Function

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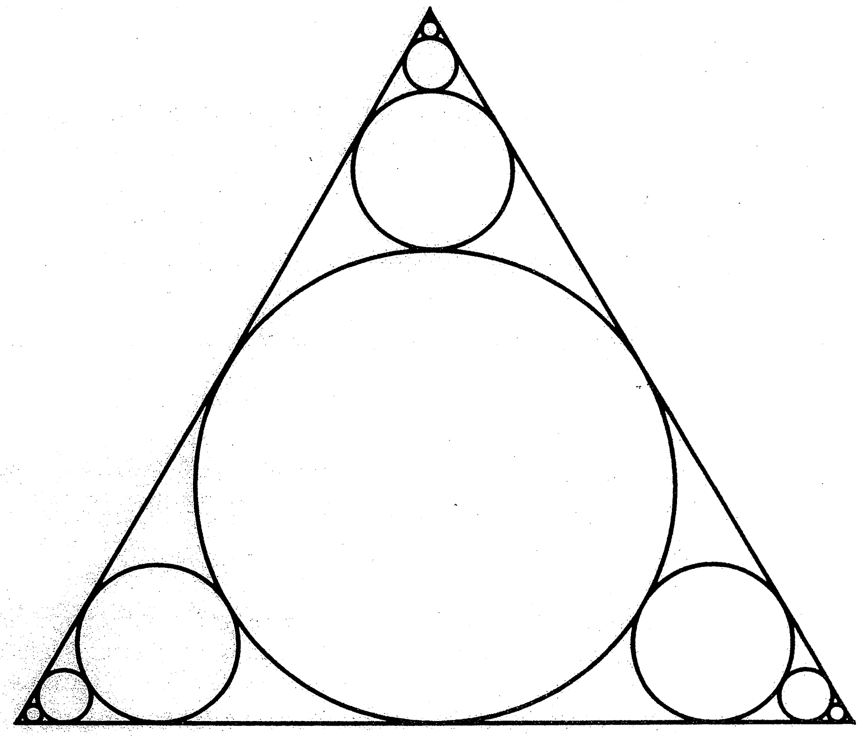

Mathematics Department - Monash University

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Function is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors, *Function*
Department of Mathematics, Monash University
900 Dandenong Rd
Caulfield East VIC 3145, Australia
Fax: +61 (03) 9903 2227
e-mail: function@maths.monash.edu.au

Alternatively, correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

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EDITORIAL

Welcome to this fourth issue of *Function* for this year.

The three feature articles included in this issue cover different areas of mathematics: number theory, geometry, and logic. Michael Deakin explains the patterns of certain sequences originating in irrational numbers and uses one of them to present a winning strategy for a Nim-type game. R Soemantri, extending an earlier article, looks at the design of a road that would allow a polygonal wheel to roll along, keeping its centre on a level. Peter Grossman takes us to the world of logic to solve a problem in a community of monks.

The topics in our regular *History* and *Computers and Computing* sections are related. Paul Grossman gives a recollection of the different ways people performed arithmetic calculations before calculators and computers became available. Michael Deakin focuses on one necessary skill before the advent of calculators: he explains some simple calculating techniques for mental arithmetic used by ordinary people, and other more sophisticated ones developed by some gifted mathematicians.

Peter Grossman has prepared a selection of new problems to challenge your mind. Solutions to the problems published in the April issue are also included.

As usual, we encourage our readers to send letters, solutions, problems, and articles, and we thank all who have made contributions to *Function* in one of those forms.

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THE FRONT COVER

Converging Circles

Cristina Varsavsky

The geometric construction on the front cover consists of an equilateral triangle with an inscribed circle tangent to the three sides, and sequences of circles approaching the vertices, each of them touching the preceding circle and two sides of the triangle.

It is a simple task to see how fast these circles converge to the vertex, or in other words, how fast each sequence of radii approaches zero.

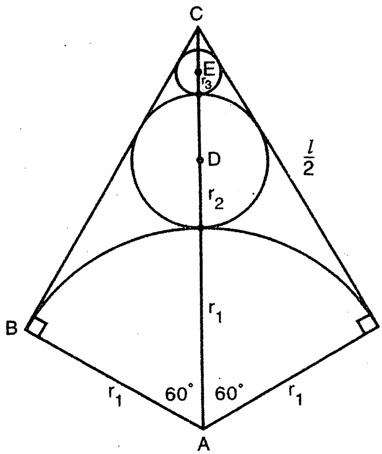

Figure 1

We can work this out from Figure 1 which represents one third of the original figure determined by two medians of the triangle. Since the angle BAC of the right angle triangle ABC measures 60°, we have

$$\cos 60° = \frac{1}{2} = \frac{r_1}{AC}$$

Then $AC = 2r_1$. By a similar analysis, we obtain $DC = 2r_2$. Therefore $2r_1 = AC = r_1 + r_2 + 2r_2$ from which we conclude that $r_1 = 3r_2$. The same relationship can be established for any pair of consecutive radii, giving the general relationship

$$r_k = 3r_{k+1}\ ,\ k = 1, 2, 3, 4, \ldots$$

Thus the sequence of radii is a geometric sequence with ratio $q = 1/3$ and first radius $r_1 = l/(2\sqrt{3})$, with l being the side length of the triangle.

A natural extension is to start with a regular n-sided polygon and inscribe circles in the same way with n sequences converging to the corresponding vertices. Figure 2 depicts one n-th of that polygon.

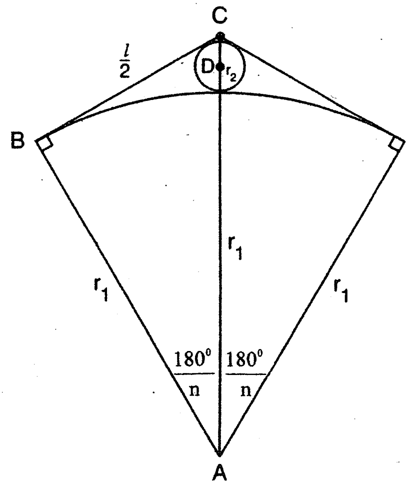

Figure 2

We have now

$$\cos(180°/n) = \frac{r_1}{AC}$$

Thus $AC = r_1/\cos(180°/n)$. Also, $DC = r_2/\cos(180°/n)$. Then, as before,

$$\frac{r_1}{\cos(180°/n)} = AC = r_1 + r_2 + \frac{r_2}{\cos(180°/n)}$$

which results in

$$r_1 = \frac{1 + \cos(180°/n)}{1 - \cos(180°/n)} r_2$$

indicating that the sequences of radii again form a geometric sequence, with ratio $q_n = [1 + \cos(180°/n)]/[1 - \cos(180°/n)]$. As the number of sides increases the denominator of this last expression will approach zero, and the ratio will become larger. The ratio q_8 is already about 25; most of the octagon is occupied by the first circle inscribed in it.

* * * * *

NUMBER PATTERNS AND WYTHOFF'S GAME

Michael A.B Deakin

I learned of these rather delightful number patterns by attending a lecture by Professor A M Vaidya of Gujarat University, India. I hope you too will enjoy and learn from them.

First, start with the number $\sqrt{2}$ and multiply it successively by $1, 2, 3, 4, \ldots$. This gives:

$$1.4142\ldots\ ,\ 2.8284\ldots\ ,\ 4.2426\ldots\ ,\ 5.6568\ldots\ ,\ \text{etc.}$$

However, in each case we keep only the part that comes *before* the decimal point: $1, 2, 4, 5, \ldots$. (This is the function represented mathematically by $[n\sqrt{2}]$, where the square brackets mean "the integral part of". Many calculators and computer packages use the notation INT( ).)

