Function

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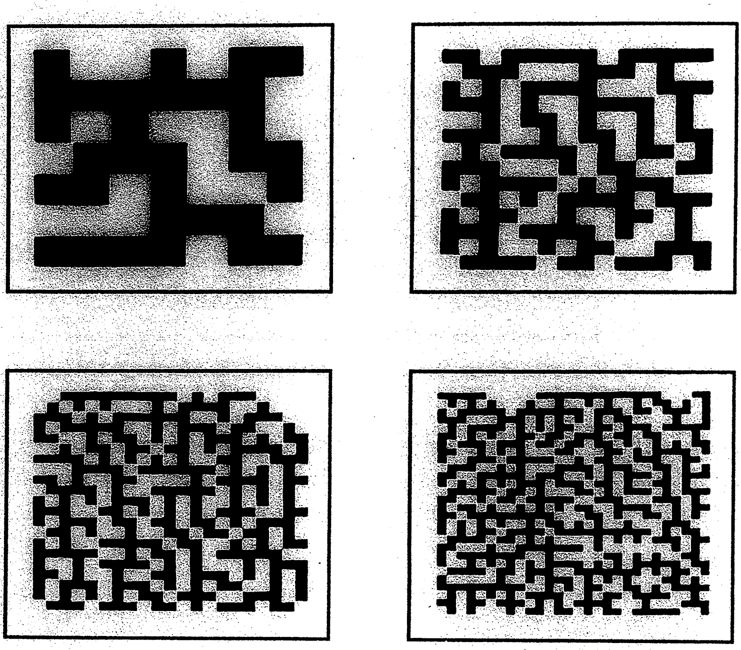

Mathematics Department - Monash University

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Function is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively, correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

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EDITORIAL

Fractals are very popular mathematical objects and they can be produced in many different ways. Benito Hernández-Bermejo presents in this issue a very simple and interesting random process for generating them. A sequence of four stages of the construction of one such fractal is depicted on the front cover.

This issue includes a variety of feature articles. G P Speck presents a procedure for obtaining "nice cubics", that is, cubic polynomials with integral coefficients which also have integral roots, and integral coordinates for turning and inflection points – cubics which come in handy especially for teachers. Rik King looks at the famous *Pell's equation* and presents a technique for finding its solutions. In a joint article, Janice Tynan and Michael Deakin make a statistical analysis of an interesting experiment to test the claims of people who believe in water divining.

In our regular *History of Mathematics* column you will find an account of how the famous Irish mathematician William Hamilton came to invent the quaternions. Read the *Computers and Computing* column to get an understanding of why an algorithm (or a computer program) shouldn't be considered good just because it carries out a predefined task, but rather because it does it more efficiently than other algorithms.

The *Problem Corner* includes, as usual, solutions to earlier problems, and a few new ones you can try out. You can also try the problems faced by the Australian team at the XXXVI International Mathematical Olympiad in Toronto.

In 1995 we have included a rich variety of interesting articles from many different authors, news, letters, problems, etc. We look forward to continuing to do so in 1996, which will be the 20th year of *Function*.

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THE FRONT COVER

Fractal Structures from a Simple Model[1]

Benito Hernández-Bermejo
Universidad Nacional de Educación a Distancia, Madrid

Fractals are probably the most popular objects of modern mathematics. Such acceptance comes, among many other causes, from their beauty, infinite assortment and similarity with countless natural patterns.

One of the causes of this diversity is the possibility of randomness in the algorithm for generating the fractal. In this article we shall deal with a simple example of a random fractal.

We start with an $n \times n$ square lattice where all the cells except one are empty. I shall call this nonempty site the *seed* of the fractal. We can locate the seed in any desired position (see Figure 1).

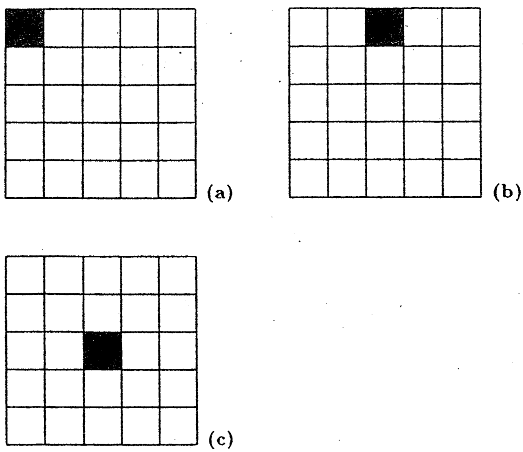

Figure 1: Three different starting configurations.

[1]This work is dedicated to my parents, for all the years of patience and encouragement.

Now the structure can grow through the occupation of any cell which is a nearest neighbour of a filled place, provided the following condition is fulfilled: the number

$$B = P - N \tag{1}$$

must strictly increase during the process. B is a *balance function* between the number of sides P of the figure and the number N of occupied lattice sites. Note that B will be an integer function, since both P and N are defined as natural numbers. Thus, one nearest neighbour is chosen randomly for the growth of the fractal among all those that augment B when added. The idea of the continued increment of the balance function and its form (1) aim at obtaining some kind of ramified arrangement, since the result will tend to have maximum perimeter and minimum inner surface. Of course, (1) is not the only possible definition of B, but it is a very simple one.

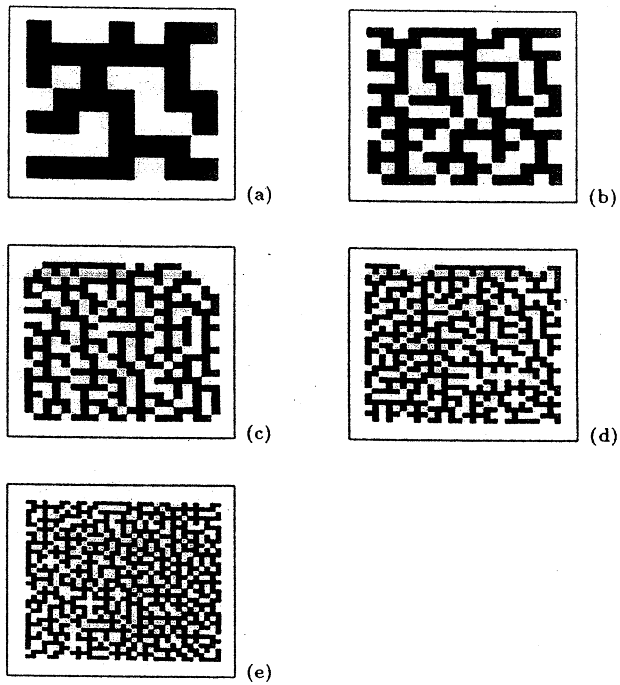

Figure 2: Result of the first iteration for: (a) $n = 7$, (b) $n = 14$, (c) $n = 21$, (d) $n = 28$, (e) $n = 35$.

This process must continue until no suitable cell is left for further development of the structure. Then we say that the *first iteration* towards the fractal is complete. Some examples of the outcome for different values of n are displayed in Figure 2 and on the front cover. It can be observed that different branches never interlace (this is easier to check visually for small values of n, although it always holds). Moreover, no clusters of occupied cells are present. Both phenomena are typical of ramification processes and our earlier reasoning is thus confirmed.

We can carry out this simulation for different values of n up to a sufficiently large number of times, for example 100 (a good strategy is to examine the full range $2 \leq n \leq 35$, using each value three times). When this is done we obtain some quantitative statistical results about this kind of growth. The most important one connects the number n^2 of cells that are initially available with the total number F of them which are actually occupied at the end of the process, through the approximate relation:

$$F(n) = a \times n^2, \tag{2}$$

where a is an empirical constant with value $a \simeq 0.7$. This means that on the average about 70% of all positions will be occupied. We shall make use of (2) later.

