repetend in the expansion of 1/14,
a sixteen(!)-digit repetend in the expansion of 1/17,
an *eighteen*-digit repetend in the expansion of 1/19.

[4]This is equivalent to the technique explained on page 67.

This is clearly much, much worse than base sixteen.

However, there are other considerations to be taken into account. In base ten we have easy rules for divisibility by the primes two, three, five and eleven and reasonably good ones for seven, thirteen, thirty-seven and one hundred and one. Hexadecimal gives easy rules for two, three and five and reasonably good ones for seven, thrun, wunty-wun, feenty-wun and wundred and wun. It misses out badly on eleven. If anything, base ten wins this round, not perhaps by a knockout, but very convincingly on points.

Finally, our base ten multiplication table, which caused so much misery in primary school to me and to many of my contemporaries, has only thirty-six non-trivial number facts to be learned. The hexadecimal table, by contrast, has a hundred and five. This would seem to be impractical. Bowden must surely have known of this difficulty, but perhaps in his day there was less sympathy for the plight of the primary student. Certainly he must have discounted the objection.

* * * * *

More on sundials

Ian Rae, formerly Dean of Science at Monash and now Deputy Vice-Chancellor at Victoria University, has sent us some further material on sundials. Monash's Clayton campus has a vertical "analemmic" sundial on the north wall of the Union building. This gives remarkably accurate readings of both time and date. We believe that in this combination of features it is unique. For more details, see *Function*, *Vol* 5, *Part 5*; *Vol* 14, *Part* 4; *Vol* 17, *Part* 4; and for a related item, *Vol* 18, *Part* 2.

Among the material Professor Rae sent there was an article from a recent issue of *Oxford Today*. This shows several sundials to be seen in various places around Oxford. One of these, the "Fellows' Quad Dial" at Merton College, is a vertically mounted dial giving both time and date. Time is however indicated by means of straight lines rather than the more accurate curves called "analemmas".

* * * * *

HISTORY OF MATHEMATICS

The Very Pure Mathematician

Michael A B Deakin

On the 7th of February 1877, a boy, Godfrey Harold, was born to Isaac and Sophia Hardy. Isaac was a school master and both he and his wife had a deep interest in mathematics, but neither of them had the chance to attend university themselves. However, they provided a warm and supportive climate in which the young boy was raised, and encouraged his obvious mathematical bent. This appeared at an early age and by the time he was thirteen he had won a scholarship to the famous Winchester College, one of England's top private schools and then as now regarded as particularly strong in mathematics.

From there he went on to Trinity College, Cambridge, which (also then as now) was seen as providing a firm education in mathematics. Actually, however, this reputation was not at that time entirely deserved. The final examination in mathematics was known as the *Tripos* and, by 1898 when the young Hardy sat it, it had become a rather stylised set of "drill exercises", and students were prepared for it by various tutors who were not unlike sporting coaches.

In point of fact this situation was symptomatic of the state of mathematics in England as a whole, for Trinity College was its most eminent institution of mathematical learning. Indeed, it set the tone for the practice of mathematics throughout the entire English-speaking world. The really good and exciting mathematics was being done in continental Europe at this time.

When Hardy first reached Cambridge, he was assigned to one of the best and most famous of the "coaches", one R R Webb. The routine and repetitive character of much of the work assigned to him bored and repelled him and he was eventually assigned to another supervisor, Professor A E H Love. This probably meant that he did less well than he might have done in the *Tripos*. (He only came fourth in this highly competitive exam.) But it gave him something much more valuable. Love suggested to him that he use as his textbook a French work, C Jordan's *Cours d'analyse*. This was not merely an undergraduate text in the usual sense, but rather a systematic exposition of calculus right up to very advanced material, recent research results, material available in fact in no English-language source at that time.

Thus the young Hardy was able to see real and exciting mathematics. "I shall never forget", he was later to write, "the astonishment with which I read that remarkable work, the first inspiration of many mathematicians of my generation, and learnt for the first time what mathematics really meant."

The mathematics that so excited the budding mathematician was "pure mathematics" in which the focus of interest lies in the mathematical objects themselves. These mathematical objects are abstract things like numbers, functions, integrals and the like. Pure mathematics does not envisage any particular application or even physical interpretation for its results. This situation contrasts sharply with "applied mathematics" in which mathematics (often advanced and difficult mathematics) is used, but used in the quest to understand some extra-mathematical situation arising from (e.g.) physics or perhaps economics, etc.