Continuing in this way, we set up the first two lines of Table 1, where I have written a_n for $[n\sqrt{2}]$.

n	1	2	3	4	5	6	7	8	9	10	11	12
a_n	1	2	4	5	7	8	9	11	12	14	15	16
b_n	3	6	10	13	17							
B_n	3	6	10	13	17	20	23	27	30	34	37	40
c_n	2	4	6	8	10							

Table 1

The third line of the table lists in order the numbers not contained in the second. These are to be called b_n.

For the moment, skip the next line (B_n) and go to the bottom line which lists c_n, where $c_n = b_n - a_n$. But this looks very simple. We seem to have

$$b_n - a_n = 2n. \tag{1}$$

In fact we do, and now let us see why.

Let α be any irrational number bigger than 1 ($\sqrt{2}$ is one such) and form the sequence of numbers $[n\alpha]$ just as we formed the numbers a_n. Call this sequence A.

Now consider another number β satisfying the equation $\alpha + \beta = \alpha\beta$, that is to say, $1/\alpha + 1/\beta = 1$. We readily find that $\beta = \alpha/(\alpha - 1)$. This new number gives rise to some nice patterns. To see this, form the sequence B by calculating the numbers $[n\beta]$. In the case of Table 1, this gives the fourth line, the values B_n.

We seem to have, in our example, $B_n = b_n$, and indeed this is so.

To show this, i.e. that the sequence of numbers not in A is precisely the sequence B, we need to prove two things:

(i) Every number is in either A or B.

(ii) No number can be in both sequences.

These statements will be proved in the Appendix. So, accepting these results, we now have $B_n = b_n$ and so the result (1), which we want to establish, may now be written

$$B_n - a_n = 2n \tag{2}$$

in the case where $\alpha = \sqrt{2}$.

But now

$$B_n = [(n\sqrt{2})/(\sqrt{2}-1)] = [(n\sqrt{2})(\sqrt{2}+1)]$$
$$= [2n + n\sqrt{2}] = 2n + [n\sqrt{2}] = 2n + a_n$$

where the step $[2n + n\sqrt{2}] = 2n + [n\sqrt{2}]$ follows because $2n$ is an integer. This proves the result (1).

Now take a different value of α. We choose

$$\alpha = \tau = (1+\sqrt{5})/2 = 1.618\ldots > 1.$$

(This value of α, i.e. τ, is the so-called "golden ratio"; see *Function, Vol* 16, *Part* 5.) Now construct a table (Table 2) similar to Table 1.

n	1	2	3	4	5	6	7	8	9	10	11	12
a_n	1	3	4	6	8	9	11	12	14	16	17	19
b_n	2	5	7	10	13	15	18	20				
B_n	2	5	7	10	13	15	18	20	23	26	28	31
c_n	1	2	3	4	5	6	7	8				

Table 2

Here the general features are the same as for Table 1 but instead of $c_n = 2n$, we now have $c_n = n$. The details of the proof in this case are left to the reader.

This brings us to Wythoff's game. It is one of those Nim-like games you play with matches, pebbles, counters or the like, and it is named after the mathematician who invented it early this century. It is an example of what a recent article (*Function*, *Vol* 19, *Part* 1) called an "impartial game", and the general remarks of that article all hold in this special case. Here are the rules.

We take two piles of matches with N matches in one pile and M in the other. Players move alternately and the player whose turn it is may take either:

any number of matches from one or the other pile,

or else

equal numbers of matches from both piles.

The objective is to take the last match: whoever does so wins.

The winning technique is this: so adjust the piles that for some value of n in Table 2 you leave your opponent with a_n matches in one pile and b_n in the other. E.g., suppose that when you come to move there are 19 matches in the first pile and 20 matches in the second. You take 7 matches from the first pile and leave your opponent with a $(12, 20)$ situation – the case $n = 8$ in Table 2. It will always be possible to do this, unless the situation you begin with is already one of the pairs from Table 2. (This you may prove for yourself.)

The interesting point, and I leave this as a project for the reader, is that *if the situation in front of the player is that of a number pair from Table 2 (i.e. if $N = a_n$ and $M = b_n$ for some value of n), then no single move can produce another such number pair; conversely, any other situation can be converted in a single move to such a pair.*

The rule is thus that *the player whose opponent is faced with one of the Table 2 pairs will be the winner.*

To continue the above example. Suppose you have left your opponent with $(12, 20)$ as envisaged above. The opponent takes (say) 2 matches from the larger pile. You now face (12, 18). Take 1 match from the smaller pile to set up (11, 18) – notice that this pair occurs in Table 2. Suppose your opponent takes one match from the larger pile. You now face (11, 17). This time take 2 matches from each pile and so set up (9, 15). Etc.

Play a few such games against yourself, or with a friend.

The key to the success of the method is that the loser is faced with (1, 2), and no matter what move is then made, the situation is hopeless.

Appendix

We now need to prove the main postponed result, namely that for any α as defined in the body of the article, every number lies in precisely one of the sequences A, B defined there. First to prove the truth of the statement labelled as (ii) in the body of the text.

No number can lie in both sequences.

For suppose some number k was a member of both sequences A and B. Then because k is in A there is a natural number such that

$$0 < k - n\alpha < 1. \tag{1}$$

(This is to say $k = [n\alpha]$ in a slightly different way – note that we can never have $k = n\alpha$, as α is irrational.)

Similarly, as k is supposed also to be in B, there is a natural number m such that

$$0 < k - m\beta < 1. \tag{2}$$

Multiplying (1) throughout by β and (2) throughout by α and adding, we find

$$0 < k(\alpha + \beta) - (n + m)\alpha\beta < \alpha + \beta. \tag{3}$$

and we recall that $\alpha + \beta = \alpha\beta$, and so deduce

$$n + m < k < n + m + 1. \tag{4}$$

This is impossible, as no integer k can possibly lie between the two consecutive integers $n + m$ and $n + m + 1$.

Thus we have proved the second of our requirements. Now to the first.

Every number must lie in one of the sequences.

The proof uses the same ideas just employed.

What we must show is that there is no number k that is "skipped over" in the sequence produced by combining A and B. If the number k is skipped over in A, then for all n, $n\alpha - k$ is either negative or greater than 1. (Thus, in Table 1, $n\sqrt{2} - 10$ is negative if $n < 8$, but $8\sqrt{2} - 10 = 1.313\ldots > 1$, and $9\sqrt{2} - 10$, etc. are correspondingly larger.)

Thus we have, for some n depending on the value of k,

$$n\alpha < k \text{ and } (n+1)\alpha > k+1. \tag{5}$$

The first of the inequalities in (5) may be written as $-k < -n\alpha$ and thus we find $k\alpha - k < k\alpha - n\alpha$. This last may be rewritten as $k(\alpha - 1) < (k-n)\alpha$, so we deduce

$$k < (k-n)\frac{\alpha}{\alpha - 1} = (k-n)\beta. \tag{6}$$

Similarly, from the second inequality in (5) we have $k+1 < \alpha n + \alpha$; then $k - \alpha + 1 < \alpha n$ and so $-k + \alpha - 1 > -\alpha n$. We add $k\alpha$ to both sides to obtain $(k+1)(\alpha - 1) > \alpha(k-n)$. Then,

$$(k-n)\beta < k+1 \tag{7}$$

(6) and (7) may be combined to give

$$k < (k-n)\beta < k+1.$$

This last inequality tells us that $[\beta(k-n)] = k$, which is to say that k must be a term of the sequence B.