We can now proceed to consider the *second iteration* towards the fractal. The rule is simple: first discard all those cells which are empty after the first run has been completed, since there will be no points of the fractal inside them. The process will only take place at the occupied positions. Then each occupied site should be divided again in an $n \times n$ lattice, just like a small copy of the original square. Thus, from (2) the total number of minor cells now will be of the order of $a \times n^4$. This irregular set of cells must be considered afresh empty, with the exception of a single seed which can be placed anywhere (for instance, inside the seed of the first iteration). The entire operation is then repeated identically as in the first step, until the structure completes its development and the process stagnates. The second iteration will be ended at that time. From relation (2) it is easily seen that the number of occupied places will now be approximately $a^2 \times n^4$.

Third and later iterations continue the process *ad infinitum* in a similar way. From (2), the number of occupied cells after the k-th iteration is

$$F_k(n) = a^k \times n^{2k} \tag{3}$$

In Figures 3 and 4 this is illustrated for $n = 2$ and $n = 3$, respectively. In these examples seeds have always been positioned at the top left cell of the array. Note that the vacant sites of each iteration appear as forbidden regions for the following ones.

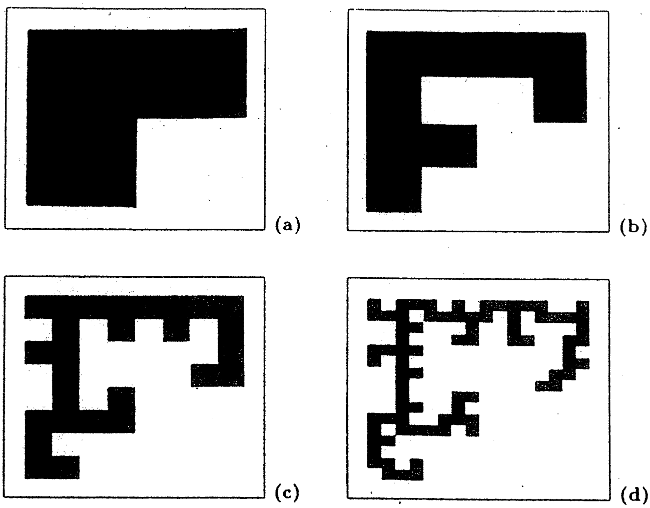

Figure 3: Case $n = 2$: (a) - (d) display iterations 1 - 4 respectively.

To convince ourselves that we really obtain a fractal by following this procedure, it is of interest to compute the fractal dimension of our constructions. This dimension, which I shall call D, displays noninteger values for fractals and can be introduced in the following way: suppose that your object is "self-similar", that is, it is composed of K subsets which have the same structure but scaled down in size by some factor r, and these subsets in turn have similar scaled subsets, and so on. Then it can be seen that the fractal dimension satisfies the equation:

$$K = \frac{1}{r^D} \tag{4}$$

I shall state this expression without proof. You can find a nice deduction of it in the article "The Sierpinski Carpet", published in the April 1994 issue of *Function*.

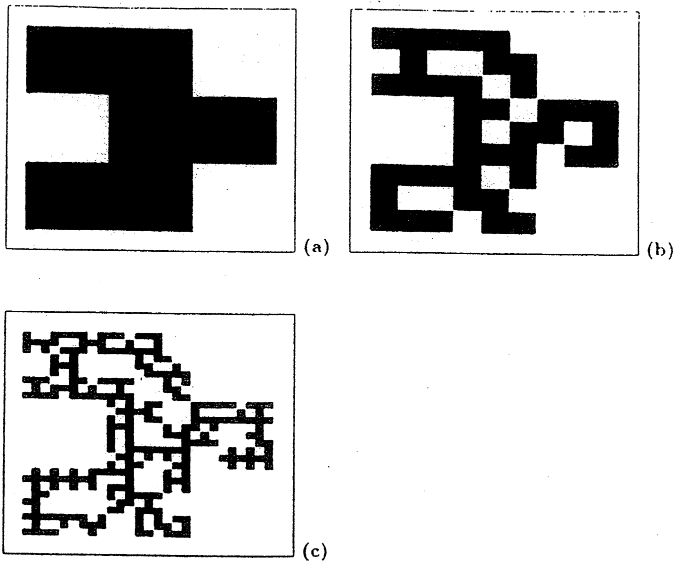

Figure 4: Case $n = 3$: (a) - (c) display iterations 1 - 3 respectively.

We can now concentrate on the application of (4) to our problem. Notice that the objects under study are self-similar in a statistical sense. This means that the subsets possess the same average properties as the main set, but they are not exact copies of it. From (2), which is one of these average properties, we have $K = F(n) = a \times n^2$. If we also observe that $r = 1/n$, then equation (4) can be written as $a \times n^2 = n^D$. Taking logarithms of both sides we finally obtain:[2]

$$D = 2 + \frac{\log a}{\log n} \tag{5}$$

Since $0.5 < a < 1$ and $n \geq 2$ it follows that $1 < D < 2$, and we really do have a fractal. The fact that D increases with n is understandable: the reason is that the original square is more uniformly covered for larger values of n. In the case $n = 2$ it is straightforward to see that the exact value of a is 0.75. Thus $D = 1.584...$. This is a well-known value of D: a famous fractal, the Sierpinski triangle, possesses this same dimension. This is not fortuitous: although the probability is negligible, the Sierpinski triangle can be generated using our procedure with $n = 2$ (Figure 5).

[2]Note that logarithms to any base can be used.

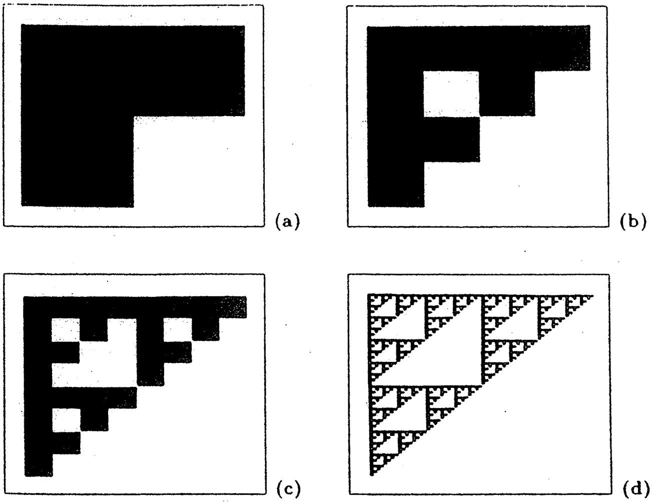

Figure 5: The Sierpinski triangle as a possible but unlikely sequence: (a) - (c) display iterations 1 - 3 respectively; (d) shows the sixth iteration.

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Benito Hernández-Bermejo is a young physicist working in Madrid. His interest in mathematics is related to his research in the area of mathematical methods used in non-linear dynamics. He is also interested in recreational mathematics and in the history of mathematics. His hobbies include movies, theatre, archaeology, and billiards.

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Nibble or nybble?

Derek Garson (Lane Cove, NSW) e-mailed to us the following: "In his article *Thrun, Fron, Feen, Wunty* the author states that there seems not to be a special word for half a byte of information or one hexadecimal digit. In keeping with the doubtful humour prevalent amongst computer designers, half a byte is called a *nibble*."

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NICE CUBICS

G P Speck, Wanganui City College, New Zealand

This article came out of an investigation that I was led to by my teaching of cubics and their properties.[1] I wanted examples of cubic polynomials such that:

1. All their coefficients were integers.
2. All their zeros were integers.
3. Both their turning points had integral coordinates.
4. The point of inflection also had integral coordinates.