In the early years of the new century, Hardy began to produce research papers, mostly in the field of analysis (i.e. advanced calculus). However, he also wrote his own textbook *A Course of Pure Mathematics* during this period. This was a text designed to show to an English-speaking readership some of the insights and the results of the Continental mathematicians and to show the way in which *they* viewed the mathematical enterprise. Hardy's *Course* was the first thoroughgoing exposition in English of the precise concepts of function, limits, and other such important (pure) mathematical notions. These are now seen as basic to a full understanding of the calculus; before the careful examination of such fundamental ideas, much mathematical writing had been somewhat vague and imprecise.

Much of Hardy's later and best research work was the result of collaborative effort. Three of his collaborators deserve mention. The first was J E Littlewood, a somewhat younger man with interests very similar to Hardy's own. The two men worked together for many years, even when for eleven years Hardy went to Oxford while Littlewood remained in Cambridge. Their main results were in the fields of analysis and number theory, especially that branch of mathematics ("analytic number theory") where the two fields overlap.

This was a most fruitful collaboration, but it was of a type that characterises the professional life of many research mathematicians. The second such collaboration was anything but. It was early in 1913 that Hardy opened a letter that had arrived unexpectedly. It was of a mathematical nature and had some similarities with those letters that many mathematicians receive. Indeed, the editors of *Function* quite often get letters of a "crank" nature, or devoted to very specialist or to merely recreational topics. Hardy's first inclination was that the letter he received that morning was in some such vein. It had been mailed in India, and came from a poor, and in part self-educated, clerk in Madras.

His name was Srinivasa Ramanujan, and he was no crank; moreover, the results he sent Hardy were not trivial but instead quite profound. (It is probably no underestimate to say that the chances of such a letter's being really significant are less than one in a million!) The results were unproved, but they sufficiently excited Hardy that he raised money to bring Ramanujan to England in 1914. The Indian had a magnificent intuitive grasp of mathematics that filled Hardy with admiration and awe. All this despite his unusual education having left him with a very poor overall grasp of many basic aspects of mathematics. The two continued to collaborate for the remainder of Ramanujan's life, each supplying aspects of expertise that the other lacked. Sadly, Ramanujan's life was a short one; he took ill in 1917, returned home sick in 1919 and died in 1920. (One tends to forget how recent all this really is; I was in India last year attending a meeting of the Indian Mathematical Society when the news came through of the death of Ramanujan's widow.)[1]

One story of Ramanujan's life is, however, well known and bears repetition. During his illness he was at one stage hospitalised and Hardy came to visit him, taking a taxi to do so. Back in those early days of motoring, taxis had relatively simple number-plates and that of this particular one was 1729. Hardy, recounting this fact to his sick friend, remarked that this did not seem a particularly interesting number. Ramanujan contradicted him, stating that 1729 is indeed a very interesting number, being the smallest number that can, in two different ways, be represented as the sum of two cubes. What Ramanujan had in mind was that

$$1729 = 12^3 + 1^3 = 10^3 + 9^3$$

and that no smaller number can be broken down like this in two different ways. The claim is correct and impressed Hardy greatly. On the other hand, had the taxi had any other number, Ramanujan could probably

[1]For more on Ramanujan, see *Function*, *Vol* 1, *Part* 3.

have come up with some other such remark. It was said of him that every one of the natural numbers was his personal friend.

The third collaborator was George Pólya, a Hungarian mathematician, whose popular books I can recommend to *Function* readers. His *How to solve it* and *Mathematics and Plausible Reasoning* are both good reads. Not only are they full of good advice for both teachers and students, but they also convey much of the essence of mathematical thought. Pólya and Hardy enjoyed a great friendship but were not above a bit of good-natured banter. Once Hardy visited the Stockholm zoo in company with the mathematician Marcel Riesz. Pólya got to hear of the episode, and here is his version. "In a cage there was a bear. The cage had a gate, and on the gate there was a lock. The bear sniffed at the lock, hit it with his paw, then he growled a little, turned around and walked away. 'He is like Pólya', said Hardy. 'He has excellent ideas, but does not carry them out.' "

Hardy, Littlewood and Pólya all collaborated on a very great textbook, *Inequalities*, that appeared in 1934. This text remains an excellent reference to this day; in its own time it was the first really systematic exposition of its subject, and it included results that up till then were only known in the very technical literature.

Everyone who knew Hardy speaks of his habit of joking, almost in the same breath as they tell of his deep love of mathematics and his abiding concern for it. The two aspects of his personality could and indeed did become intertwined. An example is the title and the topic of Hardy's last book, completed shortly before his death late in 1947, and published posthumously the following year, *Divergent Series*.

Infinite series (sums of terms) are usually classified as being either "convergent" or "divergent". Convergent series are sums like

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$$

while divergent series are like either

$$1+2+4+8+16+\dots$$

or else

$$1-1+1-1+1-\dots$$

In the first of these examples, as we take more and more terms we seem to be approaching closer and closer to a sum, in this case 2; with the others, on the one hand the sum seems to run away from us, on the other to oscillate back and forth between 1 and 0.