This completes the proof.

Problem: What values of α would you need to give $c_n = 3n, 4n$, etc.?

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Subversive Sets

Eberhardt Schmidt (born 1876 in Dorpat Tartu, capital of Estonia, then Russia) lived in Germany, but once visited his family in tsarist Russia. His book on set theory was confiscated by the police because it mentioned "Mächtigkeit der Menge" which means "Powers of Sets" but also "Powers of Masses".

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SQUARE AND POLYGONAL WHEELS

R Soemantri, Universitas Gadjah Mada, Indonesia

In *Function*, *Vol* 16, *Part* 2, there was an article that discussed, among other things, the design of a road that allowed a square wheel to roll along it, so that its centre stayed on a level, neither rising nor falling as the motion progressed. Figure 1 shows the result. It also depicts the coordinate system used to analyse the problem. Note that the y-axis points *down*:

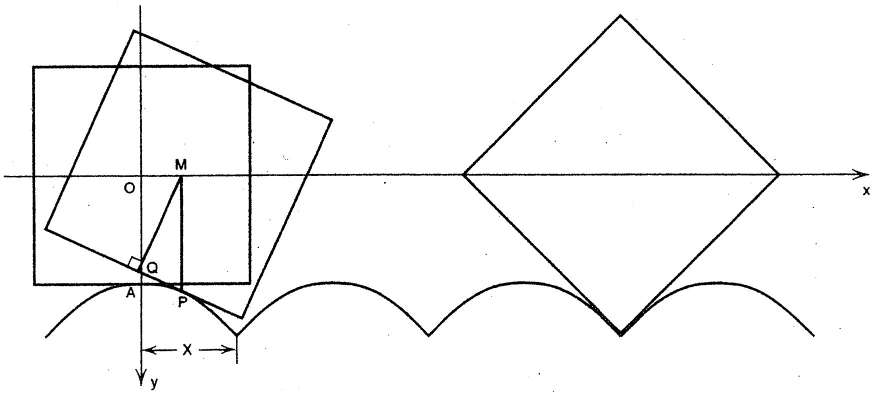

Figure 1

Before extending the analysis of that earlier article, I will revise for the reader's benefit some of the mathematical background. We require two mathematical functions that readers may not have met. These functions are called "cosh x" and "sinh x", where the latter is pronounced as "shine x"; they are defined as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2},$$

where e^x is the exponential function. Many calculators have these functions as part of their "library", and they are very like the more familiar trigonometric functions. (E.g. $\cosh^2 x - \sinh^2 x = 1$, and there are other such similarities.)

Now look again at Figure 1, and let $OA = a$. The result is then that the road is to have the shape depicted: a succession of arcs. The first of these has the equation[1]

$$y = a\cosh(\frac{x}{a}), \tag{1}$$

and this equation is valid for a range of x about $x = 0$. In fact we have

$$|x| \leq X,$$

where X satisfies the equation

$$\sinh(\frac{X}{a}) = 1. \tag{2}$$

(Approximately this gives $X = 0.88a$.)

For values of x outside the range from $-X$ to $+X$, we reproduce this primary arc as shown in the diagram. It is equation (2) that tells us when to finish the first arc and to start the second. Now we need to see where this equation has come from and for this purpose, look again at Figure 1. When the wheel has rolled through an angle of $\pi/4$ radians (i.e. 45°), the line PQ must make an angle of $\pi/4$ radians with the horizontal; this ensures that the right angle at the point P fits snugly into the depression in the road.

Thus at the point P, the slope[2] of the curve (1) must be $\tan(\pi/4)$, i.e. 1. The slope of the curve (1) is to be found by differentiating it, and this gives

$$\frac{dy}{dx} = \sinh(\frac{x}{a}).$$

Setting this equal to 1 gives equation (2).

Now the angle in a square, at the point where two adjacent sides meet, is $\pi/2$, i.e. twice $\pi/4$, and we may similarly analyse any other regular polygon. The regular n-gon, the polygon with n sides, has its adjacent sides meet at an angle of $2\pi/n$. It thus follows that if an n-sided wheel is to roll smoothly along a bumpy road, then equation (2) is to be replaced by

$$\sinh(X/a) = \tan(\frac{\pi}{n}), \tag{3}$$

where the right-hand side has been chosen to give the correct slope as the corner fits the depression in the road.

[1]This equation is discussed in more detail in the article alluded to above.

[2]Notice that the y-axis is down.

Furthermore, the "radius" of the wheel, let us call it r, is defined as the distance from the centre to a vertex. It is not difficult to show that

$$r\cos(\frac{\pi}{n}) = a,$$

where a is the distance from the centre to the mid-point of a side (as above in the case $n = 4$).

Thus in the case of a square wheel, $r = a\sqrt{2}$ and so $X \simeq 0.88a \simeq 0.62r$.

In the case of a triangular wheel, $n = 3$, equation (3) has the approximate solution $X = 0.93r$. All other values of n will be necessarily larger than 4. When $n = 5$, we have $X \simeq 0.48r$ and for $n = 6$, $X \simeq 0.39r$, etc. Notice that as n increases X becomes smaller and smaller. Look again at Figure 1. The individual arcs become smaller as n increases, and they also crowd closer and closer together. Thus as n gets larger and larger, the depressions become less and less pronounced; in other words, the road gets smoother.

In the limit as $n \to \infty$, the road becomes perfectly straight and smooth, for by this time the wheel has become circular!

* * * * *

Dr R Soemantri is a lektor kepala *(principal lecturer) in mathematics at Universitas Gadjah Mada (UGM). UGM is one of Indonesia's oldest and best universities; it has a special twinning relationship with Monash University.*

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My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all because I have followed its roots, so to speak, to the first infallible cause of all created things.

Georg Cantor

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THE PROBLEM OF THE MONKS

Peter Grossman

The following problem in logical reasoning, by Monique Parker of the Université Libre de Bruxelles, Belgium, appeared in a recent issue of the Belgian school mathematics magazine *Math-Jeunes* [1].

In a faraway country there lives a community of monks. The monks have very strict rules: they may not communicate with each other, either by talking or in any other way, and no monk may look at his own image (in a mirror or otherwise). Once each day they meet around a table. One day, the abbot (who is permitted to speak if the matter is serious – yes, I know this is all very contrived, but bear with me) says to the monks:

"A terrible disease has struck some of our members. This disease can be recognised by the appearance of a black spot on the forehead. As soon as any one of you knows he has the disease, he must leave immediately and seek treatment." (In the original version of the problem, the monk is told to commit suicide for the sake of the rest of the community. With due apologies to Monique Parker, I have taken the liberty of allowing the monks to follow a less drastic course of action.)

All the monks are experts at logical reasoning, and know that their fellow monks are as expert as they are themselves. Determine the number of days before the monks with the disease leave, as a function of the number N of monks with the disease (where $N \geq 1$).

At this point, you may wish to try to solve the problem before reading further.

The solution provided by Monique Parker runs as follows.

Suppose just one monk has the disease. On the first day, he sees that no-one else has a spot on his forehead. Since he knows that at least one monk has the disease, he deduces that he must have the disease himself, and leaves.

Now suppose that two monks have the disease, say A and B. On the first day, one of them, say A, reasons as follows: "Either B alone has the disease, or B and I both have it. If B is the only one with the disease, then by the reasoning in the previous paragraph he won't be here tomorrow." The next day, A sees that B is still there, and deduces that he himself must have the disease. He therefore leaves. Since B reasons in the same way, both monks will leave on the second day.