A cubic satisfying all these requirements is a *nice cubic.*

Clearly any cubic of the form

$$f(x) = k(x - a_0)(x - b_0)(x - c_0)$$

will satisfy the first two requirements if k, a_0, b_0, c_0 are integers. Thus if requirements 3 and 4 are to be satisfied, we need integers x_1, x_2, x_3 such that

$$f'(x_1) = f'(x_2) = 0 \quad \text{and} \quad f''(x_3) = 0.$$

Now at least two of a_0, b_0, c_0 must have the same parity, i.e. be both even or both odd. Assume, without loss of generality, that b_0 and c_0 are two such integers. Then $b_0 + c_0$ must be even and equal to $2h$, say, and similarly $b_0 - c_0$ will be even and equal to $2b$, say. Now put

$$q(x) = f(x + h).$$

We have then that $b_0 = h + b$ and $c_0 = h - b$. Then

$$\begin{aligned}(x + h - b_0)(x + h - c_0) &= (x + b_0 - b - b_0)(x + c_0 + b - c_0) \\ &= (x - b)(x + b) \\ &= x^2 - b^2\end{aligned}$$

Then we find that

$$q(x) = k(x + h - a_0)(x^2 - b^2)$$

[1]For another article on cubics and their properties, see *Function, Vol* 17, *Part* 2.

and so

$$q(x) = k(x-a)(x^2-b^2),$$

where $a = a_0 - h$. The new cubic $q(x)$ is a standard form of the old one and still satisfies requirements 1 and 2. We now seek integers x_4, x_5 and x_6 such that

$$q'(x_4) = q'(x_5) = 0 \quad \text{and} \quad q''(x_6) = 0.$$

Now x_4 and x_5 are the roots of

$$3x^2 - 2ax - b^2 = 0,$$

that is

$$(a \pm \sqrt{a^2+3b^2})/3$$

and these values are to be integral. Furthermore, we may readily find that $x_6 = a/3$, and this too must be integral. Thus a must be a multiple of 3.

Now if x_4 and x_5 are to be integral, we require that $\sqrt{a^2+3b^2}$ be an integer, so that a^2+3b^2 must be a perfect square, i.e.

$$a^2 + 3b^2 = c^2. \tag{1}$$

Since a is a multiple of 3, so is c and so x_4 and x_5 are integers.

Equation (1) is a standard one in the theory of numbers. It is a particular case of *Pell's equation* and its solution is known. However, for our purposes, it is not necessary to give this solution in its full generality.[2] It suffices for the present purpose to find a family of solutions and this can be achieved by putting

$$a = 9m^2 - 3n^2,\ b = 6mn \ \text{ and } \ c = 9m^2 + 3n^2,$$

where m and n are integers. It is readily seen that these values satisfy the basic equation (1).

We may now generate infinitely many nice cubics. E.g., if $m = 1$ and $n = 2$, we find

$$q(x) = k(x+3)(x-12)(x+12)$$

We may make such a cubic "look different" by replacing x by $x+p$ for any integral p. So in this last example, for instance, put $p = 6$ and say $k = 1$. This gives

$$f(x) = (x+9)(x-6)(x+18).$$

[2]For that, see (e.g.) L E Dickson, *Introduction to the Theory of Numbers* (New York: Dover, 1957), pp. 40-42, 115; or R D Carmichael, *The Theory of Numbers and Diophantine Analysis* (New York: Dover, 1959), pp. 26-33. See also the article in this issue titled *Small Equation – Big Numbers.*

If we multiply this out, we find $f(x) = x^3 + 21x^2 - 972$.

In this way we can generate "nice cubics" to our hearts' content!

* * * * *

G P Speck has been teaching mathematics for many years. He taught analysis and topology for twenty years at various universities in the United States. After migrating to New Zealand in 1973, he taught for another fifteen years at Wanganui Boys College (now Wanganui City College) where he also served as head of the mathematics department. He has published pedagogical and research articles in the United States, New Zealand, England, Germany, and Austria.

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Are all numbers equal?

Take two numbers a and b and suppose that $b > a$. Then we can write

$$b = a + m \quad , \quad m > 0$$

Now multiply both sides by $b - a$,

$$b^2 - ab = ab - a^2 + bm - am$$

Subtract bm from both sides,

$$b^2 - ab - bm = ab - a^2 - am,$$

and factorize as follows

$$b(b - a - m) = a(b - a - m).$$

After dividing both sides by $b - a - m$ we conclude that

$$b = a.$$

Therefore all numbers are equal.

You surely do not accept this conclusion. So, what is wrong with this argument?

* * * * *

SMALL EQUATION – BIG NUMBERS

Rik King, University of Western Sydney

The seventh-century Indian mathematician Brahmagupta is supposed to have said that a person who can within a year find integer solutions to

$$x^2 - 92y^2 = 1 \tag{1}$$

is a true mathematician. The smallest positive values for x and y for the problem he proposed are 2 649 601 and 276 240, respectively! This is an example of a Pell's equation (more correctly, however, attributed to Pierre Fermat, 1601-1665), the general form of which is[1]:

$$x^2 - dy^2 = 1 \tag{2}$$

where d is a positive integer which is not a perfect square, and it is required that x and y also be integers. The reason that d may not be a perfect square may be seen as follows: suppose d were a square ($= D^2$); then equation (2) would become

$$x^2 - (Dy)^2 = 1$$

which would mean that 1 would have to be the difference of two squares, which is clearly impossible.

There are always two easy (or trivial) solutions to equation (2) which may be found by inspection of the equation itself. It may easily be seen that if $y = 0$, then $x = \pm 1$. It turns out that if there exists *one* non-trivial solution of (2), then there are, in fact, an *infinite* number of solutions. Leaving aside the trivial cases, nothing is lost by considering only the instances in which $x > 0, y > 0$; this is known as the *positive solution.* Also, the smallest solution, defined as the one for which $x + y\sqrt{d}$ is a minimum, is called the *fundamental solution.* It is well named, for from it all other positive solutions may be generated.

It is quite intriguing that in the case when $d = 60$, the smallest value of y is 4; when $d = 61$, the smallest value of y is 226 153 980; but when $d = 62$, the smallest value of y is merely 8. There is, however, a systematic scheme for finding the smallest solution to Pell's equation and for generating others as required, which comes from the theory of continued fractions, but that is beyond the scope of this article.

[1]Books do not agree on what is really the general Pell's equation. The most common definition is $x^2 - dy^2 = N$, $d > 0$, but in some books there are no restrictions imposed on the sign of d.

Without the advantage given by that theory, the most difficult task is to obtain just *one* solution of Pell's equation to begin with. The following rudimentary program in QBASIC may be of some assistance in detecting the fundamental solution for equations in which d is not too large.

```
REM This program gives values for X and Y
REM satisfying Pell's equation
INPUT; "Enter a value for d"; D
INPUT; "Maximum number of steps? (< 6000)";MAXNUM
DEFLNG Y-Z
Y = 1
DO UNTIL Y = MAXNUM
        Z = 1 + D * (Y^2)
        IF INT(SQR(Z)) = SQR(Z) THEN EXIT DO
        Y = Y + 1
LOOP
IF INT(SQR(Z)) = SQR(Z) THEN PRINT "X=", SQR(Z), "Y=", Y
ELSE PRINT "No result yet: increase number of steps"
```

The user is required to input the value of d from (2), and then the number of search terms which, within the limits of QBASIC, may not exceed 6000; despite this restriction, a large number of equations may still be solved. (Depending on the computer's capacity, the user may need to be patient!)

The process for deriving further solutions from the fundamental one, once that has been found, may be accomplished using the following identity:

$$(x_1 + y_1\sqrt{d})(x_2 + y_2\sqrt{d}) = (x_1x_2 + dy_1y_2) + (x_1y_2 + x_2y_1)\sqrt{d}, \qquad (3)$$

which may be verified by multiplying out the left side of the equation and collecting rational and irrational terms.

Suppose that (x_1, y_1) and (x_2, y_2) are solutions of (2). Then from the right-hand side of (3) we select the rational part to define

$$x_3 = x_1x_2 + dy_1y_2,$$

and the irrational part to define

$$y_3 = x_1y_2 + x_2y_1.$$

Now it can be verified that the new number pair (x_3, y_3) is also a solution of (2) in the following way:

$$\begin{aligned} x_3^2 - dy_3^2 &= (x_1x_2 + dy_1y_2)^2 - d(x_1y_2 + x_2y_1)^2 \\ &= x_1^2x_2^2 + 2dx_1x_2y_1y_2 + d^2y_1^2y_2^2 - dx_1^2y_2^2 - 2dx_1x_2y_1y_2 - dx_2^2y_1^2 \\ &= x_1^2(x_2^2 - dy_2^2) - dy_1^2(x_2^2 - dy_2^2) \text{ (cross terms cancel out)} \\ &= (x_1^2 - dy_1^2)(x_2^2 - dy_2^2) \\ &= (1)(1) \\ &= 1 \end{aligned}$$

So, just as stated above, (x_3, y_3) is also a solution of equation (2).