So the view grew up that convergent series were somehow "good" and "useful", while divergent series were "bad" or "useless". Thus to write a book on the latter seemed a little perverse or slightly shocking. In point of fact, however, the simple perception is not accurate. Very often, sums *can* be assigned to divergent series, and these sums can be most useful computationally. It is just that the mathematics of these cases is harder than that for the simpler convergent series.

Divergent Series was not the first of Hardy's books to be given a provocative title. He had earlier produced a short autobiographical work, *A Mathematician's Apology.* This is a lovely book to read, and tells us a lot about mathematics and what it meant to Hardy. On first hearing its title, however, one might be inclined to think that Hardy had some regrets about mathematics, that he was in some way "apologising" for being a mathematician. This is far from being the case. The word "apology" has an older meaning, one more closely related to its Latin origins, that approximates to "justification", "vindication" or perhaps "explanation".

So while Hardy was undoubtedly conscious of the initial impact his title would have, he was in fact concerned to vindicate and indeed to celebrate the mathematical endeavour. He was also at some pains to sort out those characteristics that make for "good mathematics" as opposed to the trivial or the merely incidental.

As examples of "good mathematics" incorporating important insights and creative ideas, he quotes two theorems. One is the "evenness-oddness" proof of the irrationality of $\sqrt{2}$. (See *Function, Vol* 17, *Part* 5, p. 145.) The other is Euclid's proof that there are infinitely many primes. This proceeds as follows.

If there were only *finitely* many primes, $p_1, p_2, p_3, \ldots, p_n$, say, then we could list them in order of size, i.e. $p_1 < p_2 < p_3 < \ldots < p_n$. Furthermore, we could form the number $p_1 p_2 p_3 \ldots p_n + 1$ ($= N$, say). This new number is clearly larger than any of the primes on our list, and thus larger than the "largest prime", p_n. Now let us ask about the factors of N.

Now p_1 cannot divide N, because N is larger by 1 than an exact multiple of p_1. (In fact, $p_1 = 2$, and N is odd.) Similarly p_2 cannot divide N, nor can p_3, etc. Thus N must either itself be prime or else be divisible by primes not on our list, primes larger than p_n. Hence our assumption that there were only n primes has contradicted itself and so is inconsistent. It follows that there must be infinitely many primes. (For more on this *method* of argument, see *Function*, *Vol* 17, *Part* 3, pp. 83-88.)

Hardy was at some pains to distinguish such mathematics from other less worthy deductions. As an example of the latter he lists the result that 153, 370, 371 and 407 are (as we could now say) the only Armstrong numbers of order 3. (See *Function*, *Vol* 17, *Part* 5 and *Vol* 16, *Part* 4.)

Hardy was very much concerned that the valuable, the deep, the unexpected, the beautiful aspects of mathematics be appreciated in their own (pure mathematical) right. He had very little truck with the notion that mathematics had to be useful, to be practically, socially or economically applicable or relevant.

Hardy expressed his preference by contrasting two writers: Whitehead and Hogben. Whitehead had been a collaborator of Bertrand Russell (in fact, earlier than that, his teacher) and had conducted much front-line research in mathematics before turning his attention to philosophy and to popular writing. Hogben was not a research mathematician but was a populariser of some renown (and indeed of considerably more insight than Hardy allowed to him) whose background was Marxist and who saw mathematics as being validated in the main by its social utility.

He saw these two writers as exemplifying "two [different] mathematics", the one (Whitehead's and Hardy's also) real, the other (Hogben's) trivial. But Hardy's standards transcended social utility. "I have never", he wrote, "done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. ... The case for my life, then, or for that of any one else who has been a mathematician in the same sense as I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of the other artists, great or small, who have left some kind of memorial behind them."

Hardy is too modest to include himself among the "great mathematicians" but many would place him in their midst. There is, however, a further irony in this passage. For Hardy did once write an "applied" paper; it was a contribution to the field of genetics and is now seen as providing a crucial insight.

The question had been asked in particular of a genetic malformation known as brachydactyly – stubby fingers and toes seemingly lacking one of their joints – whether or not it would be likely to spread in the community. By 1908, when Hardy was writing, Mendel's laws were known, and it was also known that brachydactyly was inherited as a "simple Mendelian dominant". That is to say that the relevant gene comes in two forms: A and a. If a person inherits the A form from *either* parent, then that individual will exhibit the deformity. Only if *both* parents contribute the a form of the gene will the child be normal.