Similar reasoning can be applied for $N = 3, 4, \ldots$ to show that if there are N monks with the disease then they will leave on the N-th day. A rigorous proof proceeds by induction. We have shown that the result is true when $N = 1$. Now suppose the result is already known to be true for $N - 1$. The N monks with the disease can each see $N - 1$ of their fellow monks with spots on their foreheads, and (by assumption) they will deduce that if they themselves do not have the disease then the other $N - 1$ monks will leave on day $N - 1$. When they see that they are all still there on day N, they each know then that they have the disease.

That is the end of the problem as far as the article in *Math-Jeunes* is concerned. There is, however, a minor variation of the problem which leads to a curious and somewhat paradoxical result.

Suppose, instead of saying the disease *has* struck some of the monks (implying that at least one monk has the disease), the abbot merely says that it *may* have struck. This leaves open the possibility that none of the monks have the disease, and changes the problem completely.

Suppose one monk has the disease. He sees no other monk with the disease, but he is now no longer able to deduce that he has it himself. The second day provides him with no further information, nor does any subsequent day. Therefore he has no way of knowing whether he has the disease or not.

Since the case $N = 1$ provided the base for the entire chain of reasoning in the original problem, we see that the deduction falls down completely in the modified version. If two monks have the disease, it is no longer possible for one of them to reason, as he did before, that the other monk will leave if he himself is free of the disease. If he waits another day, he is none the wiser. Neither of the monks will leave. Continuing in this way, we see that none of the monks will leave, regardless of how many have the disease.

We now have the following strange situation. Suppose, for example, that ten monks have the disease. If the abbot merely announces that some of the monks *might* have the disease, nothing happens. If he adds that at least one of the monks actually has the disease, then the ten monks with the disease will leave ten days later. What is curious is that the additional information affects the outcome, even though the abbot is only telling the monks something they already know! The world of logic can be strange indeed.

References

[1] M Parker, "Pour ceux qui aiment raisonner ...", *Math-Jeunes*, 62, November 1993, pp. 3 & 22, Société Belge des Professeurs de Mathématique d'expression française.

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Peter Grossman is a lecturer in mathematics at Monash University, and the Problems and Solutions editor of Function.

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Population implosion

We hear a lot these days about how fast the earth's population is growing. Mr Ninny, president of the League Against Birth Control, disagrees. He thinks the world's population is *decreasing* and soon everyone will have *more* space than he or she *needs*. Here is his argument:

> "Every person alive has two parents. Each parent had two parents. That makes four grandparents. And each grandparent had two parents, so that makes eight great-grandparents. The number of ancestors *doubles* for each generation you go back. If you go back for twenty generations to the Middle Ages, you would have 1 048 576 ancestors! And this applies to every person alive today, so the population of the Middle Ages must have been a million times what it is now!"

Mr Ninny cannot be right, but where is the flaw in his reasoning?

From M Gardner *aha! Gotcha: paradoxes to puzzle and delight*
1982, W H Freeman and Company

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HISTORY OF MATHEMATICS

Mental as Anything

Michael A B Deakin

Nowadays, anyone faced with even a moderately involved calculation would automatically reach for a calculator, but it has only been in recent years that these have become routinely available.[1] Early electronic calculators (as distinct from computers) appeared in the late 1960s, and the hand-held models a little after that. Prior to this there were mechanical and electro-mechanical calculators, but these were more than somewhat unwieldy (besides being extremely noisy). The abacus, used widely in the East, seems to have dropped out of service in the West.

Here, people used pencil and paper or else their heads. When I went for my first job (in Woolworths' as a casual employee over the Christmas rush), I had to pass an exam in arithmetic, both written and oral. Commercial arithmetic in those days was complicated by the fact that money and also weights and measures were not metricated.

As to money, 12 pennies made a shilling and 20 shillings made a pound. A large part of the primary school arithmetic course was made up of learning to do addition, subtraction, multiplication and division in this rather complicated system. In the course of this education I and my classmates learned that 1/3d (i.e. one shilling and three pennies) was $\frac{1}{16}$ of a pound, while 1/4d was $\frac{1}{15}$. Many other such items were memorised, so that we could do "sums" like "How much should you pay for 17 items @ 1/3d each?" Answer: £1/1/3 (one pound, one shilling and three pennies). It was widely expected that ordinary shop assistants could do such arithmetic (and indeed much harder things) in their heads.

Then there were the measures. Twelve inches made a foot and three feet made a yard; 22 yards made one chain (the exact length of a cricket pitch) and ten chains made a furlong. Horse racing was conducted over distances of so many furlongs; a furlong is almost exactly 200 metres. Eight furlongs made a mile.

Now suppose that someone came into Woolworths' and wanted $5\frac{1}{2}$ yards of cloth @ 9/11d per yard. Suppose further they tendered £2/10/- (2 pounds

[1] This article is inspired by Paul Grossman's article in this issue of *Function*.

and 10 shillings). What change should be given? The answer is $5/5\frac{1}{2}$d, and the assistant was expected to know this. How? Well, if the cloth had cost 10/- (ten shillings) per yard, the cost would have been exactly 55 shillings, i.e. £2/5/-, and the change 5/-. The actual cost per yard was 1 penny less than this and this accounts for the extra $5\frac{1}{2}$ pennies.

The trick was to break larger, more complicated, calculations down into smaller, simpler ones. This can also be done with "straight" arithmetic computations – just involving ordinary numbers. The key to success is to size up the situation and then to seek features of the problem that are likely to assist the calculation (much as in the example above).

Suppose you want to multiply 47×52. You might first notice that 48×52 is

$$(50-2) \times (50+2) = 50^2 - 2^2 = 2500 - 4 = 2496.$$

The number we want is 52 less than this, namely 2444.

A number of people got very good at this sort of thing; one of my primary schoolmates, son of a prominent country bookmaker, was legendary. Yet others do even better.

India in particular has a tradition of what might be termed "folk mathematicians", amateurs with a great knowledge of and love for numbers. Ramanujan (who appeared briefly in my previous column) was the greatest ever to emerge from this tradition, so much so that he completely transcended it. (Every so often, claims are made that another of the tradition has reached the same heights, but these are not capable of being sustained. The name most usually mentioned is that of D R Kaprekar; Kaprekar once contributed to *Function* (*Vol* 9, *Part* 1), but the alert reader will notice that *that* contribution, like all of Kaprekar's work, is essentially "recreational mathematics" of no very great significance. Ramanujan, by contrast, spent most of his time on topics that lie well beyond the scope of *Function.*)

When I was an undergraduate, an Indian woman (whose name I forget) visited Melbourne and gave stage performances involving mental arithmetic. Like many stage magicians, she claimed to have "mystic powers" and some people took this stuff seriously. This led to some controversy when the student newspaper carried an article showing how some of the "tricks" were done.

For example, it is not particularly difficult to take the fifth root of any number under 10 000 000 000 (ten billion) *provided it works out exactly* ! Here's how the thing is done.

Suppose the number is 2 073 071 593. This is less than 10 000 000 000 and so its fifth root comprises two digits. The second digit is easy; it is the final digit of the given number. To find the first is a little harder and requires some memory. But $70^5 = 1\ 680\ 700\ 000$ and $80^5 = 3\ 276\ 800\ 000$, so any number in the 2 billion range must have a fifth root with initial digit 7. Thus the required answer is 73. For numbers somewhat smaller than this, up to about 750 million (750 000 000) the method is even easier. Try 601 692 057. To find the initial digit, *count the number of digits in the given number and subtract* 4. This gives 5 in our case. The second digit is again the final digit of the given number, namely 7. The answer is 57. There are similar but more complicated ways to do cube roots.