The significance of the identity (3) is that the product of two numbers of the form $x + y\sqrt{d}$ is once again a number having exactly the *same* form. So, suppose that x_1 and y_1 are integers. Then repeated use of (3) makes it clear that

$$(x_1 + y_1\sqrt{d})^n = x_n + y_n\sqrt{d} \tag{4}$$

where x_n and y_n are the new integers produced. Infinitely many solutions can be generated with equation (4).

Consider, as an example,

$$x^2 - 2y^2 = 1. \tag{5}$$

Inspection, or use of the program above, finds that $(x_1, y_1) = (3, 2)$ is the fundamental solution. We can find a second solution (x_2, y_2) by putting $n = 2$ in equation (4). Then $x_2 + y_2\sqrt{2} = (x_1 + y_1\sqrt{2})^2 = (3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2}$. Equating the rational parts gives $x_2 = 17$, and equating the irrational parts, $y_2 = 12$. This, then, is the second solution of equation (5). Similarly, from a further application of equation (4), $x_3 = 99$, $y_3 = 70$ may be found. As an investigation, the reader is encouraged to generate a few more solutions, and then to calculate each of the ratios $x_n/y_n = 3/2, 17/12, 99/70, \ldots$. What approximation is being made by these values? If you are uncertain, a similar problem is to find the fundamental solution of

$$x^2 - 3y^2 = 1 \tag{6}$$

and then find several more solutions, calculating the sequence of values x_n/y_n . To what number does the sequence x_n/y_n converge in this case?

A famous ancient problem (called Archimedes' cattle problem, described in *Function* 16, *Part* 3, p. 68) resulted in the need to solve the equation

$x^2 - 4\,729\,494\,y^2 = 1$. In fact, Pell's (or Fermat's) equation, known from the time of Diophantus (250 AD), has been of considerable importance in number theory, its properties having been studied in detail by many mathematicians; it has been especially useful in coming to grips with the nature of the irrational number $\sqrt{d}$. As for more recent times, Pell's equation, as mentioned earlier, is intimately connected with the theory of continued fractions, which has been employed in connection with the computer factorization of large integers.

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Dr Rik King teaches mathematics at the University of Western Sydney (Macarthur). He is interested in mathematics education and the solution, using computers, of equations that don't have exact solutions.

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An idea which can be used only once is a trick. If you can use it more than once it becomes a method.

George Pólya and Gábor Szegö
Problems and Theorems in Analysis, Vol 1
Springer-Verlag, 1972.

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SAND OR WATER: TELLING THEM APART

Janice C Tynan, ACT
and
Michael A B Deakin, Monash University

During the Science Festival held in 1994 at the National Science and Technology Centre (Questacon) in Canberra, a "hands-on (but actually not *exactly* on)" experiment was conducted with the aid of visitors, members of the public, who came through the doors while the investigation was in progress.

Each participant was confronted by ten closed and identical boxes, some of which (five actually, but the participants did not know this) contained sand and the others water. The idea was to see if, without actually interfering with the boxes, people (or perhaps some people) could distinguish which were which.

Most readers will fill in the background, but perhaps we should not be too lazy and assume that everyone will know it. So let us briefly explain. It is widely believed, both in this country and overseas, that it is possible to detect the presence of underground, or otherwise hidden, water by various techniques of "water divining", or "dowsing". Especially, and topically as we now write, in times of drought, farmers will employ diviners to find such water on their properties. In fact, most of our farmer friends would regard this as quite natural. (Many arid regions do in fact contain artesian basins, whose presence or position is not obvious to an observer on the ground above.)

On the other hand, no one has ever produced any adequate scientific explanation of water divining, and so there have been those who question whether the practice has any rational basis whatsoever. This controversy gives the background to the Questacon experiment.

The usual methods of dowsing involve various techniques of handling either pieces of wire (say coat-hangers or lengths of fencing wire), twigs or other forked sticks, or else pendulums, etc. The dowser will then use these in an attempt to detect the presence of water by watching the behaviour of, for example, the pendulum and trying to detect some change in its motion.

The Questacon experiment was designed and mounted by *Canberra Skeptics*, whose official view is that the claims of water diviners are spurious. This article will describe and analyse the results of their brief experiment.

The actual conduct of the experiments was primarily the responsibility of the first author (JCT) of this paper, but we should stress that she herself did not know which boxes contained which substance: water or sand. This is a most important point in all statistical trials. The technical name is "double blind" (neither the person administering the experiment nor the person on whom it is being administered knows the "right answer"). This precaution prevents the very real possibility of "body language" and other such subtle (but still important) clues influencing the outcome.

As volunteers passed through the exhibition and took part in the experiment, they were asked to nominate which of three groups they belonged to. Of the 151 volunteers, 54 nominated as "**D**owsers" (in other words, they claimed to be able to tell the difference between the water and the sand by use of the various pieces of equipment made available to them), 56 nominated as "**S**keptics" (in other words they claimed to believe that such claims as outlined above are spurious); the remaining 41 had "no opinion" (the "general **P**ublic").

		D	**S**	**P**
0	1	0	0	0
1	10	1	0	0
2	45	2	1	6
3	120	5	9	3
4	210	12	9	9
5	252	15	11	8
6	210	10	14	11
7	120	6	7	1
8	45	2	3	3
9	10	1	2	0
10	1	0	0	0
	1024	54	56	41

Table 1

For each of these three groups, their success rate was recorded. Table 1 displays the results. Here the column labelled **D** gives the results achieved by those claiming to be **D**owsers; **S** refers to **S**keptics, **P** to members of the general **P**ublic. The first two columns are, in order, the number of correct "guesses" and the theoretical numbers of each of these based on pure chance. The latter is based on the binomial distribution with $p = 0.5$

and $q = 1 - p = 0.5$, the chance of one or the other, water or sand, being the right answer to any single response. The bottom row gives the sums of the numbers in the various columns. For columns 3, 4, 5, these sums are the actual numbers of respondents in each category.

The question is what to make of this set of data. To answer this question we need to clarify what exactly is the question we want it to answer.

The most obvious thing to ask is whether any of the groups did any better than could be achieved by mere guesswork. To check this out, we first calculate the mean number of correct answers achieved by each of the three groups. For someone relying on pure guesswork, the expected value of this mean is, of course, 5. (Because exactly half of the 10 contained water and the other half sand.)

For the **D**owsers, the **S**keptics and the general **P**ublic the means turn out to be respectively 5.000, 5.268 and 4.732 respectively. (You may readily check these figures as an exercise.)

The first thing to strike our notice from the three numbers is that the **D**owsers, as a group, did *precisely* as well as expected of someone relying on guesswork. (This is in itself a considerable fluke. Its probability of occurrence is about 0.015; can you deduce this result? However, no significance should be read into it!) It thus follows that for this group no further statistical analysis is called for.

This is not, however, the general case, nor will it apply to the other two groups in the Questacon experiment. Indeed, it looks as if the **S**keptics, who did rather better than expected, might be in danger of disproving their own case and displaying hidden and unsuspected talents.

To test such cases, we set up a figure known as a statistic. The statistic relevant to this case is known as z, and it will be subjected to a test known as the z-test. Let μ be the expected mean number of correct responses achieved by pure guesswork, and let $\overline{x}$ be the mean found in the experiment. Further, let σ be the standard deviation expected on the hypothesis of pure guesswork. Finally, let n be the number of persons in the group. Then σ is known from theory to be $\sqrt{npq}$, where in this case $p = q = 0.5$. The statistic z is defined as

$$z = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}}.$$

For the **S**keptics, the value of z is thus $(5.268 - 5.000)/0.5 = 0.536$. (The value of $\sqrt{n}$ cancels out.) If n is sufficiently large (and "sufficiently large" is usually taken to mean more than 30, so it applies here), then z has a standard normal distribution, which means we can use tables of the normal distribution to determine if the value of z is "significant".

