Function

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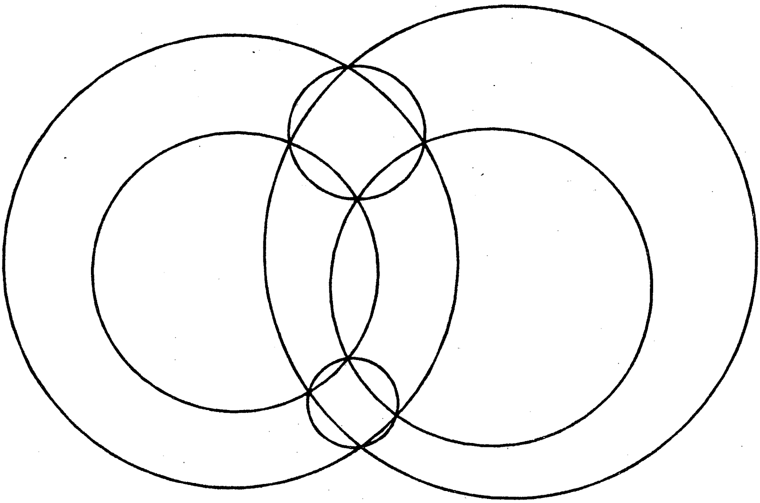

Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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For more information about *Function* see the journal home page at `http://www.maths.monash.edu.au/~cristina/function.html`.

* $10 for *bona fide* secondary or tertiary students.

EDITORIAL

In this issue of *Function*, Ken Evans gives us an account of Ptolemy's Theorem which relates the lengths of sides and diagonals of a cyclic quadrilateral. You will find here the proof of the theorem as well as its interpretation in some special cases. This theorem was used by Ptolemy in the *Almagest* and, even though it bears his name, the theorem was known well before Ptolemy's time.

We continue with the *Function* tradition of showing the use of mathematics in the modelling of real-life situations. This time we include an article by Michael Deakin and Charles Hunter which analyses in terms of game theory the immunisation dilemma faced by many parents: immunisation is desirable to prevent diseases, but is it worth taking the risk of the immunisation procedure to avoid what is believed a low risk disease?

We finally received a reply from Kim Dean regarding the result provided by his eccentric friend Dr Dai Fwls ap Ryll which was published in our April issue in 1997 and later challenged by one of our readers. We invite you to read his interesting letter.

In the regular *History of Mathematics* column you will find the second part of the story of logarithms. The first part — included in the previous issue— presented the mathematics needed to understand the related history presented in this issue. In the *Computers and Computing* column you will find instructions and explanations for the construction of a paper computer that will help you understand the basic principles behind a Turing machine.

Finally, we include, as usual, problems and solutions. You will also find a report of our special correspondent on Olympiads which includes the problems of the 1998 Australian Mathematical Olympiad and the Tenth Asian Pacific Mathematics Olympiad.

We hope you enjoy this issue of *Function*.

* * * * *

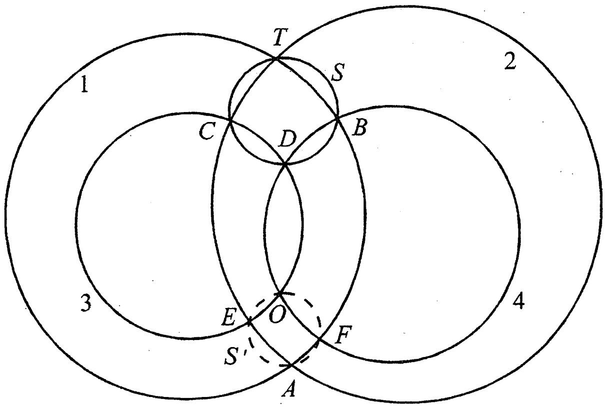
THE FRONT COVER

Michael A B Deakin, Monash University

The Front Cover depicts a very little-known theorem in Euclidean geometry. It comes from a classic text on the subject, H F Baker's *Principles of Geometry.* This is a multi-volume compendium of geometrical knowledge, published in 1922 by Cambridge University Press.

The cover diagram comes from p. 72 of the second volume, devoted to plane Euclidean geometry. Let us explore what it says.

We start with four circles, any four circles in the same plane, and which are labelled 1, 2, 3, 4 in the diagram.



Circles 1, 2 intersect twice at the points labelled T and A; circles 2, 3 likewise intersect at C and E; circles 3, 4 at D and O, and finally circles 4, 1 at B and F.

The theorem then says that:

> *If the points T, C, D, B lie on a circle (S say) then the points A, E, O, F also lie on a (different) circle (S' say).*

We won't offer a proof here — it's actually quite difficult and requires very sophisticated concepts. However we may note that in 1922, this was an *exercise.*

The reader was expected to prove the result with only minor hints from Baker, the author.

This theorem is one of a series of related theorems all to be found on pp. 68–75 of Baker's book. They were first proved by Jakob Steiner a nineteenth–century mathematician whose work did much to revive interest in geometry at that time.

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ERRATUM

In the Front Cover of the February 1998 issue of *Function*, page 2, the wrong name was typed in the 4th paragraph: it was Ramanujan who had a deficient formal education — not Hardy.

* * * * *

National and International Informatics Competitions

Informatics is the European word for Computer Science. The Australian Informatics Olympiad Committee officially formed early in 1998 with a view to organising an National Australian Informatics Competition (the first provisionally planned for late 1998), and selecting Australian secondary students for the International Olympiad in Informatics (the first provisionally planned for 1999). Of the six current international science olympiads, the International Olympiad in Informatics is the third youngest and the second largest, the only larger one being the oldest one, Mathematics.

Interested secondary students should, at this initial stage, monitor the site `http://www.csse.monash.edu.au/~dld/IOI/InformaticsOlympiad.html` for updates and further information.

* * * * *

I love mathematics not only for its technical applications, but principally because it is beautiful; because man has breathed his spirit of play into it, and because it has given him his greatest game — the encompassing of the infinite.

—Rózsa Péter

Playing with Infinity, 1962, New York, Simon and Schuster.

* * * * *

PTOLEMY AND HIS THEOREM

Ken Evans

The golden age of Greek mathematics reached its peak in the 3rd century BC with the work of Archimedes and Apollonius. In the next 400 years there was little theoretical progress in geometry, but significant progress in its application to astronomy, especially by Hipparchus (2nd century BC) and Claudius Ptolemy (2nd century AD).