Another good trick uses the "magic number" 142 857 143. We can multiply this by any nine-digit number the audience may provide, writing down the numbers from left to right! Here's how *this* is done.

Suppose the number provided is 123 456 789. Now think of the number 123 456 789 123 456 789, and divide this by 7, working from left to right of course. This gives 17 636 684 160 493 827. I leave you to check that

$$142\ 857\ 143 \times 123\ 456\ 789 = 17\ 636\ 684\ 160\ 493\ 827$$

and to see how the trick works.

These tricks, of course, do require considerable practice and also good facility at mental arithmetic. But it is not difficult to teach yourself to do them if you really want to learn. Other people, however, go beyond this stage and become genuinely good at mental calculation in quite practical contexts. The basis is still practice and memorisation, but there do seem to be people with quite amazing gifts also.

At the relatively simple level, anyone can learn to multiply two-digit numbers mentally. Often there are available special tricks like the 47×52 calculation illustrated above, and the more algebra one knows and the more one practises the more these come into play. However, if all else fails, one proceeds like this:

$$\begin{array}{rrl} & 40 \times 50 = & 2000 \\ 7 \times 50 = & 350, & \text{running total } 2350 \\ 2 \times 40 = & 80, & \text{running total } 2430 \\ 7 \times 2 = & 14, & \text{grand total } 2444. \end{array}$$

Now suppose that one knew the multiplication table up to 99×99. Then numbers up to 9999 would all be, in effect, 2-digit numbers in base a hundred and so all four digit numbers could be so multiplied. When Gordon Preston, the founder and first editor of *Function*, took his first job as a code-breaker as part of the British war effort in the 1940s, his first task was to learn the multiplication table in base a hundred. D R Kaprekar, mentioned above, always worked in base a hundred, and I remember seeing many of his audience puzzled by this during one demonstration he gave. It is widely believed that G P Bidder, an English calculating prodigy who lived early last century, worked in base a thousand.

The Nobel laureate Richard Feynman had and used very considerable abilities in mental calculation. For some of this material, see his witty autobiography *Surely you're joking, Mr Feynman.* However, when they worked together, he was not regarded as being as good as John von Neumann, a mathematician credited, among other things, with the invention of the computer.

There is a famous problem in which two locomotives, 30 km apart and each travelling at 30 km/hr, are approaching one another on parallel tracks. A bird flying at 60 km/hr flits backward and forward between them. How far has the bird flown by the time the locomotives pass one another? The answer (30 km) may be given instantaneously if you see the right approach, but may also be found by summing a series. Have a go at this yourself.

The story goes that someone tried this problem on von Neumann, who gave the right answer instantly. "Oh!" they said, "you know the trick and didn't sum the series." "No," replied von Neumann, puzzled. "I summed the series; how else would you do it?"

I don't really believe this story, mainly because I think that von Neumann would see the simple way, but such stories are told of people who are already famous for the skill they illustrate.

Other great and famous mathematicians have been similarly gifted. C F Gauss (1777-1855) was one of the greatest mathematicians of all time and also a calculating prodigy of skill and note. He seems to have evidenced this as a very young boy. Although it seems hard to believe, it has been authoritatively stated that Gauss could calculate before he could talk. At the age of 3, he corrected a mistake in his father's account-book. When he first went to school (at the age of 8), the teacher, apparently to keep the class busy, had them add up all the numbers from 1 to 100. The young Gauss simply wrote down 5050 and showed this to the teacher.

What he had realised was that 1+100 = 101; 2+99 = 101; etc. As there were 50 such pairs, the answer was obvious. He had, in effect, rediscovered the formula

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Gauss seems to have remained extremely adept at such calculations throughout his life, but of course this aspect of his skill pales into insignificance beside his mathematical output. He contributed to mathematics in both breadth and depth to an extent rarely if ever matched. Algebra, number theory, calculus, geometry, astronomy, geodesy and physics all benefited enormously from the insights he brought and the research he conducted. He was one of the founders of non-Euclidean geometry, he showed how to compute the date of Easter (see *Function, Vol* 17, *Part* 4), a number-theoretic result he derived (the "quadratic reciprocity law") has been described as the most beautiful theorem in all of mathematics, and there is much else besides. His collected works occupy some 20 large volumes.

Calculational skills of the sort here under discussion have now almost completely died out. The computer and its little brother the calculator have killed them. Indeed, as we lean on these aids, we lose former skills. In 1958, I had sufficient such ability as to be employed one summer in this area. The job was to check all the field calculations that had been made during the course of an antarctic geological survey. Each page contained some eighty relatively simple calculations, including some twenty standardisations involving multiplication by factors like 37.8. At the bottom of the page a mean was taken of the twenty numbers so produced. After a week or so of this work, I could check a page, correcting any mistakes, and do so in just under 2 minutes. Even using my trusty calculator, I could not do anything like as well today.

The last of the really great calculators was a New Zealand mathematician, Alexander Craig Aitken. Aitken was born in 1895, so this is his centenary year. He grew up in Dunedin and his immense powers of memory were evident at an early age. It has been claimed of him that he was the very greatest of all the calculating prodigies, although such judgements are hard to make.

Aitken made his career in mathematics and was a worthy contributor. Following the intrusion of World War I (he fought as an Anzac, and was subsequently wounded on the Somme) he completed his degree, doing better in languages than in mathematics (he excelled in both and later became a Fellow of the Royal Society of Literature as well as of the Royal Society, the scientific body). However, after some difficulties he obtained a post at the University of Otago as an assistant in the department of mathematics.

His career in mathematics really began, however, when he relocated to Edinburgh, where he worked for the years 1925 onward. He became professor there in 1948. He lived, despite increasingly severe illness, till late 1967.

His area of speciality was what we now call numerical analysis, the study of computational efficiency, and this fitted in well with his superb personal computational gifts. However, he was very broadly capable, having taught actuarial mathematics before being appointed to his chair in pure mathematics. His textbook *Determinants and Matrices* was in use in the 1950s and was a prescribed text on this topic when I first met it.

There are many stories of Aitken's amazing facility at mental arithmetic. Here is how to compute 777^2:

$$a^2 = [(a+b) \times (a-b)] + b^2$$

$$777^2 = [(777+23) \times (777-23)] + 23^2 = [800 \times 754] + 529$$
$$= 603\,200 + 529 = 603\,729.$$

Every one of these steps would have been automatic, of course. The answer would have been given instantaneously.

A more involved computation was the method for deducing the decimal expansion of 1/59. I won't explain it in full; explore it for yourselves. The basic idea is that

$$\frac{1}{59} = \frac{1}{60-1} \text{ and } \frac{1}{60} = \frac{0.1}{6}.$$

The computation now proceeds in terms of a modified division algorithm for this last quantity. 6 into .10 goes .01 times, with a remainder of 4 (now ignoring decimal points) and we now take this 4 and the 1 just produced and make 41. 6 into 41 goes 6 times, with a remainder of 5. Enter the 6 as the next term of the answer and also after the 5 to make 0.016 and also 56. 0.016 is the answer so far, and 56 is the new number to be divided by 6. 6 into 56 goes 9 times, with a remainder of 2. Enter the 9 as the next digit of the answer and also following the 2 as the next number to be divided by 6. 29 is the new number to be divided by 6. The answer is 4 with remainder 5. The 4 goes as the next number into the answer, and so on. The answer is 0.016 949 152 ... .