The argument goes like this. Imagine two people, one of whom says: "Skeptics actually have the ability to divine water"; the other says "Give me your evidence". "Well", says the first, "they are right 0.5268 of the time instead of only 0.5". "That's nothing", retorts the second. "That could have been caused simply by chance."

Our first speaker is announcing what we call a *hypothesis*. The second speaker is enunciating what is referred to as the *null hypothesis*. The null hypothesis is the assertion that the evidence advanced by the first speaker (in support of the hypothesis) is really no big deal. The method of testing hypotheses is that of determining how likely the result would be if the null hypothesis were true. We continue to accept the null hypothesis until the evidence builds up to such an extent that we find ourselves believing somewhat improbable things.

From tables of the normal curve, we can read off that a z-value of 0.536 or *even more* will be achieved by pure chance 29.6% of the time. This can hardly be accounted significant. For significance the result would have to be much less likely. The usual figure statistical convention accepts is 5%.

But now note a further consideration. A value of -0.536 *or even less* will also be achieved by pure chance 29.6% of the time. Thus a *deviation* of the magnitude observed in the Questacon tests on Skeptics (or even greater) will in fact occur in 59.2% of such trials, even if nothing but pure chance is involved. If our hypothesis had been that Skeptics were likely to obtain results *different* from those produced by pure chance (but without specifying whether they were to be better or worse), then the evidence is even weaker.

The first of these figures (29.6%) applies to the so-called *one-sided test*: relevant to the hypothesis that Skeptics did significantly *better* than pure chance. The second (59.2%) applies to the so-called *two-sided test*: relevant to the hypothesis that Skeptics did significantly *differently* from pure chance.

We thus see that there is no compelling evidence for either of these hypotheses, and so we continue to hold the null hypothesis. In other words, being a skeptic does not help you to divine water! In order to shake us from this conviction, we need much more compelling evidence.

The reader may care to duplicate the above calculations with the third category: the members of the general Public. Their "poor performance" is also best explained as a purely chance fluctuation. (In fact the level of significance is, to the accuracy given here, *exactly* the same as in the previous case; this again is coincidence and again nothing should be read into it! A somewhat more difficult question: Can you calculate the probability of this coincidence?)

So, until someone can come up with better, more convincing, figures than these, we continue to think that there is nothing in claims to be able to divine water.

* * * * *

Janice Tynan is the immediate past president of the Canberra Skeptics; Michael Deakin is on the editorial board of Function and a regular contributor.

* * * * *

The holiest number

Young Irving Joshua Bush, who later took the name of Matrix ... grew up a devout believer in the biblical prophecies of his parents' faith, and, owing to a natural bent in mathematics, was particularly intrigued by the numerical aspects of those prophecies. At the age of seven he surprised his father by pointing out that there was 1 God, 2 testaments, 3 persons in the Trinity, 4 Gospels, 5 books of Moses, 6 days of creation, and 7 gifts of the Holy Spirit.

"What about 8?" his father had asked.

"It is the holiest number of all," the boy replied, "The other numbers with holes are 0, 6, and 9, and sometimes 4, but 8 has two holes, therefore it is the holiest."

Martin Gardner
The Magic Numbers of Dr Matrix, 1985, New York: Prometheus

* * * * *

HISTORY OF MATHEMATICS

A Walk in Dublin

Michael A B Deakin

Last year I visited Dublin, and thought to check up on the story of Hamilton and the invention of Quaternions. (See *Function Vol* 18, *Part* 3, and *Vol* 5, *Part* 3.) Hamilton was William (indeed, in later life Sir William) Rowan Hamilton (1805-1865), a brilliant Irish mathematician and astronomer. At the age of 22, before he had in fact taken a degree, he was appointed Professor of Astronomy at Trinity College, Dublin and to the position of Astronomer Royal at the Dunsink Observatory on the outskirts of the city.

A problem that for a long while occupied and defeated Hamilton was that of devising an algebraic representation of three-dimensional space. The standard number line gives a way to represent a one-dimensional space in numerical terms, and the Argand diagram shows how to represent the points of a plane in numerical form as complex numbers. The need existed for a similar extension of the number system to enable the representation of points in three-dimensional space.

As pointed out, however, in the two previous *Function* articles on this topic, there are major problems with this endeavour. This question greatly occupied Hamilton and he ultimately solved it by inventing the algebra of *quaternions*. Quaternions in fact represent a *four*-dimensional space, but then it would be possible to ignore one of these dimensions and concentrate on the other three. This in fact led to our present-day vector algebra and calculus.

A quaternion was a number of the form

$$q = a + bi + cj + dk,$$

where a, b, c, d were real numbers and i, j, k were three different square roots of -1. For the most part the normal rules of algebra apply, but with one important exception. Suppose that q_1 and q_2 are two different quaternions, then we would expect that $q_1q_2 = q_2q_1$, but this is not possible. The two products, in general, have to be different.

We add quaternions exactly in the natural way, and similarly subtraction holds no surprises. Multiplication, however, has

$$i^2 = j^2 = k^2 = -1 \qquad (1)$$

and also

$$ij = k;\ ki = j;\ jk = i;\ ji = -k;\ ik = -j;\ kj = -i. \tag{2}$$

Note the way in this last table that $ij \neq ji$, etc.

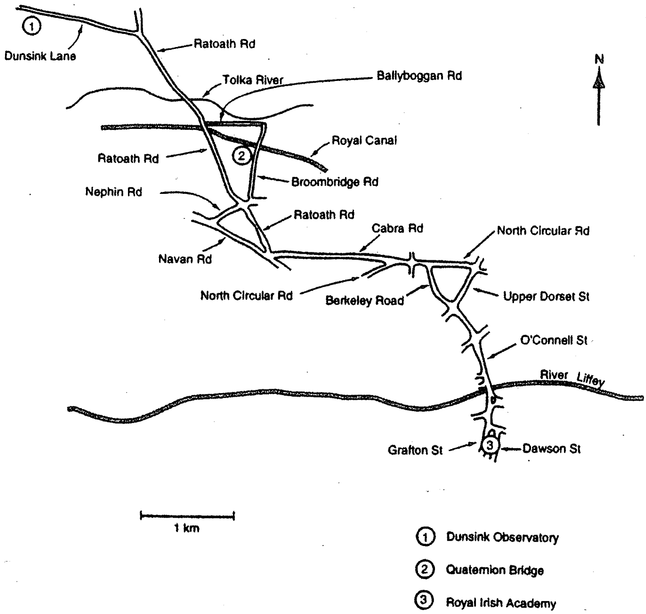
What has long interested historians and mathematicians alike about Hamilton's discovery of the quaternions and their algebra is the circumstances of the discovery.

> "[Quaternions] started into life, or light, full grown ... . That is to say, I then and there felt the galvanic circuit of thought close, and the sparks that fell from it were the fundamental equations between i, j, k; *exactly such* as I have used them ever since. I pulled out, on the spot, a pocketbook, which still exists, and made an entry, on which, *at that very moment*, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come. But then it is fair to say that this was because I felt a *problem* to have been at that moment *solved*, an intellectual want *relieved*, which had *haunted* me for at least *fifteen* years before."

This is one account Hamilton later gave of his discovery. Here is another.

> "... on the 16th day of [October 1843] – which happened to be a Monday and a Council day of the Royal Irish Academy – I was walking to attend and preside, and [my wife] was walking with me, along the Royal Canal, to which she had perhaps been driven; and although she talked with me now and then, yet an undercurrent of thought was going on in my mind, which gave at last a result whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should ever be allowed to live long enough to communicate the discovery. I pulled out a pocket-book, which still exists, and made an entry there and then. Nor could I resist the impulse – unphilosophical as it may have been – to cut with a knife on the stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k;
>
> $$i^2 = j^2 = k^2 = ijk = -1, \tag{3}$$

which contains the solution of the Problem, but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy ... which records the fact that I then asked for and obtained leave to read a paper on Quaternions, at the First General Meeting of the Session: which reading took place accordingly on Monday the 13th of November following."