Hardy supposed that the three possible combinations of the genes (AA, Aa and aa) were distributed in the community in the proportions (respectively) of $p : 2q : r$. That is to say, a fraction p of the population were genetically AA, etc. He went on to make a number of relatively straightforward further assumptions and then calculated the frequencies of the three types in the next generation, using "a little mathematics of the multiplication-table type". The proportions turn out to be $p_1 : 2q_1 : r_1$, where $p_1 = (p+q)^2$, $q_1 = (p+q)(q+r)$ and $r_1 = (q+r)^2$.

He then goes on: "The interesting question is – in what circumstances will this distribution be the same as that in the generation before? It is easy to see that the condition for this is $q^2 = pr$. And since $q_1^2 = p_1 r_1$, whatever the values of p, q and r may be, the distribution will in any case remain unchanged after the second generation."[2]

This same conclusion was reached independently in the same year by the German researcher W Weinberg, and is today known as the "Hardy-Weinberg Law"; it is regarded as one of the fundamental principles of theoretical genetics.

So Hardy's work was in fact *not* all theory and non-utilitarian. However, now that the ethic of practicality and greed once again rides high, we would do well to look to those other values which he held so dear and which he expressed so eloquently.

Further Reading. The best place to start on the very much more that could be said is with Hardy's *A Mathematician's Apology* itself. Read the second edition (Cambridge University Press, 1967) which contains a useful biographical sketch by C P Snow, who in fact had been shown the original manuscript and had suggested changes to it, changes that for the most part Hardy adopted. Beyond this, the entry in the *Dictionary of Scientific Biography* and the references listed there provide much further material.

[2]Readers may like to duplicate the calculation for themselves. It is not difficult.

COMPUTERS AND COMPUTING

Creating Three-Dimensional Graphs

Cristina Varsavsky

Functions are truly fundamental in mathematics. Often they are given in the form of a mathematical formula where one variable, usually called y, is expressed in terms of another variable, x. We use a pair of cartesian axes to represent graphically the x-y relationship expressed in the formula: the graph consists of the set of points (x, y) in the plane where x is in the "domain" of the function and y is the corresponding value for x calculated with the formula. You are probably already familiar with linear functions (straight lines), quadratic functions (parabolas), cubic functions, trigonometric functions, and perhaps also exponential and logarithmic functions. They are all useful for modelling physical situations.

In some instances modelling needs to be done with functions of more than one variable. For example, we can think of the volume of a cylinder as a function of two variables: the radius of the circular base and the height. The corresponding formula is

$$Volume = \pi \times radius^2 \times height.$$

One way of representing graphically a two-variable function is by using a set of three cartesian axes: two for the variables, say x- and y-axes, and a third one, the z-axis, for the value of the function at each point (x, y). The graph of the function will consist of triples (x, y, z) which will form a surface in three dimensions. Figure 1 shows one such triple, where the independent values are represented as the point $(x, y, 0)$ on the xy-plane and the corresponding point on the surface (x, y, z) is directly above it at a distance z.

Plotting two-variable functions is a bit harder than plotting one-variable ones. We have to represent a surface in three dimensions on a two-dimensional piece of paper or computer screen. In this article we will present a simple program for plotting two-variable functions, a program which could be a starting point for you to do more sophisticated plots.

Say we want to plot $z = \cos x \cos y$ on the computer screen. By giving values to x and y we can find some points on the graph of the function. For example $(0, 0, 1)$, $(\pi/2, 3, 0)$, $(0, \pi, -1)$ all belong to the surface; but we have to find a method for searching for a selection of points that would give us a good visual idea about the shape of the surface. One way is to plot the points on the graph of the function corresponding to points on the x-y plane lying on lines parallel to the x- and y-axes. In other words, for lines parallel to the y-axis, we keep the x-value constant and take equally spaced y-values to generate a curve – or rather points of that curve – on the surface. This is called a *cross-sectional plot* as we plot curves which are the result of the intersection between the surface and planes parallel to the z-y plane.

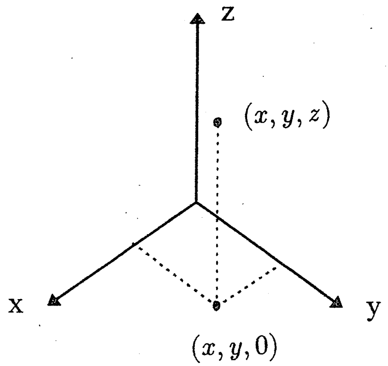

Figure 1

We will design a program for obtaining the cross-sectional plot for the function $z = \cos x \cos y$ for $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. As usual, we will do this with QuickBasic; if you prefer another programming language you will need to make the appropriate "translation".