Ptolemy lived in Alexandria in Egypt which became, perhaps, the greatest centre of learning in the Greek empire. There, as an eminent scholar, he wrote "Syntaxis Mathematica", a collection of 13 books which was called by other scholars "magiste", meaning "greatest". Since Alexandria was a convenient meeting place for Greeks, Romans, Arabs and others, Greek mathematics gradually spread to other civilisations. Ptolemy's works, as well as those of other Greek mathematicians, were translated into Arabic. The Arabs attached the definite article "Al", so Ptolemy's works became known as "Al-magest" (the greatest), and later in Europe as the "Almagest".

In Ptolemy's time there were several theories of planetary motion: Aristarchus, a contemporary of Archimedes, had suggested that the Earth and other planets travel round the sun, but Ptolemy chose the theory in which the earth is fixed in space, and the sun and other planets move around Earth. He used a complicated series of circles and epicycles which "fitted" the observations of Hipparchus and others. Not until the time of Copernicus (1473–1543) and Kepler (1571–1630), who had access to more accurate observations, was the theory of Aristarchus restored.

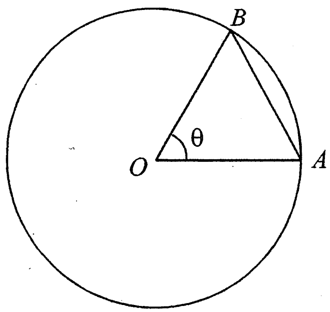


Figure 1

The first book of the Almagest contains a table of values of the length of chord $\overline{AB}$ (Figure 1) of a circle centre O, radius length $OA = 60$ units, the chord subtending an angle of magnitude θ at O. The length, AB, is given as θ increases from $0°$ to $180°$ in steps of $\frac{1}{2}°$. The use of the table of chord length in astronomy occurs in later Almagest books. It is, essentially, the use of what is now called trigonometry.

The first book of the Almagest also gives the methods of calculation of chord lengths. One of the theorems used is now known as Ptolemy's Theorem. Ptolemy must be one of the few mathematicians to have theorems named after them which they did not discover. It is now known that Ptolemy's Theorem was known before Ptolemy's time.

Ptolemy's Theorem: *In a cyclic quadrilateral, the sum of the products of the lengths of each pair of opposite sides is equal to the product of the length of the diagonals, ie. in Figure 2*

$$AB \cdot CD + DA \cdot BC = AC \cdot DB$$

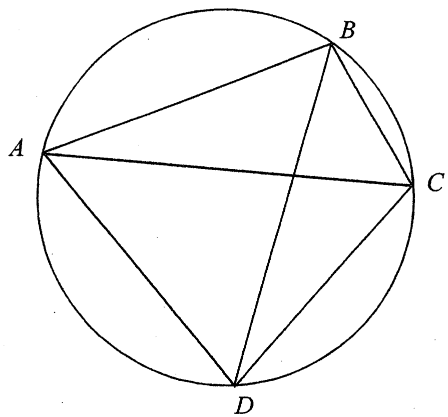


Figure 2

Proof: Denote angle magnitudes by letters shown in Figure 3.

Draw $\overline{CP}\left(P \in \overline{DB}\right)$ so that $\theta_2 = \theta_1$. In ΔACD and ΔBPC,

$$\alpha_1 = \alpha_2 \quad \text{(angles in same segment)}$$
$$\theta_1 = \theta_2 \quad \text{(construction)}$$
$$\therefore \quad \Delta ADC \sim \Delta BPC$$
$$\frac{DA}{PB} = \frac{AC}{BC}$$

$$\Leftrightarrow DA \cdot BC = AC \cdot PB \qquad (1)$$

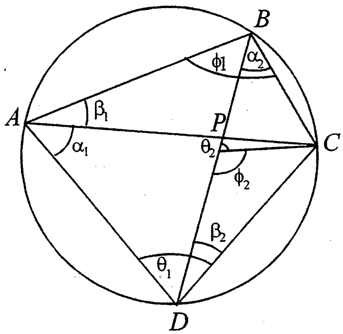


Figure 3

Also, in ΔABC and ΔDPC,

$$\phi_1 = 180° - \theta_1 \quad \text{(opposite angles of cyclic quadrilateral)}$$
$$\phi_2 = 180° - \theta_2 \quad \text{(adjacent angles)}$$
$$\phi_1 = \phi_2 \quad (\theta_1 = \theta_2 \text{ above})$$
$$\text{and} \quad \beta_1 = \beta_2 \quad \text{(angles in same segment)}$$
$$\therefore \quad \Delta ABC \sim \Delta DPC$$
$$\therefore \quad \frac{AB}{DP} = \frac{CA}{CD}$$
$$\Leftrightarrow AB \cdot CD = CA \cdot DP \qquad (2)$$

From (1) and (2)

$$\begin{aligned} AB \cdot CD + DA \cdot BC &= CA \cdot DP + AC \cdot PB \\ &= AC\,(DP + PB) \\ &= AC \cdot DB \qquad (3) \end{aligned}$$

Some special cases:

(i) a) Two adjacent sides are of equal length (Figure 4). If $AB = BC$, equation (3) becomes

$$AB \cdot CD + DA \cdot AB = AC \cdot DB$$
$$\Leftrightarrow AB(CD + DA) = AC \cdot DB \qquad (4)$$

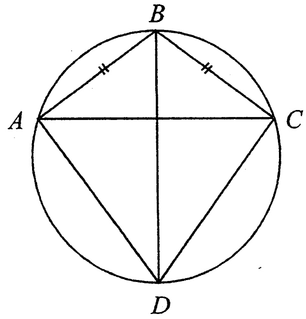


Figure 4

b) If in addition $AB = AC$, equation (4) simplifies to

$$CD + DA = DB. \qquad (5)$$

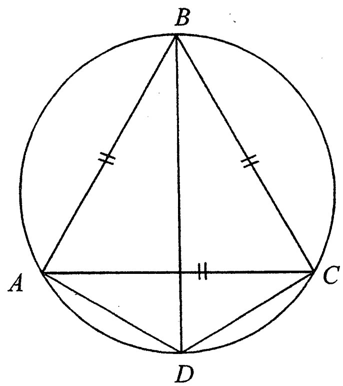


Figure 5

Thus if ΔABC is equilateral (Figure 5), and D is any point on the minor arc AC of the circumcircle, the sum of the distances of D from A and C is equal to the distance from D to B.

(ii) a) Two opposite sides are of equal length. If, as in Figure 6, $BC = DA$, cyclic quadrilateral $ABCD$ is an isosceles trapezium with diagonals $\overline{AB}$, $\overline{BD}$ of equal length. Equation (3), Ptolemy's Theorem, becomes

$$AB \cdot CD + BC^2 = AC^2 \tag{6}$$

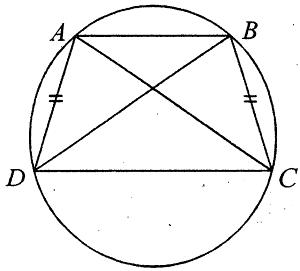


Figure 6

b) If in addition $AB = CD$, $ABCD$ is a rectangle (Figure 7), and equation (6) becomes : $AB^2 + BC^2 = AC^2$, which is the Theorem of Pythagoras.