The place at which these events took place has long been a point of pilgrimage for mathematicians. Hamilton must have told the story quite often and his students are said to have renamed Brougham Bridge "Quaternion Bridge". Last year I was offered the opportunity to take a week off in Ireland from my labours in the British Library, and my first port of call was Dublin. While there I visited Quaternion Bridge, whose official name is now "William Rowan Hamilton Bridge". The accompanying map shows how one gets there.

I set out from Dawson St, the site of the building where the Royal Irish Academy used to be housed, and walked to the point where Broombridge Rd crosses the Royal Canal. It took me 45 minutes, walking quite fast, but if you ever get the chance to go you can instead take a No. 11 or No. 22

bus. Probably I had to contend with more crowded footpaths and heavier traffic than would Hamilton and his wife, but back then the road surface would probably have been more uneven and much heavier going. It was a long walk, and it occurred to me to wonder why Hamilton was so far away from his destination.

It was my host, at the bed & breakfast where I was staying, who pointed out to me that it would have been natural for Hamilton to be going from Dunsink Observatory to the Academy and that this point was on the way. Even today, the Royal Canal marks the edge of Dublin at this point; north of this is open country. The area south of the canal is occupied by a relatively modern housing estate, so in Hamilton's day the entire locale would have lain outside the city proper.

The bridge itself is a stone one, but it is rather run down and graffiti-covered (which is unusual for Dublin, a very clean city) and furthermore the stone has weathered. Thus Hamilton's original scratchings have indeed "long since mouldered away". They were replaced by a concrete plaque which is rather more imposing, but sadly succumbing to a similar fate. To see the plaque, you need to walk toward the bridge from its western side. (A railway line now runs beside the canal; I don't know if this railway was there in Hamilton's day.)

The plaque reads as follows.

Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication $i^2 = j^2 = k^2 = ijk = -1$ and cut it on a stone of this bridge

Problem: Can you, from the concise form (3), deduce the full multiplication table (2)? It is interesting to note that Hamilton's "flash of genius" extended to the elegant form, rather than the more explicit!

Further Reading: A pilgrimage similar to my own was undertaken by Joseph Ayton, then a teacher at Fairlawn Senior High School, Fair Lawn, New Jersey, and his account published in the periodical *The Mathematics Teacher, Vol* 62, *Number* 6 (October 1969), pp 479-480.

COMPUTERS AND COMPUTING

Fast and Slow Algorithms

Cristina Varsavsky

Often two or more algorithms exist for performing the same task. A natural question to ask is which is more efficient. This leads us to the question of how to judge the efficiency of an algorithm. The same algorithm will most likely run faster on a faster machine. Also, the speed of an algorithm may heavily depend on the particular programming language in which it is written, but it is convenient to be able to determine the efficiency of algorithms before time is invested in their implementation.

The amount of memory used by an algorithm is an important aspect to be considered when analysing and comparing algorithms. An even more important consideration is its running time, which is directly related to the number of operations performed during its execution. In this article we will focus on the running time, that is, we will compare and judge the efficiency of algorithms by looking at the number of operations involved.

Suppose we need to design an algorithm to search for a word in a list of words. Some people need to do this quite often to find in the phone book the telephone number of a person, or search through a bank data base for the account number of a particular customer. The following algorithm does the job:

Algorithm NaiveSearch

(Search for X in the list $n(1), n(2), n(3), ..., n(k)$)

Step 1. Set $i = 1$.

Step 2. If $n(i) = X$ then output i and stop.

Step 3. Set i to $i + 1$.

Step 4. If $i > k$ then output "X is not in the list" and stop.

Step 5. Go to step 2.

This algorithm performs a *sequential search:* it simply goes through the list starting from the beginning, comparing each element with the search word until that word is found.

Now consider this other algorithm:

Algorithm BinarySearch

(Search for X in an ordered list $n(1), n(2), ..., n(k)$)

Step 1. Set $first = 1$ and $last = k$.

Step 2. Set $mid = \lfloor (first + last)/2 \rfloor$.

Step 3. If $n(mid) = X$ then output mid and stop.

Step 4. If $mid = first$ then go to step 7.

Step 5. If $n(mid)$ precedes X then set $first = mid + 1$ and go to step 2.

Step 6. Set $last = mid - 1$ and go to step 2.

Step 7. Set $mid = last$.

Step 8. If $n(mid) = X$ then output mid and stop.

Step 9. Output "X is not in the list" and stop.

This algorithm compares the search word with the word in the middle of the list, and discards the half of the list not containing the search word. This technique is called *binary search* and works only on a sorted list (**NaiveSearch** doesn't need the list to be arranged in alphabetical order; but lists are usually built up in such a way). The brackets $\lfloor\ \rfloor$ in step 2 indicate the *floor function*, which gives the highest integer less than or equal to its argument.

Although it is clear that **BinarySearch** is faster, we need to support this with a mathematical argument, comparing the number of operations involved in each algorithm.

Let us start with **NaiveSearch**. The operations involved are assignments (steps 1 and 3), additions (step 3), and comparisons (steps 2 and 4). We will assume that assignments happen instantaneously, so we will only count the number of sums and comparisons. Obviously that number will very much depend on the position of the word we are searching for. Computer scientists and mathematicians usually look at the *worst-case* situation; in other words, they are mainly interested in an upper bound on the number of operations for a fixed input size. For this algorithm, the worst case occurs when we need to go to the end of the list (the search word was the last one, or was not in the list at all). By the time we get to the end of the list, we have made k comparisons in step 2, k additions in step 3, and k comparisons in step 4 – a total of $3k$ operations.

Now we look at **BinarySearch**. Again, we consider the worst-case scenario. Steps 2 to 6 define a loop. Each time the loop is executed we have, in the worst situation, a sum and a division in step 2, a comparison in each of the steps 3, 4, and 5, and either an addition in step 5 or a subtraction in step 6 – a maximum of 6 operations per loop. Next we need to determine the number of times the loop is executed. Each time through the loop we reduce the length of the search interval to a half of its previous value; therefore after the first pass we have at the most $k/2$ words left, after the second pass we have $k/4$, after the third $k/8$, and so on. So the maximum number of passes is a number j such that $k/2^j \leq 1$ or equivalently, $k \leq 2^j$. Taking log_2 of both sides of this last inequality, we have $log_2(k) \leq j$. Thus, we make at most $\lceil log_2(k) \rceil$ passes through the loop ($\lceil\ \rceil$ denotes the *ceiling function*, i.e. the smallest integer greater than or equal to its argument); with the comparison made outside the loop (step 8) the worst-case count of operations comes to $6\lceil log_2(k) \rceil + 1$.

If, for example, we use these algorithms to search through a list with four words ($k = 4$), the maximum numbers of operations performed to carry out the task for **NaiveSearch** and **BinarySearch** are 12 and 13 respectively. For $k = 20$, the respective numbers of operations are 60 and 31, making it already evident that **BinarySearch** carries out the search task more efficiently. Taking a much larger list, for example $k = 2\,000\,000$ (quite possible for a bank customer list), the upper bound for **NaiveSearch** is 6 000 000 operations while for **BinarySearch** it is only 127 operations. To get some feeling for how these numbers translate into running times, suppose we do the search with a computer that can perform 5000 operations per second. **BinarySearch** will perform the search in a small fraction of a second while the upper bound for **NaiveSearch** is approximately 20 minutes. If we go through a list of 10 000 000 words, then **BinarySearch** would still need only a fraction of a second while **NaiveSearch** would take more than one and a half hours if the word searched for happens to be the last on the list. A time no bank customer can afford!

The field in computer science that deals with the analysis of algorithms is called *complexity theory* and many mathematicians and computer scientists around the world are devoted to studying it.