First we convert the screen into a cartesian plane, using the WINDOW command. Second, for each pair of x and y values we need to find the corresponding coordinates, A and B on the screen; we also calculate the z-value for that pair. Third, the corresponding point on the screen will have coordinates $(A, B + z)$; we plot this point.

The second step is the one that needs most attention. Look at Figure 2. The angle between each pair of axes is 120°. The pair of axes x and y are the transformations of the horizontal and vertical axes of the screen (dotted lines). Therefore for each pair (x, y), the corresponding position (A, B) on the screen is given by the following equations:

$$A = -x\cos(30°) + y\cos(30°)\ ,\ B = -x\sin(30°) - y\sin(30°)$$

That is,

$$A = -\frac{\sqrt{3}}{2}(x - y) \ , \ B = -\frac{1}{2}(x + y)$$

Transformations of this type were explained in *Function, Vol* 18, pp. 147-153.

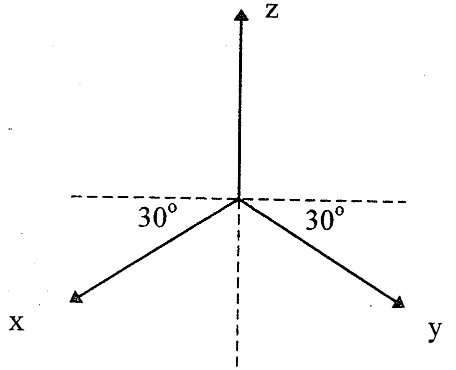

Figure 2

Now that we have observed this, the program is very simple:

```
SCREEN 9
WINDOW (-6, -4)-(6, 4)

REM Draw the axes
    LINE (0, 0)-(-2*SQR(3), -2)
    LINE (0, 0)-(2*SQR(3), -2)
    LINE (0, 0)-(0, 4)

FOR x = 3 TO -3 STEP -.1
    FOR y = 3 TO -3 STEP -.1
        A = -SQR(3)/2*(x-y)
        B = -1/2*(x+y)
        z = COS(x)*COS(y)
        PSET (A, B + z)
    NEXT y
NEXT x

END
```

The output of the program is shown in Figure 3.

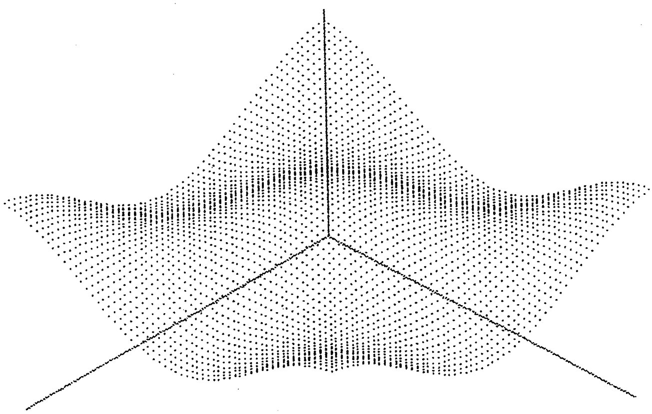

Figure 3

You could modify and improve this program in several ways:

- Change the perspective from which you plot the surface (i.e. change the transformation from (x,y) to (A,B)).
- Put scales on the axes.
- Change the direction of the cross-sections.
- Add colour.

And of course, there are many functions that would produce really nice surfaces. Try the following:

- $z = -2x^2 - y^2$
- $z = e^{-x^2-y^2}$
- $z = \sin(xy)$

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 19.1.1 (Originally from the *New Zealand Mathematics Magazine*, 30(1), 1993; reproduced in *Mathematical Digest*, 94, January 1994, University of Cape Town)

It is possible to choose four lattice points in the plane (i.e. points (x, y) for which x and y are both integers), and connect each of these points to each of the other three points with a line segment, in such a way that none of the line segments passes through any lattice point.

Suppose we now choose *five* lattice points in the plane, and connect each of these points to each of the other points with a line segment. Prove or disprove: there must be at least one line segment that goes through a lattice point.

SOLUTION

The set of all lattice points in the plane can be partitioned into four disjoint subsets as follows:

$$\begin{aligned} S_1 &= \{(x,y) : x \text{ and } y \text{ are both even}\} \\ S_2 &= \{(x,y) : x \text{ is even and } y \text{ is odd}\} \\ S_3 &= \{(x,y) : x \text{ is odd and } y \text{ is even}\} \\ S_4 &= \{(x,y) : x \text{ and } y \text{ are both odd}\} \end{aligned}$$

If five lattice points are chosen, then, by the pigeonhole principle, at least one of these four sets must contain at least two of the chosen points. For two points (x_1, y_1) and (x_2, y_2) in any one of the sets, the x coordinates have the same parity (both even or both odd), as do the y coordinates. Therefore the point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ halfway between the two points is a lattice point, and this point is on the line segment joining the two points. Hence there must be at least one line segment that goes through a lattice point.