All this was done mentally and almost immediately. What the alert reader will notice is that *mathematical* insights are being pressed into service here as aids to efficient computation. There is also the phenomenal memory, of course. Aitken was not above "cheating" by using this. It is a bit of a trick question to ask for the decimal value of 1/97. This runs to 96 digits[2] before it repeats; Aitken simply memorised them.

When he memorised these vast accumulations of data, Aitken made no use of memory aids ("mnemonics"; this reminds me of this and that reminds me of that, etc.). "Mnemonics I have never used," he said, "and deeply distrust. They merely perturb with alien and irrelevant association a faculty that should be pure and limpid."

Aitken lived long enough to see the dawn of computational machinery and some of the stories of the last years of his life are impressive, if also a little sad. The Scottish mathematician Thomas O'Beirne recalled accompanying Aitken to a demonstration of some then top-of-the-range desk calculators. "The salesman-type demonstrator," he subsequently wrote, "said something like 'We'll now multiply 23 586 by 71 283.' Aitken said right off 'And get ...' (whatever it was). The salesman was too intent on selling even to notice, but his manager, who was watching, did. When he saw Aitken was right, he nearly threw a fit (and so did I)."

But like my own much, much lesser skills, Aitken's began to deteriorate as he used calculators. "Aitken confessed ... that his own abilities began to deteriorate as soon as he acquired his first desk machine and saw how gratuitous his skill had become. 'Mental calculators' [he said], 'may be doomed to extinction. Therefore, ... [some of those whom I've met] may be able to say in the year AD 2000, "Yes, I knew one such." ' " One demurs a little, however. Try some of the calculations in this article in BASIC or on your favourite calculator!

I conclude with a very famous Aitken story, but one which puzzled me for a long while. Aitken was asked by his children to multiply 123 456 789 by 987 654 321. Here is his account of the event.

"I saw in a flash that 987 654 321 by 81 equals 80 000 000 001; and so I multiplied 123 456 789 by this, a simple matter, and divided the answer by 81. Answer: 121 932 631 112 635 269. The whole thing could hardly have taken more than half a minute."

[2]It is a theorem that the decimal representation of $1/p$ (where p is a prime) repeats after $p-1$ digits, or else some divisor of $p-1$. See *Function*, *Vol* 9, *Part* 1, pp. 8-12.

This all gels, but it took me a while to see how he could "see in a flash" that 987 654 321 by 81 gives 80 000 000 001. This is how I think he would have done so:

We look at the number patterns involved. We can readily see that

$$\begin{aligned} 81 &= 9 \times 9 \\ 801 &= 89 \times 9 \\ 8001 &= 889 \times 9 \\ &\vdots \end{aligned}$$

and these facts and their underlying principle would surely have been known to Aitken.

Now move on to a second interesting pattern. Divide 9 into a string of 8's:

$$9\,\overline{)888888888888888\ldots}$$

The first division $9\overline{)88}$ gives a quotient of 9 with remainder 7. This carries and so the next step is $9\overline{)78}$, which gives 8 with remainder 6. This carries and so the next division is $9\overline{)68}$, which gives 7 with remainder 5. This carries and so the next step is $9\overline{)58}$, etc. We finally reach a zero remainder when there are nine 8's, i.e.

$$888\,888\,888 = 9 \times 98\,765\,432 \qquad (1)$$

and thus if there is an extra 9 on the left of the equation (1), there will be a final 1 on the quotient.

Further Reading

The main sources for this article are the columns by Martin Gardner in *Scientific American* in April and May of 1967. The "Aitken confessed ... " quotation towards the end comes from the first of these. Gardner was a journalist and amateur mathematician who for many years ran a column, *Mathematical Games*, in that journal. He was also (and this is relevant to parts of the present article) an accomplished magician. His articles were anthologised into many popular books, and it may well be that your school library has some of these. The other source I drew upon is a recently published short biography of Aitken in the *Australian Mathematical Society Gazette* (March 1995).

* * * * *

COMPUTERS AND COMPUTING

Calculating without Calculators

Paul U A Grossman

We often read about the computer revolution and its impact. Associated with the development of computers was the spread of the electronic pocket calculator and this caused a particularly rapid change in the attitudes and practices of most people. I am of the age of the grandparents of today's students, but even their parents and many of their teachers remember the days before the common use of calculators. It may be of interest to readers to look at fairly recent history and the way people went about obtaining arithmetic results. Recollections from my youth may differ somewhat from the Australian experience since I grew up in central Europe.

Mental arithmetic

Additions, subtractions and multiplications involving moderate numbers of digits were often performed in the head and skill in mental arithmetic was fostered in schools. Many people proudly displayed the ability to perform calculations with numbers of many digits or with a sequence of numbers and would show off at parties or at fairs. Nobel laureate Richard Feynman relates in his autobiography how he baffled fellow-scientists in the 1940s by rapidly working out in his head powers of e, surds or other evaluations of functions.

Calculations with pen and paper

More commonly in everyday life, people would write down the numbers and work out the results on paper. Much time was devoted in schools to achieve accuracy and speed. With processes such as long division students soon learned to round off, determining the number of digits required for their purpose and saving themselves unnecessary work. Today's user of the calculator is often tempted to record as many decimal places as the calculator lists, coming up with answers that are not only clumsy but often meaningless and misleading.

I remember learning in school not only the four basic operations but also procedures to determine square and cube roots on paper.

As a teenager I once helped the accountant of a small factory to determine the firm's income and expenditure for the annual financial statement.

Small firms could not then afford calculating machines. There were pages of columns of figures to be added and the sums cross-checked. Fortunately the amounts were given in decimal currency and I did not have to cope with the complication of converting every 12 pence to a shilling and every 20 shillings to a pound, as people had to in Australia and other countries using pounds, shillings and pence.

Tables

The procedures mentioned earlier to obtain square or cube roots were not used in practice. Many such results could be looked up in tables. Generally, for operations involving powers, for divisions and even sometimes for multiplications we would use logarithmic tables. Books of tables listed logarithms to base 10 (known as Briggs' logarithms after Henry Briggs, 1561-1630).

Since

$$\log(ab) = \log a + \log b,$$
$$\log(a/b) = \log a - \log b, \text{ and}$$
$$\log a^n = n \log a,$$

knowledge of logarithms and their inverses can replace the operations of multiplication and division by respectively additions and subtractions, while powers, including fractional powers (roots), can be solved using multiplication and division.

The advantage of using logarithms to base 10 is that only numbers between 1 and 10, with the corresponding logarithms between 0 and 1, need to be listed. Any other number may be expressed as the product $a \times 10^n$, where $1 < a < 10$ and n is an integer; the corresponding logarithm is $n + \log a$ and is readily found when $\log a$ is known. Commonly the tables listed numbers in steps of 10^{-3}, with the logarithms given to five decimal places. If more precision was needed, one would interpolate linearly, accepting small errors.

Trigonometric, exponential and other functions were also found in books of tables. Often the logarithms of the trigonometric functions were given directly to enable the user to manipulate these functions without further recourse to tables.

Working with tables was slow but could be speeded up with practice. I used my personal volumes of log-tables into the early 1970s, although I had access to a big computer. For minor calculations it was not worth the effort of punching a program onto cards for the computer, and in earlier years even waiting a day for the results to be returned.