* * * * *

NEWS: Carmichael Numbers

A proof appeared recently in the technical literature[1] establishing that there are infinitely many Carmichael numbers. We use this opportunity to introduce these interesting numbers to our readers.

We begin with a theorem of Pierre Fermat (1601-1665) which can be found in many introductory texts on number theory:

If p is any prime number, and a is a natural number coprime with p, then $a^{p-1} \equiv 1 \pmod{p}$.

For example, if $p = 5$ then the theorem asserts that the remainder in the division of a^4 by 5 is 1, provided that a is coprime with 5. You can check this for $a = 1, 2, 3$, and 4.

It is natural to ask whether the property of prime numbers asserted in the theorem *characterises* prime numbers. In other words, if n has the property that $a^{n-1} \equiv 1 \pmod{n}$ for any natural number a coprime with n, does it follow that n is prime?

An initial exploration of the problem with small numbers might suggest that an even stronger result is true, namely that it is sufficient for the property to hold for just one such value of a. For example, if we put $a = 2$ and start checking each odd number in turn as a value of n, we seem to find that if $2^{n-1} \equiv 1 \pmod{n}$ then n is prime in every case. However, this eventually breaks down when we reach the composite number $n = 341$. We express this by saying that 341 is *pseudoprime* with respect to the base 2. Similarly, 91 is pseudoprime with respect to the base 3.

The existence of pseudoprimes with respect to a fixed base does not resolve our earlier question, which can now be restated as follows: do there exist numbers n that are pseudoprimes with respect to *every* base coprime with n? The answer turns out to be yes; such numbers are called *Carmichael numbers*, and they appear to be very rare. The first few Carmichael numbers are 561, 1105, 1729, ...

The question whether there are infinitely many Carmichael numbers had been an unresolved problem in number theory for a considerable time, so its solution is a significant achievement.

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[1]The proof is given by Alford et al in *Ann Math* 140 (1994), pp 703-722.

PROBLEM CORNER

SOLUTIONS

PROBLEM 19.3.1

You have access to an n-storey building, and two eggs. Assume there is a *critical storey*, such that an egg dropped from that storey or higher will break, but an egg dropped from any lower storey will not. (We will also admit the possibility that an egg dropped from the top storey does not break, in which case we will say that $n+1$ is the critical storey.) A *drop* consists of choosing a storey and dropping an egg from it. Your task is to find the critical storey by performing a sequence of drops. Of course, if an egg survives a drop intact then it can be retrieved and used in another drop.

(a) If $n = 10$, what strategy should you adopt in order to ensure that you will find the critical storey in at most four drops?

(b) For arbitrary n, find the smallest number k (as a function of n) such that the critical storey can always be found in at most k drops.

SOLUTION

Assume the storeys are numbered from 1 to n. Firstly, observe that if only one egg were available, it would be necessary to drop it successively from the first, second, third, etc. storeys until either it broke or it had been dropped from the top storey, requiring up to n drops.

Now consider the problem with two eggs and $n = 10$. If the egg breaks on the first drop, from storey k, say, then the critical storey must be less than or equal to k, and the only way to find it using the remaining egg is by dropping the egg successively from storeys $1, 2, \ldots, k-1$. Since only four drops in total are allowed, k cannot be greater than 4. If the egg does not break on the first drop, then the critical storey is greater than k, and we effectively now have the original problem but with a $(10-k)$-storey building (i.e. storeys $k+1$ to 10), and with only three drops available. We would gain nothing by making k less than 4, as this would just make $10-k$ larger than necessary. Therefore we choose $k = 4$.

The problem can now be solved by reasoning recursively. If the first egg breaks when it is dropped from the fourth storey, drop the second egg from storeys 1, 2 and 3 until it breaks. If the first egg doesn't break, then go up another three storeys (to the seventh storey) and drop it a second time. If the egg breaks this time, drop the second egg from storeys 5 and 6, otherwise go up another two storeys (to the ninth) for the third drop. If the egg breaks on this drop, drop the second egg from the eighth storey, otherwise go up another storey (to the tenth) for the final drop.

The solution to the n-storey problem for arbitrary n should now be clear. If n is the kth triangular number, $n = \frac{k(k+1)}{2}$, then the critical storey can be found in k drops: drop the first egg successively from storeys $k, k + (k-1), k + (k-1) + (k-2), \ldots, n$, until it breaks (or the nth storey is reached), then make a series of drops with the second egg, starting from the storey immediately above the highest storey from which the first egg survived its drop. If n is between the $(k-1)$-th and kth triangular numbers, the same strategy can be used with minor modifications. The smallest value of k is therefore the (positive) solution to $n = \frac{k(k+1)}{2}$, rounded upward to the next integer, i.e. $k = \left\lceil \frac{-1+\sqrt{1+8n}}{2} \right\rceil$ (where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x).

PROBLEM 19.3.2 (Ian Hawthorn, University of Waikato, New Zealand)

Let a, b, c be the lengths of the sides of a triangle, let A be the area of the triangle, and let R be the radius of the circumcircle (the circle passing through the vertices). Prove that $abc = 4AR$.

SOLUTION

Let the vertices opposite the sides with lengths a, b, c be X, Y, Z respectively. Let O be the centre of the circumcircle, let $\theta = XZY$, and let P be the bisector of $\overline{XY}$ (see Figure 1). Since the triangle ΔOXY is isosceles, OPX is a right angle and $\overline{OP}$ bisects XOY. Therefore $XOP = \theta$ and so $PX = R\sin\theta$. Hence:

$$c = 2R\sin\theta \tag{1}$$

The area of the triangle is given by the formula

$$A = \frac{1}{2}ab\sin\theta \tag{2}$$

Eliminating $\sin\theta$ from (1) and (2) gives the required result.

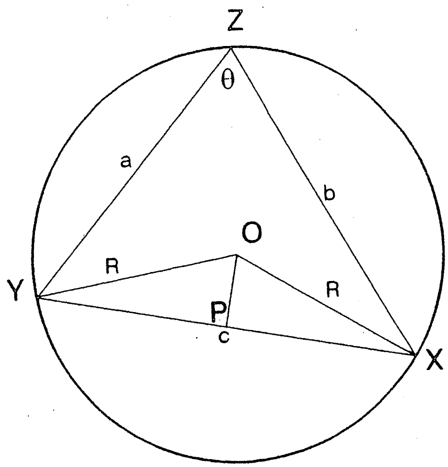

Figure 1

PROBLEM 19.3.3 (from *Mathematics and Informatics Quarterly*, 4/93)

Find all non-negative integral solutions of $3 \times 2^m + 1 = n^2$.

SOLUTION by John Barton (Carlton North, Vic.)

Subtract 1 from both sides and factorise the right-hand side, to obtain

$$3 \times 2^m = (n-1)(n+1)$$

Therefore 3 divides $n-1$ or $n+1$, so $n = 3s \pm 1$ for some non-negative integer s. Hence $2^m = s(3s \pm 2)$ and now s is non-zero. Since, now, s divides 2^m, we write $s = 2^i$ for some integer i such that $0 \leq i \leq m$, and so $2^m = 2^i(3 \times 2^i \pm 2)$. Dividing both sides by 2^{2i}, we get

$$2^{m-2i} = 3 \pm 2^{1-i} \qquad (3)$$

Now $(3s \pm 2) - s = 2s \pm 2 \geq 0$, since s is greater than or equal to 1. Hence $3s \pm 2 \geq s$, therefore $2^m = s(3s \pm 2) \geq s^2 = 2^{2i}$, so $m \geq 2i$ and thus $m - 2i \geq 0$. Therefore the left-hand side of equation (3) is an integer, so 2^{1-i} must be an integer. Hence $i = 0$ or $i = 1$. We now have $s = 1$ or $s = 2$, so n must belong to the set $\{2, 4, 5, 7\}$. These are easily tested in turn, and we arrive at the only solutions:

$$(m, n) \in \{(0, 2), (3, 5), (4, 7)\}$$

S I B Ayeni (Popondetta, Papua New Guinea) used a computer to investigate a more general class of equations, $ab^m \pm 1 = n^2$, for several different values of a and b and a range of values of m.