PROBLEM 19.1.2 (from IX Mathematics Olympiad "Thales", in *Epsilon*, 26, 1993, p.107)

What is the minimum number of cubes required to construct the building in Figure 1?

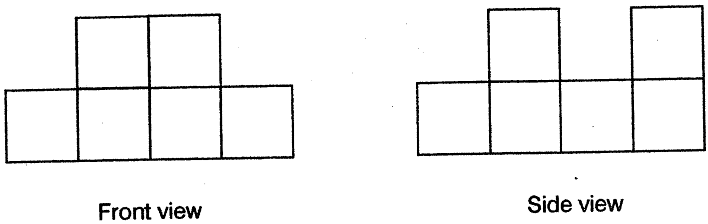

Figure 1

SOLUTION

There are four different ways of arranging six cubes (two "towers" of two cubes each, and two other cubes) to give the front and side views shown, but none of them qualifies as a single building. With a seventh cube, a building can be constructed in two ways, one of which is shown in plan view in Figure 2. (The other is its mirror image with respect to a vertical axis through the centre of the diagram.)

1			
	2		
		1	1
		2	

Figure 2

PROBLEM 19.1.3 (K R S Sastry, Dodballapur, India)

$ABCDE$ is a convex pentagon in which each diagonal is parallel to a side. The sizes of the angles EAB, ABC, BCD, CDE and DEA form an increasing arithmetic progression. Show that the angles BCA, ACE and ECD are also in arithmetic progression.

SOLUTION by K R S Sastry

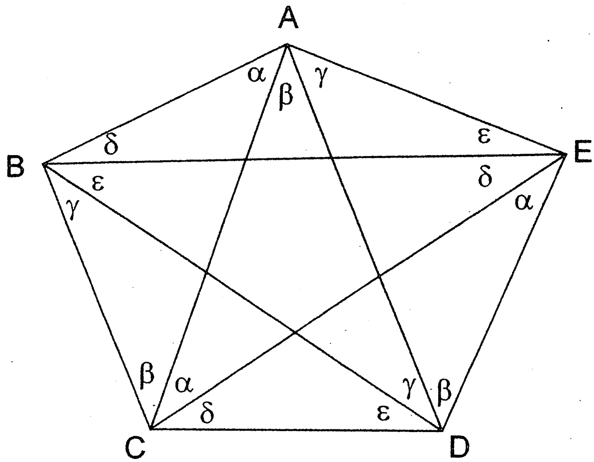

Figure 3

Let the angles $\alpha, \beta, \gamma, \delta, \varepsilon$ be as shown in Figure 3. Let d be the common difference in the arithmetic progression of the angles EAB, ABC, BCD, CDE, DEA of the pentagon. Then:

$$(BCD - 2d) + (BCD - d) + BCD + (BCD + d) + (BCD + 2d) = 540^\circ$$

so $BCD = 108^\circ$. Therefore:

$$\begin{aligned} EAB &= \alpha + \beta + \gamma = 108^\circ - 2d \\ ABC &= \gamma + \delta + \varepsilon = 108^\circ - d \\ BCD &= \alpha + \beta + \delta = 108^\circ \\ CDE &= \beta + \gamma + \varepsilon = 108^\circ + d \\ DEA &= \alpha + \delta + \varepsilon = 108^\circ + 2d \end{aligned}$$

Then:

$$216^\circ = 2\alpha + 2\beta + 2\delta = (\alpha + \beta + \gamma) + (\alpha + \delta + \varepsilon) = (\gamma + \delta + \varepsilon) + (\beta + \gamma + \varepsilon)$$

Hence $\beta + \delta = \gamma + \varepsilon$ and $2\alpha = \gamma + \varepsilon$, so $216^\circ = 2\alpha + 2(\beta + \delta) = 2\alpha + 2(\gamma + \varepsilon) = 6\alpha$ and thus $\alpha = 36^\circ$ and $\beta + \delta = 72^\circ$. Therefore $BCA = \beta, ACE = \alpha = 36^\circ$, and $ECD = \delta = 72^\circ - \beta$; so these angles are in arithmetic progression with a common difference of $36^\circ - \beta$.

This proof answers the problem as it was stated. However, it turns out that rather more can be proved. P U A Grossman (Nunawading, Vic) writes:

"So far so good. We have proved a property for a class of geometrical figures, namely pentagons with diagonals parallel to the sides and with angles increasing in arithmetic progression. However, on closer examination it transpires that this class is so closely limited by its definition that its only members are regular pentagons."

An outline of the proof follows; we leave it to the reader to complete the details.