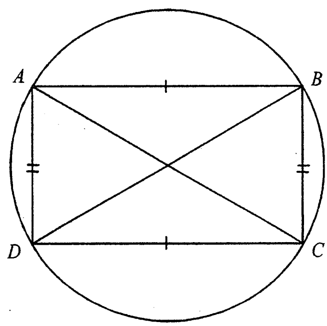


Figure 7

(iii) One side of cyclic quadrilateral $ABCD$ is a diameter of the circle. In this case Ptolemy's Theorem is used to prove a trigonometric formula. Figure 8 is constructed as follows: Draw diameter $\overline{CD}$ of circle, centre O, radius length measure R. From C draw $\overline{CB}$, $\overline{CA}$ so that the magnitude of $B\hat{C}D$ is α and the magnitude of $A\hat{C}D$ is β.

$$0^\circ < \alpha < 90^\circ \;,\; 0^\circ < \beta < 90^\circ \;,\; \beta < \alpha$$

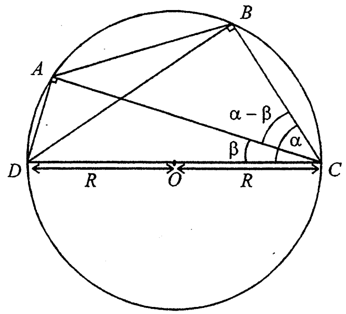
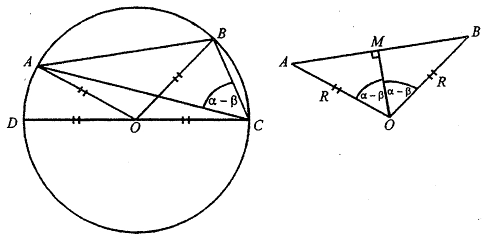


Figure 8

Complete the quadrilateral $ABCD$ and draw $\overline{BD}$. To apply Ptolemy's Theorem, trigonometry is used to calculate side lengths and diagonal lengths in terms of α, β, R.

In the right triangle ΔBCD, $BC = 2R\cos\alpha$ and $DB = 2R\sin\alpha$.
In the right triangle ΔACD, $DA = 2R\sin\beta$ and $AC = 2R\cos\beta$.
Also $CD = 2R$, but it remains to find AB.



(a) (b)

Figure 9

Using Figure 9 (a), the magnitude of $A\hat{O}B = 2(\alpha - \beta)$ [magnitude of angle in a segment is half corresponding central angle magnitude.] Hence (Figure 9 (b)) the magnitude of $A\hat{O}M = \alpha - \beta$. In the right triangle ΔAOM, $AM = R\sin(\alpha - \beta)$, so $AB = 2R\sin(\alpha - \beta)$.

Now, by Ptolemy's Theorem

$$AB \cdot CD + BC \cdot DA = AC \cdot DB$$

Hence

$$\begin{aligned}
&2R\sin(\alpha - \beta)\cdot 2R + 2R\cos\alpha \cdot 2R\sin\beta = 2R\cos\beta \cdot 2R\sin\alpha \\
\Leftrightarrow\ &\sin(\alpha - \beta) + \cos\alpha\sin\beta = \sin\alpha\cos\beta \quad \text{(dividing by } 4R^2\text{)} \\
\Leftrightarrow\ &\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad (7)
\end{aligned}$$

Because $\cos\theta = \pm\sqrt{(1 - \sin^2\theta)}$, equation (7) can be used to calculate $\sin(\alpha - \beta)$ given $\sin\alpha$ and $\sin\beta$. Hence, given chord lengths corresponding to two central angle magnitudes, Ptolemy was able to calculate the chord length corresponding to the difference in central angle magnitudes.

As another application of Ptolemy's Theorem, the following problem is posed: Is it possible to construct an isosceles trapezium in which the length measures of all sides and diagonals are integers? ie. Can Figure 10 be constructed with *a, b, c, d* all integers?

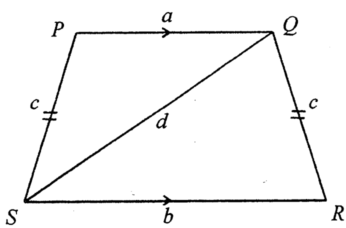


Figure 10

To attempt to solve this problem, choose positive integers *a, c, d* and construct ΔPQS. Note, however, that *a, c, d* cannot be chosen arbitrarily. To construct a triangle at all, the inequalities

$$c + a > d \;,\; a + d > c \;,\; d + c > a \tag{8}$$

must all hold. Once ΔPQS is constructed, a line through S is drawn parallel to $\overline{PQ}$. R is then determined on this line by using the fact that $QR = PS$ (but $\overline{QR}$ is not parallel to $\overline{PS}$). Clearly b is determined by a, c, d. Furthermore, b is calculated using Ptolemy's Theorem because any isosceles trapezium is a cyclic quadrilateral. Equation (3) gives

$$ab + c^2 = d^2 \text{ , hence } \quad b = \frac{d^2 - c^2}{a}$$

If b is to be positive, it is necessary to have $d > c$: the trapezium cannot be drawn if $d \le c$. (Furthermore, if $d > c$, the second inequality in (8) is automatically satisfied.) As a, c, d are positive integers, it is certainly, and remarkably, true that b is a rational number, but may not be an integer as shown in Example 1 below.

Example 1

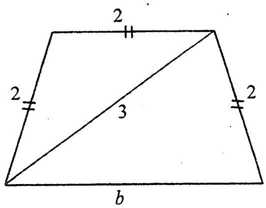


Figure 11

Choose $a = 2,\ c = 2,\ d = 3$

Then $b = \dfrac{3^2 - 2^2}{2} = \dfrac{5}{2}$

which is not an integer.

Example 2

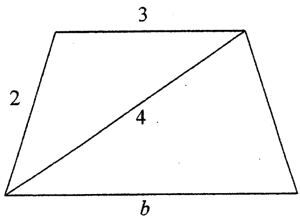


Figure 12

Choose $a = 3,\ c = 2,\ d = 4$

Then $b = \dfrac{4^2 - 2^2}{3} = 4$ which is an integer,

and shows that a solution to the problem exists.

The reader is invited to investigate further.

Finally, notice that it is possible to state the problem in another way: Find a set of 4 points so that the distance between each pair of points is an integer. This is already a generalisation of the original problem, because the 4 points may, for example, be vertices of a kite, which is not a cyclic quadrilateral. This new formulation of the problem suggests another generalisation, namely to a set of 5 points, of 6 points, etc. Again the reader is invited to explore this problem.