The slide rule

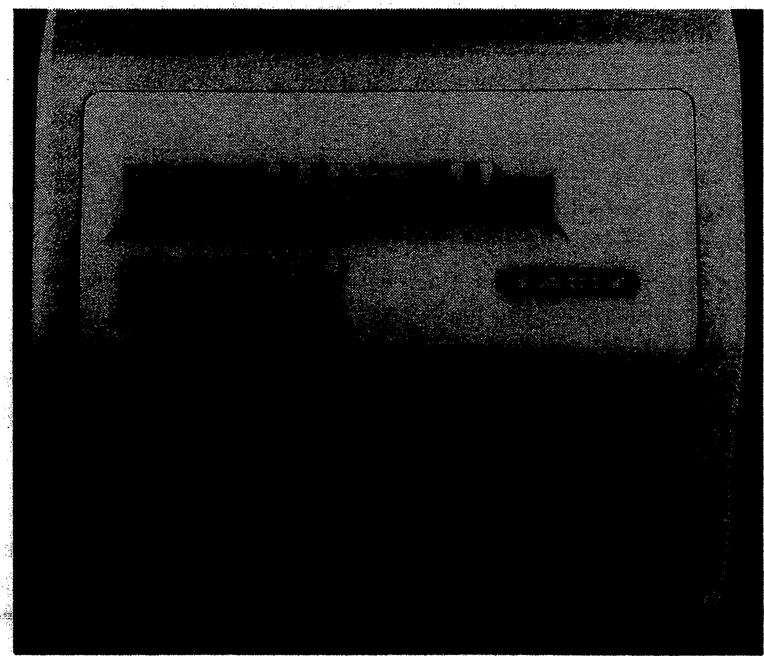
The logarithmic slide rule was very widely used in technical and commercial applications. The slide rule was a precision instrument consisting of three adjacent strips. The outer strips were firmly attached to each other and the centre one slid in grooves. Positioning marks on the centre strip with respect to marks on the outer ones is a way of geometrically adding distances; since the adjacent edges had logarithmic scales engraved, addition of distances was read as multiplication, subtraction as division. The scales on the lower strip and the adjacent edge of the centre strip ranged from 1 to 10, those on the upper strip and adjacent edge from 1 to 100 over the same length. This meant that the numbers on the upper scale were the squares of those on the lower one and these numbers could be lined up with the help of a slide known as a cursor, which had a line engraved. Slide rules usually had other scales engraved on the back and on the edges to suit particular groups of users.

The slide rule did not directly indicate the decimal power of the results, but this is readily determined. Its precision was limited by its length (the most common length was about 300 mm), its quality and condition, and the care taken by the operator. However, slide rules were adequate for many practical purposes and provided quick results.

Calculating machines

The abacus has had a long history as an aid to calculation, but has been regarded more as a curiosity in Western countries in recent centuries. In Russia, China, Japan and other Asian countries this simple frame with wires and counters was widely used before the spread of electronic calculators. Using the abacus requires dexterity and concentration, but experts are said to achieve great speed and accuracy in the four basic operations; even square and cube roots can be performed, I understand.

Mechanical calculating devices date back to the 17th century in many forms and designs but were mass-produced and came into wider use, I believe, in the 1920s. I did not get to use one until some forty years ago, a small device based on concentric cylinders. Digits were set by levers, a lateral shift placed the decimal point and a handle was turned clockwise for addition, counter-clockwise for subtraction. For multiplication one turned the handle the appropriate number of times after shifting the multiplicand laterally to allow for the power of ten. Division could be obtained by repeated subtraction until a bell indicated the machine had gone into negative numbers.



This calculating machine was state of the art in the early 1960s. It has three registers to perform a calculation of the type $a * b = c$, where $*$ could be either $+$, $-$, $\times$, or $/$. For addition and multiplication, a and b were placed in the lower and top right register respectively, and the answer appeared in the top left. Inverse operations had the answer appear in the bottom register by reversing the process. Decimal points were inserted with the manual tabs visible on all registers. (The number 6 decided to pose for this photo upside down!)

Electrical calculating machines still worked on the principle of mechanical transfer but were powered by a motor and automated to some extent. Numbers were entered by sets of buttons instead of levers. These machines became essential for anyone required to process large amounts of data.

However, they still were limited to basic operations, they were slow compared with electronic calculators, they were bulky and noisy and required regular servicing. Furthermore, they were expensive; on today's values one electrical calculating machine would cost thousands of dollars. Electric calculating machines were phased out in the early 70s when the capabilities of electronic calculators were rapidly improving and their cost decreased.

Conclusions

It is obvious that the electronic calculator has revolutionised everyday calculations over the past two decades, removing drudgery while increasing speed and reliability. However, there are also drawbacks in total reliance on these instruments. Users should not lose the facility to cope without them. They should realise the limitations of calculators and possible errors. They should keep in mind that not all decimal places given are necessarily appropriate for their purposes and should understand the full meaning of the operations initiated by buttons such as those for statistical results. Let us, with caution, take advantage of further developments in calculators.

* * * * *

Dodgy Statistics

> "Children in residential care have more than a 50 per cent chance of being below average academically and in their personal skills, a new Victorian study has found."
>
> The Age, 29 May 1995

It sounds dramatic to say that more than 50 per cent of these children are below average. But is it really saying very much? What percentage of *all* children would you expect to be below average?

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 19.2.1 (from *School Science and Mathematics*; submitted by M Deakin)

Show that $a^{a^{1/e}} \geq 1/e$ for $a > 0$.

SOLUTION (based on the solution in *School Science and Mathematics*)

Let $f(x) = x^x$ for $x > 0$. (This function was the subject of the cover story in *Function Vol* 6, *Part* 2.) Then $f'(x) = x^x(1 + \ln x)$; this result can be derived by using the technique known as *logarithmic differentiation*, or alternatively by first writing x^x as $e^{x \ln x}$ and then differentiating. The only critical value occurs where $f'(x) = 0$, for which $x = 1/e$. Since $f'(x) < 0$ for $x < 1/e$, and $f'(x) > 0$ for $x > 1/e$, f has an absolute minimum at $x = 1/e$. Therefore $f(a^{1/e}) \geq f(1/e)$ for all $a > 0$. Hence $(a^{1/e})^{a^{1/e}} \geq (1/e)^{1/e}$, so $(a^{a^{1/e}})^{1/e} \geq (1/e)^{1/e}$ and therefore $a^{a^{1/e}} \geq 1/e$, for all $a > 0$. Equality holds for $a^{1/e} = 1/e$, i.e. for $a = e^{-e}$.

Other solutions were found by M Deakin, N Cameron and K Anker, all of the Department of Mathematics, Monash University.

PROBLEM 19.2.2 (from *Parabola, Vol* 30, *Part* 3, 1994, University of NSW)

Four weary explorers have to cross a bridge over a river one night. Owing to their various degrees of exhaustion, they would individually take 5, 10, 20 and 25 minutes (respectively) to cross the bridge. However, the old and rickety bridge will take only one or two people at a time. Furthermore, it is too dangerous to cross the bridge in the dark, and the expedition has only one torch. How can all four explorers cross the bridge in the least possible total time?

SOLUTION

It is clear that the solution must consist of three forward crossings with two explorers each, and two return crossings by one explorer to bring back the torch. There are two "fast" explorers (taking 5 and 10 minutes to cross) and two "slow" explorers (taking 20 and 25 minutes). The total time taken will be dominated by the time taken by the slow explorers, so it seems reasonable to try to minimise the time they take. This can be done by ensuring that they cross the bridge together (and neither of them crosses back). The following steps will achieve this:

1. The two fast explorers cross the bridge together (10 minutes).
2. One of them (it doesn't matter which, but let's say the faster of the two) returns with the torch (5 minutes).
3. The two slow explorers cross the bridge together (25 minutes).
4. The other fast explorer returns with the torch (10 minutes).
5. The two fast explorers cross together (10 minutes).