More on some earlier problems

Andy Liu sent us an alternative solution to Problem 18.5.2 which is not only simpler than the one given in *Function*, *Vol* 19, *Part* 2, pp. 59-60, but which also does not require us to assume that the triangle is equilateral. (The problem asked for the area of the central region of an equilateral triangle subdivided in a certain way, given the areas of the other regions; see *Vol* 18, *Part* 5 or *Vol* 19, *Part* 2 for a precise statement of the problem.)

The proof is based on Figure 2 below, which is a modified version of the figure given in *Vol* 19, *Part* 2.

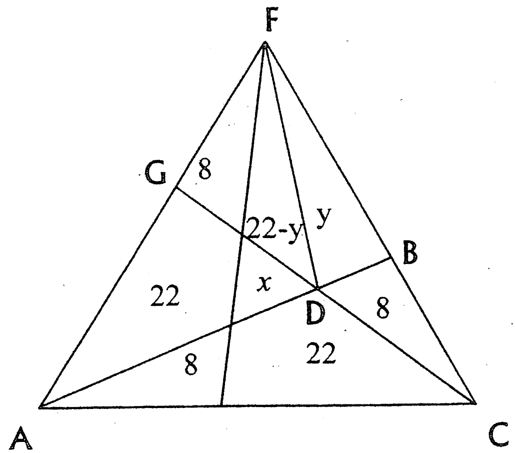

Figure 2

Using the fact that the areas of triangles with the same altitudes are proportional to the lengths of their bases, we obtain:

$$\frac{\text{Area}(\Delta FCD)}{\text{Area}(\Delta FCG)} = \frac{CD}{CG} = \frac{\text{Area}(\Delta ACD)}{\text{Area}(\Delta ACG)}$$

$$\frac{\text{Area}(\Delta CBD)}{\text{Area}(\Delta CAB)} = \frac{BD}{AB} = \frac{\text{Area}(\Delta FBD)}{\text{Area}(\Delta FAB)}$$

Therefore $\frac{y+8}{38} = \frac{30}{x+52}$ and $\frac{8}{38} = \frac{y}{x+52}$. Upon eliminating y and simplifying, we obtain the equation $(x-5)(x+147) = 0$. Hence $x = 5$.

Andy Liu also informed us that Problem 19.2.3 (the problem of the dominoes on the 6×6 array of squares) appears in Chapter 13 of *New Mathematical Diversions from Scientific American* by Martin Gardner. A three-dimensional variant appeared as Problem 5 of the Senior Spring 1988 A-Level paper of the Tournament of the Towns.

S I B Ayeni noted that the solution to Problem 19.2.4 (the "safe combination" problem) is still unique if the condition that the digits sum to 17 is dropped, provided that "increasing sequence" is taken to mean "strictly increasing sequence". If repeated digits are allowed, there are two other solutions: $(1,1,3)$ and $(1,1,8)$.

At this time of the year, when many of our readers are preparing for final exams, we thought it would be appropriate to provide something in lighter vein. The solutions to the two problems below have not previously appeared in *Function.*

PROBLEM 16.2.5 (*Mathematical Spectrum*, Vol 24, Number 2)

A bath takes 3 minutes to fill and 4 minutes to empty. How long does it take to fill the bath with the plug out?

SOLUTION

If we may assume that both the rate of filling and the rate of emptying are uniform, we can reason as follows. Using the volume of the bath (one "bathful") as our unit of capacity, we can say that the bath fills at a rate of $1/3$ bathful per minute, and empties at a rate of $1/4$ bathful per minute. The rate of filling with the plug out is therefore $1/3 - 1/4 = 1/12$ bathful per minute, so the bath takes 12 minutes to fill.

PROBLEM 16.3.1 (from Switzerland)

The number representing the year of birth of a famous Swiss mathematician has the following properties:

(i) the number of its digits is a perfect square;

(ii) the sum and the difference of its third and its fourth digit (counted from the left) as well as its rightmost digit are perfect non-zero squares;

(iii) the two leftmost digits, read as a decimal integer, form a perfect square.

In which year was this mathematician born? Who was he?

SOLUTION

The year is readily found; it is 1654. The mathematician was Jacob (sometimes given as Jacques or James) Bernoulli. He was one of several famous mathematicians from the Bernoulli family, and made important contributions to calculus and the theory of mathematical probability.

PROBLEMS

Problem 19.4.5 was unfortunately misprinted in the August issue. We apologise to readers who attempted it. The problem is reprinted correctly below.

PROBLEM 19.4.5 (from Trigg C W, *Mathematical Quickies*, 1967, McGraw-Hill)

During a period of days, it was observed that when it rained in the afternoon, it had been clear in the morning, and when it rained in the morning, it was clear in the afternoon. It rained on 9 days, and was clear on 6 afternoons and 7 mornings. How long was this period?

PROBLEM 19.5.1 (Garnet J Greenbury, Upper Mt Gravatt, Qld)

Prove that, in a right triangle,

$$2\tan^{-1}\left(\frac{\text{hypotenuse} - \text{base}}{\text{height}}\right) = \tan^{-1}\left(\frac{\text{height}}{\text{base}}\right)$$

PROBLEM 19.5.2

Let ABC be a triangle, and let O be any interior point of ABC. Prove that $AB + AC > OB + OC$.

PROBLEM 19.5.3

There is a unique number such that its square and its cube together use each of the 10 digits exactly once. Find the number.

(This problem could be solved by brute force with the aid of a computer. The challenge is to solve it as elegantly as possible, with a minimum of checking of particular cases.)

PROBLEM 19.5.4

Positive integers are to be expressed using the digits 1, 2, 3 and 4 exactly once each, together with the binary operations $+, -, \times, /$, the unary operations of negation and square root, and logarithm to a base, where the base must be supplied from the available digits. For example, 4 can be expressed as:

$$\sqrt{1+3}\log_2 4$$

Which positive integers can be expressed in this way?

OLYMPIAD NEWS

Hans Lausch, Special Correspondent on Competitions and Olympiads

The XXXVI International Mathematical Olympiad

Toronto was this year's venue for the IMO. Teams, most of six members, from 73 countries had to contend with six problems during nine hours spread equally over two days in succession.

Here are the problems:

Day 1

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

2. Let a, b and c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ for which there exist n points $A_1, A_2, \ldots, A_n$ in the plane, and real numbers $r_1, r_2, \ldots, r_n$ satisfying the following two conditions:

   (i) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;

   (ii) for each triple i, j, k $(1 \leq i < j < k \leq n)$ the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

Day 2

4. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

   (i) $x_0 = x_{1995}$;

   (ii) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each $i = 1, 2, \ldots, 1995$.

5. Let $ABCDEF$ be a convex hexagon with

$$AB = BC = CD,$$

$$DE = EF = FA,$$

and

$$\angle BCD = \angle EFA = 60^\circ.$$

Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^\circ$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

6. Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \ldots, 2p\}$ such that

(i) A has exactly p elements, and

(ii) the sum of all the elements in A is divisible by p.

The Australian team finished in place 21. The top five teams (in descending order of their unofficial scores) were: People's Republic of China, Romania, Russia, Vietnam and Hungary. The other countries of the Asian Pacific Mathematics Olympiad ranked as follows: 7, Republic of Korea; 9, Japan; 11, United States of America; 12, Republic of China (Taiwan); 19, Canada; 20, Hong Kong; 26, Singapore; 34, Thailand; 38, Colombia; 46, New Zealand; 53, Indonesia; 58, Mexico; 62, Macao; 65, The Philippines; 70, Chile (2 participants only); 72, Malaysia (2 participants only).

Members of the Australian team received awards as follows:

Jian He, Victoria, silver medal

Deane Gordon, Queensland, bronze medal

Nigel Tao, South Australia, bronze medal

Trevor Tao, South Australia, bronze medal

Herbert Zhi Ren Xu, New South Wales, bronze medal

Christopher Barber, Western Australia, honourable mention.

Well done!

* * * * *

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