References

1. Historical topics for the Mathematics Class-room, 31st Yearbook, NCTM, USA, 1969, pp 359–361, *The Almagest of Ptolemy* ; pp 333–339, *The History of Trigonometry.*

2. Integer Geometry: some examples and constructions, B E Peterson and J H Jordan, *The Mathematical Gazette*, March 1997, Vol 81, No 490, pp 18–28.

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An Excellence Award for our Special Correspondent on Competitions and Olympiads

Our Special Correspondent on Competitions and Olympiads, Associate Professor Hans Lausch, has been named as a 1998 recipient of the prestigious Bernhard H Neumann Award for Excellence in Mathematics Enrichment.

The Australian Mathematics Trust gives the awards annually to outstanding academics and teachers who have made a significant contribution to extending the mathematical knowledge of talented students.

Our Special Correspondent is regarded as Australia's leading authority on current trends in setting mathematical problems of an international standard, and has had a long time involvement in the training of members of the Australian International Mathematical Olympiad teams. He has been the Chair of the Australian Mathematical Olympiad Committee's Senior Problems Committee since 1986.

Congratulations!

THE IMMUNISATION DILEMMA

Michael A B Deakin & Charles R Hunter, Monash University

This year (1997, as we write) the government of Victoria has put into place the last of its step-by-step implementations of a 1991 law requiring all children[1] entering primary school to be immunised[2]. This was in response to an alarming prior development.

Australia's rates of regular immunisation against routinely preventable childhood diseases had been falling steadily and indeed are now at undesirably low levels. The matter has indeed become one of considerable concern. (Just as one example, a recent report from the Australian Bureau of Statistics on this topic contains some very alarming reading.)

There seem to be several reasons for this development. The first of these is simple complacency: many parents (wrongly) feeling that the risks of diphtheria, whooping cough, polio, measles and the like are now so slight as to be non-existent. There are also, of course, the ever–present factors of poverty and other social variables. But beyond these, there is the effect of propaganda emphasising the risks (small as in fact they are) of the immunisation procedure itself.

This situation may be analysed in terms of "Game Theory", a relatively recent branch of mathematics, which emerged definitively during World War II. It now claims many applications. But, most particularly to the point, a "game" known as "The Prisoner's Dilemma" provides an instructive analogy for the question under discussion here[3]. Although, in its simplest form, the analogy is not exact, it is sufficiently close to be instructive, and a fuller (though much more complicated) analysis would allow for more precise correspondence.

In its simplest form, the Prisoner's Dilemma has two "players", in this setting often represented as prisoners. The following account is quoted from Luce and Raiffa's book on the subject: *Games and Decisions*.

> "Two suspects are taken into custody and separated. The district attorney is certain that they are guilty of a specific crime, but he does not have adequate evidence to convict them at a trial. He points out to each prisoner that each has two alternatives: to confess to the crime the police

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[1] But an exception is made in the case of a genuine conscientious objection.

[2] See, on the Internet : `http://hna.ffh.vic.gov.au/phb/hprot/imm/imm.html`

[3] There is an account of this in *Function,* Vol 8, *Part* 1.

> are sure they have done, or not to confess. If they both do not confess, then the district attorney states he will book them on some very minor trumped-up charge such as petty larceny and illegal possession of a weapon, and they will both receive minor punishment; if they both confess they will be prosecuted, but he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor will receive lenient treatment for turning state's evidence, whereas the latter will get 'the book' slapped at him. In terms of years in the penitentiary, the strategic problem might reduce to: [1 year each if neither confesses, 8 years each if both confess, 3 months for a single confessor with 10 years for a single non-confessor]."

The point is that it is best for each *individual* to "rat" on the other, but the benefit of this treachery will not appear unless the other behaves more honourably. In terms of the total number of years, mutual loyalty is the best policy.

This description applies to the so-called "2-person game". There are more general games in which there are N "players"[4]. Such generalised situations are much more complicated but apply to many real-life situations. Luce and Raiffa give an example in which farmers either restrict or don't restrict production. The 2-person situation has also been used to model the arms race and clearly an N-person generalisation is possible in this case as well. And indeed, there have been many other applications.

Either the 2-person or the N-person game may be used as a model of what we may term the "Immunisation Dilemma" facing a parent or guardian. The N-person game provides the better model, but at the expense of much greater complexity, and thus we prefer to present in detail a simpler model based on the 2-person game and merely to make general comments on the more difficult analysis.

In the case of the 2-person game model, the players are the individual child (or its representatives) on the one hand and the community at large on the other. This is already different from the case given above and clearly more complicated. For with the two prisoners, each was "in the same boat". What applied to Prisoner A applied equally to Prisoner B. Here the individual and the community are not on the same footing; they have different status and this makes for some differences in the analysis. However, let us look at the various possibilities.

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[4] See A Rapoport, *N-person Game Theory: Concepts and Applications Resolution* (Ann Arbor: University of Michigan, 1970), pp 79-86 (which includes a very detailed analysis of the case N = 3) or D R Hofstadter's "Metamagical Themas" column in *Scientific American* Vol 248 (6), June 1983.

If the individual child is immunised in a community where immunisation is the norm, then that child is protected from disease, as indeed are all the other (immunised) children. This situation is analogous to the case in which neither prisoner confesses. Both the particular child and children in general assume the small risk of the procedure itself, but avoid the greater penalties associated with the diseases.

For any *individual* child, of course, the optimal situation is to leave *that child* un-immunised but in a community where immunisation is the rule. This way, there is none of the risk of the procedure itself and neither is there risk of disease because all the potential carriers are themselves immune. However, all the risks are borne by the other families in the community together with those risks posed by the un-immunised child. The situation the community at large faces is analogous to that in which one prisoner saves his own neck at the expense of his accomplice.

However, what happens in practice is that with *everybody aiming for this optional result, the actual outcome is in fact the worst.* No one is immunised, and although the minor risk associated with the procedure itself is avoided, the community and all its individual children are at the much greater risk of disease. This case is like that of the two prisoners both confessing, thus avoiding the "minor trumped-up charge" but going down for the crime itself.

This simple model allows us to see the difficulties, but it posits a faceless "community" that in fact does not act as a single decision-maker.

A better model recognises that there are more than two "players"; that the above model is somewhat oversimplified — that the "community" is itself composed of many other deciding parents or guardians. Because the two "players" discussed just now are not equivalent, as in the earlier case of the prisoners, the situation lacks the essential symmetry of the original.