From the equations in Sastry's proof, we can deduce that $\alpha = 36°$, $\beta = 36° + d$, $\gamma = 36° - 3d$, $\delta = 36° - d$, and $\varepsilon = 36° + 3d$.

Suppose $d \neq 0$. We will assume $d > 0$; the reasoning is similar if $d < 0$. Clearly $d < 12°$. Applying the sine rule to the triangle ADE, we obtain:

$$DE = \frac{AE \sin\gamma}{\sin\beta}$$

Since $\gamma < \beta$ and the angles are less than 90°, we must have $\sin\gamma < \sin\beta$ and hence $DE < AE$. A similar argument involving the angles δ and ε and the triangle ABE yields the result $AB > AE$. Therefore $AB > DE$.

Let h be the distance between the (parallel) lines AE and BD. Then $h = AB\sin(\delta + \varepsilon)$ and $h = DE\sin(\beta + \gamma)$. Therefore $AB\sin(\delta + \varepsilon) = DE\sin(\beta + \gamma)$, so $AB\sin(72° + 2d) = DE\sin(72° - 2d)$. But this is a contradiction, because $AB > DE$ and (with the restrictions on the range of d) $\sin(72° + 2d) > \sin(72° - 2d)$. Therefore $d = 0$.

It follows immediately that $\alpha = \beta = \gamma = \delta = \varepsilon = 36°$, and it is then a straightforward matter to show that the side lengths must also be equal. Hence the pentagon is regular.

Solution to an earlier problem

We continue our series of solutions to problems from earlier issues for which a solution has not previously been published.

PROBLEM 16.2.3

If p is a prime number, s the sum of the digits when N is expressed as a base-p numeral and h is the highest power of p contained (as a factor) in $N!$, prove that $h = \frac{N-s}{p-1}$.

SOLUTION

Let the base-p representation of N be "$d_k d_{k-1} \ldots d_0$", so that $N = d_k p^k + d_{k-1}p^{k-1} + \ldots + d_0$. The number h is the number of the factors $N, N-1, N-2, \ldots, 1$ of $N!$ divisible by p, plus the number divisible by p^2, plus the number divisible by p^3, and so on. Thus h is the number of positive multiples of p not exceeding N, plus the number of positive multiples of p^2 not exceeding N, and so on. Now, $N - d_0$ is the largest multiple of p not exceeding N, so the number of positive multiples of p that are less than or equal to N is $\frac{N-d_0}{p}$. By a similar argument, the number of positive multiples of p^2 that are less than or equal to N is $\frac{N-d_0-d_1p}{p^2}$. Continuing in this way, we obtain expressions for the number of positive multiples of $p^3, \ldots, p^{k+1}$ that are less than or equal to N. Therefore h is the sum of these expressions:

$$\begin{aligned} h &= \frac{N-d_0}{p} + \frac{N-d_0-d_1p}{p^2} + \frac{N-d_0-d_1p-d_2p^2}{p^3} + \ldots + \\ &\quad \frac{N-d_0-d_1p-d_2p^2-\ldots-d_kp^k}{p^{k+1}} \\ &= N\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots+\frac{1}{p^{k+1}}\right) - d_0\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots+\frac{1}{p^{k+1}}\right) \\ &\quad -d_1\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots+\frac{1}{p^k}\right) \\ &\quad -d_2\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots+\frac{1}{p^{k-1}}\right) \\ &\quad -\ldots-d_k\left(\frac{1}{p}\right) \end{aligned}$$

Applying the formula for the sum of a finite geometric series, we obtain:

$$\begin{aligned} h &= \frac{N(1-p^{-k-1})}{p(1-1/p)} - \frac{d_0(1-p^{-k-1})}{p(1-1/p)} - \frac{d_1(1-p^{-k})}{p(1-1/p)} - \frac{d_2(1-p^{-k+1})}{p(1-1/p)} \\ &\quad -\ldots-\frac{d_k(1-p^{-1})}{p(1-1/p)} \\ &= \frac{1}{p-1}\left[(N-d_0-d_1-d_2-\ldots-d_k) - Np^{-k-1} + d_0p^{-k-1} + d_1p^{-k} + \right. \\ &\quad \left. d_2p^{-k+1} + \ldots + d_kp^{-1}\right] \\ &= \frac{1}{p-1}\left[N - s - Np^{-k-1} + p^{-k-1}(d_0 + d_1p + d_2p^2 + \ldots + d_kp^k)\right] \end{aligned}$$

$$= \frac{1}{p-1}\left(N - s - Np^{-k-1} + Np^{-k-1}\right)$$
$$= \frac{N-s}{p-1}.$$

PROBLEMS

PROBLEM 19.3.1

You have access to an n-storey building, and two eggs. Assume there is a *critical storey*, such that an egg dropped from that storey or higher will break, but an egg dropped from any lower storey will not. (We will also admit the possibility that an egg dropped from the top storey does not break, in which case we will say that $n+1$ is the critical storey.) A *drop* consists of choosing a storey and dropping an egg from it. Your task is to find the critical storey by performing a sequence of drops. Of course, if an egg survives a drop intact then it can be retrieved and used in another drop.