The total time using this procedure is 60 minutes. Any other procedure would require at least two "slow" crossings with a time of 45 minutes altogether. The other three crossings must take at least 5 minutes each, so the total time could not be less than 60 minutes.

PROBLEM 19.2.3

A 6×6 array of squares is completely covered with 18 dominoes, in such a way that each domino covers two adjacent squares. Prove that there must be at least one line, either horizontal or vertical, that divides the array into two parts without passing through any of the dominoes.

SOLUTION

Suppose there is no such line. Consider any one of the 10 lines (5 horizontal and 5 vertical) that divide the array into two parts without passing through any of the squares. The number of squares on one side of the line is a multiple of 6 and hence is even, so those squares must be covered by a number of complete dominoes and an even number of half dominoes. Therefore the line must pass through an even number of dominoes. Since by assumption the line passes through at least one domino, it must therefore pass through at least two dominoes. But this argument applies to all 10 lines, giving a total of at least 20 dominoes through which a line must pass. This exceeds the total number of dominoes, so we have a contradiction.

PROBLEM 19.2.4 (from IX Mathematics Olympiad "Thales", in *Epsilon*, 26, 1993, p 107)

The brothers Al Caparroni are trying to open a safe in the Peseta Bank. The combination is composed of an increasing sequence of three non-zero digits. In the pocket of the cashier they have found the following information:

- The sum of the digits is 17.
- The product of any two digits added to the third digit is a perfect square.

What is the correct combination?

SOLUTION

Let the digits be a, b and c. Then $a < b < c$, $a + b + c = 17$, and the expressions $ab + c, ac + b$ and $bc + a$ are all perfect squares. The answer can be found by checking each combination of three digits in increasing order that total 17. Note that a must be less than 5, since $5 + 6 + 7 > 17$. Similarly, c must be greater than 6, since $4+5+6 < 17$. This gives just 12 combinations of a and c to check. The answer is $a = 1$, $b = 7$ and $c = 9$.

Solution to an earlier problem

We continue our series of solutions to problems from earlier issues for which a solution has not previously been published. The problem below appeared in the October 1992 issue of *Function*.

PROBLEM 16.5.2 (Republic of Slovenia – 36th mathematics competition for secondary school students, first class, Question 2)

For natural numbers $a_0, a_1, \ldots, a_{1992}$,

$$a_0{}^{a_1} = a_1{}^{a_2} = a_2{}^{a_3} = \ldots = a_{1991}{}^{a_{1992}} = a_{1992}{}^{a_0}$$

holds. Prove that $a_0 = a_1 = a_2 = \ldots = a_{1992}$.

(The problem contained a minor misprint when it originally appeared. It is shown correctly here.)

SOLUTION

If any one of the numbers a_i $(0 \leq i \leq 1992)$ equals 1, then clearly they must all equal 1. We will therefore assume that $a_i > 1$ for all i. We first establish two lemmas.

Lemma 1. If $a_i < a_{i+1}$ then $a_{i+1} > a_{i+2}$. (The subscripts are modulo 1993, so that $1992 + 1 = 1991 + 2 = 0$ and $1992 + 2 = 1$.)

Proof. $a_i{}^{a_{i+1}} = a_{i+1}{}^{a_{i+2}}$, so $a_{i+1} \log a_i = a_{i+2} \log a_{i+1}$, where the logarithms are to any fixed base. Since $a_i < a_{i+1}$, it follows that $\log a_i < \log a_{i+1}$, so $a_{i+1} > a_{i+2}$ as required.

Lemma 2. If $a_i > a_{i+1}$ then $a_{i+1} < a_{i+2}$.

Proof. As for Lemma 1, but with the inequalities reversed.

Now, if $a_0 < a_1$ then we have successively $a_1 > a_2$ (by Lemma 1), $a_2 < a_3$ (by Lemma 2), $a_3 > a_4, \ldots, a_{1992} < a_0$, and finally $a_0 > a_1$, contrary to assumption. A similar contradiction arises if we begin by assuming that $a_0 > a_1$. Therefore $a_0 = a_1$. By a similar argument, $a_1 = a_2$, $a_2 = a_3$, etc., and the result is proved.

PROBLEMS

PROBLEM 19.4.1 (K R S Sastry, Dodballapur, India)

In the convex quadrilateral $ABCD$, the diagonal AC is the bisector of angle DAB and a trisector of angle BCD, and the diagonal BD is a trisector of angle CDA and a quadrisector of angle ABC, as indicated in Figure 1. Also, $\alpha, \beta, \gamma, \delta$ are natural numbers of degree measures of the angles indicated. Find the possible degree measures of the angles of $ABCD$.

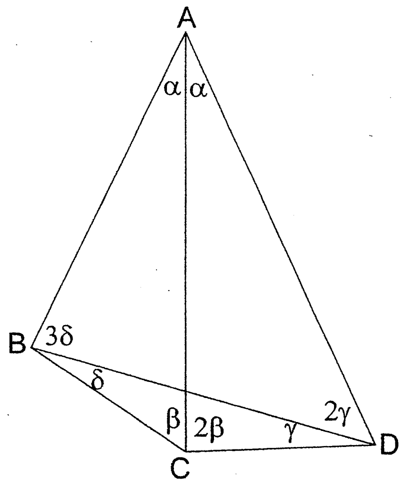


Figure 1

PROBLEM 19.4.2 (Juan-Bosco Romero Márquez, Departamento de Algebra, Geometría y Topología, Universidad de Valladolid, Valladolid, Spain)

Find all solutions in positive integers of the Diophantine equation $4xy = 9(x + y)$.

Problems 19.4.3 and 19.4.4 are taken from the South African 1993 Old Mutual Mathematics Olympiad, and are reproduced from *Mathematical Digest*, January 1994 (University of Cape Town).

PROBLEM 19.4.3

For which values of n is the number $S_n = 1! + 2! + \ldots + n!$ the square of an integer?

PROBLEM 19.4.4

For every pair of natural numbers p and q a circle is drawn above the x-axis; it has diameter $\frac{1}{q^2}$ and it touches the x-axis at the point $\left(\frac{p}{q}, 0\right)$. Prove that no two circles intersect, and determine the condition for tangency.

PROBLEM 19.4.5 (from Trigg C W *Mathematical Quickies*, 1967, McGraw-Hill)

During a period of days, it was observed that when it rained in the afternoon, it had been clear in the morning, and when it rained in the morning, it was clear in the afternoon. It rained on 9 days, and was clear on 7 afternoons and 7 mornings. How long was this period?

PROBLEM 19.4.6 (from Trigg C W *Mathematical Quickies*, 1967, McGraw-Hill)

Albert and Bertha Jones have five children: Christine, Daniel, Elizabeth, Frederick, and Grace. The father decided that he would like to determine a cycle of seating arrangements at their circular dinner table so that each person would sit by every other person exactly once during the cycling of meals. How did he do it?

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Function editor on the Queen's Birthday honours list

Ken Evans, a *Function* editor since its foundation in 1977, has been awarded a Medal of the Order of Australia for services to mathematics education and the community. Our congratulations! We also take this opportunity to express our thanks to Mr Evans for his invaluable contribution to *Function* over its 18 years of existence.

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