So the full analysis really requires the N-person game, since in real life there will be varying levels of immunisation in a population, and medically there is the very real question of threshold levels of immunisation allowing epidemic or endemic disease to take hold. This makes for much more complicated analysis and indeed, the full analysis of the N-person game presents very considerable difficulty.[5]

[5] Even the references given in Footnote 4 above fail to give full detail.

In particular, the important medical concept of "herd immunity" really requires the N-person game if it is to be fully incorporated in the model. This concept allows for the observation that a disease may be eliminated from a community even without full immunisation programmes.

This "herd immunity" is an effective threshold level that varies considerably from disease to disease. If a disease is very easily transmitted then we need to immunise almost everybody; if, by contrast, transmission is difficult, then it may suffice to immunise a smaller percentage of the population.

As examples, herd immunity is only achieved with 96% immunisation in the case of measles, which is highly contagious. However, in the case of diphtheria (which is less easily transmitted), a rate of 75% will suffice.

Clearly, the best interests of society at large are served by ensuring that the required level of immunisation is achieved. By April 1995 (the date of the Commonwealth Statistician's report), the proportions of fully immunised children in Australia between the ages of 3 months to 6 years had actually fallen below the herd immunity levels in both these cases. In the case of measles the level[6] was 91.6%, while in that of diphtheria it was 68.6%.

The "Prisoner's Dilemma Game" is often correctly presented as a conflict between the advantages of selfishness on the one hand and of cooperation on the other. (For this reason it has been the subject of several psychological studies.) In the abstract, it is not all that difficult to see where the community's best interest lies. When it comes to the nitty-gritty of decision-making, however, matters are otherwise. Not everyone plays the "game" according to rules that seem ethical to those with the overall good in their minds.

Luce and Raiffa in the book quoted above put the matter rather well. "The hopelessness that one feels ... is inherent in the situation." The matter is one of ensuring mutual cooperation in a situation where others, whose cooperation we require, have what they perceive as a strong incentive to withhold that co-operation.

We feel that these same authors also have the matter right, when they refer to a higher authority (which interposes rather than waits for the supposedly enlightened

[6] As we write, a New Zealand epidemic of measles is threatening Australia, because we do not have sufficient immunity to achieve the 96 % level.

"players" to cooperate of their own accord)[7]. As they say: "Indeed, some hold the view that one essential role of government is to declare that the rules of certain social 'games' must be changed whenever it is inherent in the game situation that the players, in pursuing their own ends, will be forced into a socially undesirable position. ... Finally the government feels as we do, steps in, and passes a law ... ."

It appears to us that the time for such action has indeed come in respect of the immunisation dilemma and for this reason we applaud the initiative taken by the Victorian Government. The cooperation they wish to enforce is clearly in the communal interest; previous selfishness on the part of a sufficiently large minority of the community reached proportions large enough to give grounds for concern.

When the actions of a minority threaten the common good, then there is need for government intervention. We might even say that that is what governments are for! Indeed, it is not particularly difficult to enforce immunisation, as in this case by making it a prerequisite of school enrolment. Here indeed is an area where the time has come for the government to "feel as we do, step in and pass a law". And of course make sure it is *enforced.*

Further Reading

The figures we use come from several sources. The Commonwealth report we cite is "Children's Immunisation in Australia" by W McLennan. See also *Immunisation* by G Dick and *Medical Virology* by D O White and F J Fenner. For a further discussion of the matter reporting recent research at Monash University, see the April 1997 issue of *Montage*, which is available also in electronic form on the Internet at

`http://www.monash.edu.au/pubs/montage_97-01/p1.html`

* * * * *

Why are *wise* few, *fools* numerous in the excesse?
'Cause, wanting *number*, they are *numberlesse*.

Lovelace
Noah Bridges: Vulgar Arithmetike, 1659, London

[7] A similar resolution may be read into the technicalities of a detailed analysis by Rapoport in a *Scientific American* article (January 1967)

TO THE EDITOR

I recently had a call from the Chief Editor of *Function*, Dr Varsavsky. She it was who asked me to write this letter. Well, not *this* letter precisely; what she really wanted was news from my eccentric Welsh correspondent Dr Dai Fwls ap Rhyll. But this causes a difficulty for I haven't heard a thing from him since March 1997.

My distinct feeling was that Dr Varsavsky was not exactly happy to be told this news — indeed I rather got the impression that she wanted me to make something up. This course of action, as I trust readers will know, is quite repugnant to me. Insofar as I can achieve it, I aim to write nothing in these reports other than the literal and exact truth!

My occasional reports to *Function* are mostly concerned with the work of Dr Fwls. He has made a name for himself by challenging existing notions, particularly in mathematics. He has conclusively demonstrated that many of what we think of as mainstream mathematical results are in fact false.

Last year he considered the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ldots$$

whose sum is usually given as $\ln 2$, the natural logarithm of 2. Dr Fwls, however, gave a demonstration that in fact the sum is 0. His proof was given in full in my last report, and so I won't repeat it here, but it was quite simple.

However it was challenged (*Function Vol* 21, *Part* 5) by Peter Bullock of Norwood Secondary College, who derived the interesting result

$$\ln 2 = \lim_{n \to \infty} \left(\frac{2}{2n+1} + \frac{2}{2n+3} + \ldots\ldots + \frac{2}{4n-1} \right)$$

(or rather this result can be deduced from what Mr Bullock wrote).

Remarkably, this result has already appeared in *Function*, or rather something to which it turns out to be equivalent. In *Vol* 6, *Part* 3, there was a brief note on "The Rainbow Series". It appeared over the initials MD, which happened to be those of the then Chief Editor, Michael Deakin. This was rather dishonest of him; the material on which his note is based was in fact provided to him by a very good friend of mine, Sue De Nimmes.

Ms de Nimmes discovered the result

$$\ln 2 = \lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n} \right)$$

(which I leave readers to show to be equivalent to that given by Mr Bullock) in an obscure pamphlet entitled *The Rainbow Series and the Logarithm of Aleph Null*. This was privately published by W P Montague in 1940. Montague was a professor of Philosophy at Barnard College in the United States and evidently was also interested in mathematics.

He called his result the "Rainbow Series" because as n goes to infinity in the formula above, both of its ends, the start and the finish, vanish, as he put it,"over the horizon", as with a rainbow.

(Readers who are curious to see Montague's work will find a copy of his pamphlet in the library of the University of Melbourne. It is bound together with various other such things in a volume called *Mathematical Pamphlets*.)

Montague went on to the discovery of many paradoxes in standard mathematics. In particular, he disputed the standard account of transfinite numbers[1]. On this account, there is a smallest such number written $\aleph_0$ (and pronounced "Aleph Null"), and representing the number of all the integers. Montague used his Rainbow Series to investigate the logarithm of $\aleph_0$ and so found a smaller transfinite number than $\aleph_0$ itself, the logarithm of this new number was an even smaller transfinite number, and so on.