(a) If $n = 10$, what strategy should you adopt in order to ensure that you will find the critical storey in at most four drops?

(b) For arbitrary n, find the smallest number k (as a function of n) such that the critical storey can always be found in at most k drops.

PROBLEM 19.3.2 (posted on the Internet by Ian Hawthorn, University of Waikato, New Zealand)

Let a, b, c be the lengths of the sides of a triangle, let A be the area of the triangle, and let R be the radius of the circumcircle (the circle passing through the vertices). Prove that $abc = 4AR$.

Ian Hawthorn also wondered whether there is a similar formula for tetrahedra. Some readers might enjoy tackling this more difficult problem.

PROBLEM 19.3.3 (from *Mathematics and Informatics Quarterly*, 4/93)

Find all non-negative integral solutions of $3 \times 2^m + 1 = n^2$.

* * * * *

OLYMPIAD NEWS

Hans Lausch

1. The Seventh Asian Pacific Mathematics Olympiad

The Asian Pacific Mathematics Olympiad (APMO), an annual competition, was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the number of participating Pacific Rim countries has grown to fifteen. Moreover, Argentina and Trinidad and Tobago were using the contest questions for their national competitions. In Australia, 29 students sat this four-hour examination on 14 March:

Question 1. Determine all sequences of real numbers $a_1, a_2, \ldots, a_{1995}$ which satisfy

$$2\sqrt{a_n - (n-1)} \geq a_{n+1} - (n-1), \text{ for } n = 1, 2, \ldots, 1994,$$

and

$$2\sqrt{a_{1995} - 1994} \geq a_1 + 1.$$

Question 2. Let $a_1, a_2, \ldots, a_n$ be a sequence of integers with values between 2 and 1995 such that:

(i) any two of the a_i's are relatively prime.

(ii) each a_i is either a prime or a product of different primes.

Determine the smallest possible value of n to make sure that the sequence will contain a prime number.

Question 3. Let $PQRS$ be a cyclic quadrilateral (i.e. P, Q, R, S all lie on a circle) such that the segments PQ and RS are not parallel. Consider the set of circles through P and Q, and the set of circles through R and S. Determine the set A of points of tangency of circles in these two sets.

Question 4. Let C be a circle with radius R and centre O, and S a fixed point in the interior of C. Let AA' and BB' be perpendicular chords through S. Consider the rectangles $SAMB, SBN'A', SA'M'B'$, and $SB'NA$. Find the set of all points M, N', M', and N when A moves around the whole circle.

Question 5. Find the minimum positive integer k such that there exists a function f from the set Z of all integers to $\{1, 2, \ldots, k\}$ with the property that $f(x) \neq f(y)$ whenever $|x - y| \in \{5, 7, 12\}$.

2. Australians at the XXXVI International Mathematical Olympiad

The performance of students at the APMO was used in selecting eleven candidates for the team, which is to represent Australia at this year's International Mathematical Olympiad (IMO). Also thirteen highly gifted students, having at least one more year of secondary education ahead of them, were singled out for further training.

These 24 students participated in the ten-day Team Selection School of the Australian Mathematical Olympiad Committee. Following a tradition, the School was held in Sydney. Participants had to undergo a day-and-evening programme consisting of tests and examinations, problem sessions and lectures by mathematicians. Finally, the 1995 Australian IMO Team was selected.

Toronto is the venue of the XXXVI IMO scheduled for July. There the Australian team will have to contend with six problems during 9 hours spread equally over two days in succession. The following students, all in year twelve, were selected for the Australian team:

Christopher Barber, Carine Senior High School, Western Australia
Gordon Deane, Kenmore High School, Queensland
Jian He, University High School, Victoria
Nigel Tao, Westminster School, South Australia
Trevor Tao, Brighton Secondary School, South Australia
Zhi Ren Xu, Canterbury Boys' High School, New South Wales.

Reserve:

Ben Lin, North Sydney Boys' High School, New South Wales.

Good luck to them all!

* * * * *

Correction: Despite our best efforts to make each issue of *Function* as free of errors as possible, mistakes and misprints still occur. In the previous issue of *Function* the name of the great mathematician Archimedes was misspelt twice. Our apologies if this caused confusion to the readers.

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