Thus Montague discovered an entire new class of numbers, transfinite but progressing backwards towards the ordinary finite numbers. He had rather a nice description of them: "the pygmy members of a giant race".

All this is very much in the spirit of Dr Fwls, so much so that I was tempted to believe at first that Professor Montague and Dr Fwls might be one and the same person. However a consideration of the likely dates puts this out of court as Dr Fwls is not old enough to have been a professor in 1940. History does from time to time produce such maverick geniuses; Montague was one, Dr Fwls is another.

So apologies to Dr Varsavsky and to my readers. There is no recent news from my wayward correspondent. This repetition of an earlier story will have to

[1] See *Function Vol* 2, *Parts* 1 and 2.

take his place this time. Perhaps by next year I will have heard again from this brilliant but eccentric genius.

Kim Dean
Union College
Windsor

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The Role of Mathematics in Australian Football

The Monash University School of Computer Science and Software Engineering's probabilistic footy-tipping competition is now being advertised amongst secondary students. The top 10 secondary students in the competition will be awarded prizes to the tune of a total of no less than $\$2^9$, or $\$512$. For each game in the footy season, tipsters choose a team they think will win and assign a probability to it between 0 and 1.

If you assign a probability of p to the team that ended up winning a game and $1-p$ to the team that ended up losing it, then your score for that game is $1+\log_2 p$. This is done for at least two reasons. The first reason is so that an ultra-conservative tipster who tips $p=1-p=0.5$ throughout the season will end up with a score of $1+\log_2 0.5=1+(-1)=0$. A tipster who has some knowledge of football but does not make unrealistically extreme tips will be expected in the long run to get a positive score, whereas a tipster whose tips are extreme will inevitably end up with a negative score. The second reason behind our scoring function becomes clear if you suppose that the results of the games really are being generated from some process giving them a probability, q. In this case one can use differential calculus to show that the optimal probability to tip will be q.

This competition was initiated by Dr David Dowe and Monash CSSE colleagues in 1995, and is open to all comers. To join in, point your probabilities to `http://www.csse.monash.edu.au/~footy/` or e-mail `footy@cs.monash.edu.au` . Techniques used by some of the more serious entrants in this competition have been used to find patterns in DNA. It's never too late to start — there always seem to be plenty of over-confident tipsters around with negative scores to ensure that late starters with a score of 0 aren't at the bottom.

* * * * *

HISTORY OF MATHEMATICS

The Story of Logarithms – Part 2

Michael A B Deakin

My previous article contained quite a lot of mathematics and very little history. This is because we can't understand the history until we see what the mathematics is all about. (This is in fact a common occurrence, and is one of the factors that makes the history of mathematics actually quite difficult to study.) In that earlier article we saw that *logarithmation* was one of the six or seven basic operations of the number system (there are seven in all, but two turn out to be the same, so six).

A logarithm is a third number calculated from two given numbers. To keep matters simple, I will only consider here the case in which both these numbers are real and positive. This will ensure that the new number, the logarithm, is itself real (but not necessarily positive). I chose before the example of 2 and 8 as the given numbers. They have different status, one (in this case the 2) being called the *base*.

The new number is usually written $\log_2 8$, and pronounced "log of 8 to base 2". It is the answer to the question that last time I wrote 2 ^ ? = 8, but which in more standard notation would be written $2^x = 8$. Clearly this answer, the new number or logarithm, is 3. It was noted in the previous article that the base cannot be 1.

I also used a personal and idiosyncratic notation. I did this to stress the fact that logarithmation was an arithmetical operation, just like the other five: $+$, $-$, $\times$, $\div$ and ^. However I will not continue here with this practice. Rather I shall use a fact noted in that earlier article: because of the "change of base formula", it is enough to consider logarithms taken to some standard base. The earlier article also noted that in practice only three standard bases are used: 10, 2 and e.

Base 2 has some specialist uses that I won't consider further. The logarithm (or more simply log for short) of a number n to base 10 will be written $\log n$. Such logarithms are referred to as "common logarithms". The log of n to base e will be written $\ln n$; such logs are called "natural logarithms". This notation is now standard on most calculators.

We today can calculate logarithms (to either of these bases) simply by pressing the appropriate button on a scientific calculator, but this is a relatively recent luxury (and besides, the *calculator* must "know" how to find the logarithm). Without a sufficiently sophisticated calculator, the calculation of logarithms is very difficult. However, once it is done and a table made out, then the use of logarithms greatly simplifies difficult calculations.

For instance, multiplication is reduced to addition. Suppose we had to multiply 1234567 by 7654321. Without a calculator, and without resorting to special tricks, the task would be quite tedious. However once we know that

$$1234567 = 10^{6.0915147} \quad \text{and} \quad 7654321 = 10^{6.8839067}$$

then we may simply add the two indices (that is to say the logarithms) and so find

$$1234567 \times 7654321 = 10^{12.975421} = 9.44977 \times 10^{12}.$$

(Because the logarithms are not themselves exact, but 8-figure approximations, this answer is not quite exact either; correct is 9 449 772 114 007, or in other words $9.449772114007 \times 10^{12}$. However the answer given is a very good approximation, good enough for most practical purposes.)

The difficult parts of the calculation, the finding of those numbers 6.0915147 and 6.8839067, and the calculation of the value of $10^{12.975421}$ were once achieved by the use of carefully constructed tables.

In a similar way, division was replaced by the subtraction of the logarithms of the two numbers, square roots could be evaluated simply by halving the logarithm, and so on. All one needed was a good table of logarithms.

This meant that there was a great demand for such tables. The first attempt to construct such a table was that of John Napier, a Scot who lived from 1550 to 1617. It is sometimes wrongly thought that Napier used natural logarithms, which are occasionally called naperian logarithms[1]. This is however based on a misunderstanding. In fact Napier came to his logarithms by a very roundabout route and did not explicitly employ the concept of a base at all. As has several times been remarked "Pioneering work is often clumsy"!

[1] They are also sometimes called hyperbolic logarithms for reasons that I won't go into. Nor is it clear why the letter *i* is dropped from the spelling of Napier's name.

So, rather than go through all the various steps Napier used, let me get right to the punch-line and say that what he did was later realised to have been equivalent to calculating logarithms to the base $\left(1 - \frac{1}{10^7}\right)^{10^7}$ a rather strange choice at first sight. The value of this number is 0.3678794 to all the accuracy my CASIO*fx* 7 will give, and this is also (to the same accuracy) the value of $\frac{1}{e}$. More accurately, we have

$$\frac{1}{e} = 0.367879441 \quad \text{and} \quad \left(1-\frac{1}{10^7}\right)^{10^7} = 0.367879423.$$

Logarithms to the base $\frac{1}{e}$ are simply the negatives of logarithms to the base e (can you prove this?), that is to say natural logarithms, and so Napier did not exactly tabulate natural logs, but rather something very similar to them.

There is a good reason for the close correspondence. One of the properties of the number e is that

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

and in Napier's case he had the large number 10^7 for n and the value -1 for x. So Napier's "base", as a result of this theoretical formula, lies close to e^{-1}, in other words to $\frac{1}{e}$, as claimed.

Napier died in 1617, leaving most of his work unpublished. The only work on logarithms[2] published in his lifetime was *Mirifici logarithmorum canonis descriptio*[3], now usually referred to as the *Descriptio*; all his other work on the topic appeared after his death.

Napier's logarithms and their close cousins, the natural logarithms, are the easier ones to calculate, but not necessarily the easiest to use. Common logarithms, using base 10, have a lot of things going for them. This is because we express

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2 There was an earlier theological tract, not here relevant.

3 Latin for *An Account of the Wonderful Table of Logarithms*.

numbers in base ten; that is to say we have a decimal system and the place notation which uses ten as the basis of its positional information.[4]

Just as an example: tables tell us that $\log 2 = 0.3010300$. It follows that, because $20 = 2 \times 10$,

$$\log 20 = \log 2 + \log 10 = 0.301030 + 1 = 1.3010300$$

and similarly $\log(0.02) = \log 2 - \log 100 = -2 + 0.3010300$, and so on. We only need to know the logs of the numbers between 1 and 10, and those of all other numbers can be deduced.

This was the insight of the English mathematician Henry Briggs (1562–1630), a somewhat younger contemporary of Napier's, and with whom he kept in touch. He managed to convert Napier to his system, but Napier died just as a potential collaboration was reaching fruition. That was in 1617, the year in which Briggs' account of his work was published. His book was called *Logarithmorum chilias prima*, Latin for "The First Thousand Logarithms": in other words he had computed one thousand entries in a logarithmic table.

In theory, Briggs could have converted Napier's already published tables and used the change of base formula to get from Napier's (almost) natural logarithms. All that is required is to multiply each entry by the factor 0.4342944 (ie. $\log e$). However Briggs did not take this route. What he did was something very clever. He had to do it longhand, but readers can use their calculators to follow very closely and easily in his footsteps.[5]

He began with the number 10. Then he took its square root: $\sqrt{10} = 3.1622777$; then he took the square root of this new number, $\sqrt{\sqrt{10}} = 1.7782794$, and then the square root of *that* and so on. This generates the sequence of numbers: 3.1622777, 1.7782940, 1.3335214, 1.1547820, 1.0746078, 1.0366329, 1.0181517, 1.0090350, 1.0045074, 1.0022511, 1.0011249, 1.0005623, 1.0002811, and so on. I have given only 13 terms in the sequence and put in only seven decimal places. Briggs went on to calculate 54 terms in the sequence. This needed great accuracy and many many more decimal places than my little pocket calculator will hold. After 24 terms in the sequence, my calculator

[4] For an account of the possibilities offered by other bases, see *Function, Vol* 9, *Part* 1.

[5] There is an excellent account readers may care to consult in Chapter 22–4 of *The Feynman Lectures on Physics*, *Vol* 1 by R P Feynman, R B Leighton and M Sands. This gives rather fuller details than I do about the actual techniques of computation.

simply keeps returning 1.0000001. This is round-off error. The sequence in fact gets closer and closer to 1, the difference from 1 being approximately halved with each successive square root. The number Briggs reached was extremely close to 1.

Using a few tricks I calculated it and got 1.000 000 000 000 000 127 819. As I say, very close to 1 (and certainly way outside my calculator's range without the tricks!). Briggs did his calculations even more accurately than I did. Where I have used 21 decimal places, he used 30.

But now Briggs was in a position to build up his table of logarithms. He now had $\log(10^{1/2^{54}}) = 1/2^{54}$. But $10^{1/2^{54}}$ is the number

$$1.000\,000\,000\,000\,000\,127\,819$$

just calculated and $1/2^{54}$ can also be calculated by taking terms from the sequence 1, 0.5, 0.25, etc to 54 terms. The result is 5.55111×10^{-17}, that is to say the number

$$0.000\,000\,000\,000\,000\,055\,511\,1,$$

and so we have

$$\log(1.000\,000\,000\,000\,000\,127\,819) = 0.000\,000\,000\,000\,000\,055\,511\,1$$

as the basic entry in the table, but along the way he had also calculated another 53 logarithms.

Then by multiplying 1.000 000 000 000 000 127 819 by itself and by adding 0.000 000 000 000 000 055 511 1 to itself the same number of times in each instance an entire table can be built up. As long as the numbers are small the multiplications are relatively easy as numbers just a little over 1 can be multiplied[6] by adding the amounts by which the numbers exceed 1. Nonetheless it is time-consuming work and indeed Briggs did not manage to complete his entire programme. The work was finished by the Dutch mathematician, Adriaan Vlacq (1660–1666 or 1667) in 1628, and all subsequent tables have made some use of this work.

The computational use of logarithms was very much a part of the Year 10 syllabus in my own schooldays. Indeed I still own a copy of the textbook we used as well as some of the tables we needed. There was an interesting peculiarity of the way arithmetic was done with logs.

[6] That is to say: if ε and η are both small then $(1+\varepsilon)(1+\eta) \approx 1 + \varepsilon + \eta$.

We saw above that $\log 2 = 0.3010300$, that in consequence $\log 20 = 1.3010300$, and similarly that $\log(0.02) = -2 + 0.3010300$, and so on. The tables give only the logs of numbers between 1 and 10. All other numbers are deduced from these. All the logarithms actually tabulated lie between log 1, which is 0, and log 10, which is 1. Any logarithm lying in this range are referred to as a *mantissa*. When it is modified by placing an integer before the decimal point (as with the 1 and the -2 in the examples above) this integer is termed the *characteristic*.

In writing the numbers arising from the process of logarithmation, a form was adopted in which *the mantissa was always positive*. Thus in the last of the examples given above, we wrote log (0.02) as given rather than simplifying it to the more usual form -1.6989700.

In practice, a shorthand was adopted. Rather than write $-2 + 0.3010300$, as in this example, the minus sign was placed *above* the characteristic so that we had

$$\log(0.02) = \bar{2}.3010300.$$

This convention was used whenever the logarithm turned out to be negative, that is to say whenever the original number lay between 0 and 1. A number written in this form was pronounced "bar two point three ... " or whatever.

Students of that time needed to know how to express negative numbers in the "positive mantissa" form. Here is an example of such a calculation. We have

$$\log 2 = 0.3010300$$

and we want to find $\log\left(\frac{1}{2}\right)$ let us say. Well

$$\log\left(\frac{1}{2}\right) = \log 1 - \log 2 = -\log 2 = -0.3010300$$

but this is not in the right form. However this is easily fixed

$$-0.3010300 = -1 + 0.6989700 = \bar{1}.6989700$$

which is right because $\frac{1}{2} = 0.5$ and $\log 5 = 0.6989700$.

There is a lot more that could be said, but I hope I have given you a taste of a subject now not studied as intensively as was once the case.

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COMPUTERS AND COMPUTING

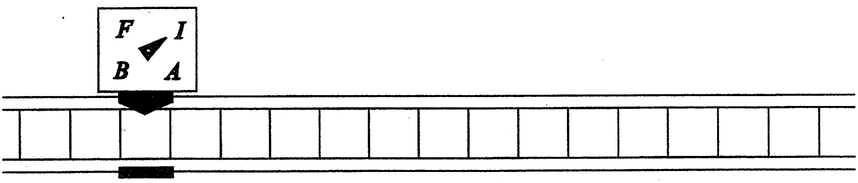
Construct your own Computer

Cristina Varsavsky

Alan Turing (1912–1954) was a very creative computer scientist and mathematician. He is best known for his invention of the concept of Turing machines for the modelling of computations.

Turing machines are very simple but extremely powerful. A Turing machine consists of a *control unit* which can be in a number of states, and a *tape* divided into cells, which in theory, is infinitely long. Each cell contains a symbol from a defined alphabet. A Turing machine performs computations by reading and writing on the tape as the control unit moves back and forth along it. For each symbol read, and depending on its state, the machine will write a symbol on the tape, possibly change to another state, and move left or right, or simply halt. The operation of the machine is defined by a function of the current state and the current symbol on the tape.

The best way to understand the concept is by constructing your own Turing machine. Let's make a paper machine that will add two non-negative numbers. You will need a long strip of paper subdivided in cells, and a slider made of cardboard that can indicate the state in which the machine is, as shown below.



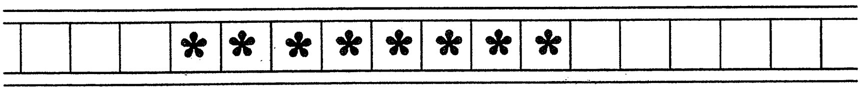
The alphabet will consist only of two symbols: * and +. The number n will be represented on the tape with $n + 1$ asterisks. So to calculate $3 + 4$ we will write the following on the tape:

	*	*	*	*	+	*	*	*	*	*					

The machine will have four states: *initial*, *A*, *B*, and *final*. The function is defined by the following table:

Current state	**Reads**	**Next state**	**Writes**	**Moves**
initial	*	*A*	*blank*	right
A	+	*final*	*blank*	right
A	*	*B*	*blank*	right
B	*	*B*	*	right
B	+	*final*	*	right

So let us follow the table to compute $3 + 4$. We start in the *initial* state with the control unit on the left-most asterisk. According to the table, the machine clicks to state *A*, erases the asterisk, and moves one cell to the right. The control unit is now in state *A* reading * on the tape; therefore it clicks to state *B*, erases the cell, and moves one cell to the right. Now your paper machine is in state *B* reading * ; so it only moves to the right, and does so three more times to reach state *B* reading +. Following the instruction in the last row of the table, the machine writes * over + and stops. The result of the calculation is shown below: 8 asterisks, meaning that $3 + 4 = 7$.



This is, in plain language, how the machine performed the addition: starting from the left, the two first asterisks were erased, and the + symbol was replaced by an asterisk. A very simple procedure but certainly not the most efficient method for adding two numbers!

But efficiency was not the purpose Turing had in mind when he invented these computers. His goal was to prove that a Turing machine can always be found to compute anything that is *computable*. However, even though we know that it is possible to construct Turing machines to carry out tasks as simple as multiplying two integers, the construction of such machines could be quite involved.

As an exercise, I suggest you to build some other simple machines. For example, one that outputs $n - 2$, for $n \geq 2$, and 0 for $n = 0,1$; or one that finds the minimum of two positive integers. It could be great fun!

PROBLEM CORNER

SOLUTIONS

PROBLEM 21.5.1 (K R S Sastry, Bangalore, India)

Show that the infinite arithmetic progression 2, 7, 12, 17, ... does not contain any triangular number. (The rth triangular number, T_r, is given by $T_r = \frac{1}{2}r(r+1)$.)

SOLUTION by K R S Sastry

The given arithmetic progression is $A_n = 5n + 2$, $n = 0, 1, 2, 3, \ldots$. This sequence may be looked upon as the sequence of positive integers that leave a remainder of 2 on division by 5.

The sequence of triangular numbers is 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, ... Let us divide these numbers by 5 and examine the pattern of the remainders:

$$1, 3, 1, 0, 0, 1, 3, 1, 0, 0, \ldots$$

To see that this pattern is periodic with period 5, let's evaluate $T_{r+5} - T_r$, $r = 1, 2, 3, \ldots$:

$$\begin{aligned} T_{r+5} - T_r &= \tfrac{1}{2}(r+5)(r+6) - \tfrac{1}{2}r(r+1) \\ &= 5(r+3) \end{aligned}$$

Hence $T_{r+5} - T_r$ is divisible by 5, so T_{r+5} gives the same remainder as T_r when each is divided by 5. From this result we conclude that when triangular numbers are divided by 5, there will never be a remainder of 2 or 4. Hence the given sequence 2, 7, 12, ... does not contain any triangular number. Of course, the sequence $A'_n = 5n + 4$, $n = 0, 1, 2, \ldots$, i.e. 4, 9, 14, 19, ... , does not contain any triangular number either.

The problem was also solved by Julius Guest (East Bentleigh, Vic), Lachlan Harris (Gisborne South, Vic), Keith Anker, and Carlos Alberto da Silva Victor (Nilópolis, Brazil).

PROBLEM 21.5.2 (K R S Sastry, Bangalore, India; modified by the editors)

Let ABC be an isosceles triangle with $AB = AC$, and let O be its circumcentre. Let D be a point on $\overline{AB}$ such that $AD = BC$. If C, O and D are collinear, prove that $\angle BAC = 20°$.

SOLUTION by Carlos Victor

Let H be the midpoint of $\overline{BC}$, let $\angle BAH = \theta$, and let P be the point on $\overline{OH}$ such that $\angle OBP = \theta$ (see Figure 1). Since O is the centre of the circumcircle, $OA = OB$, so the triangle OAB is isosceles and hence $\angle OBA = \theta$. Therefore $\angle BOH = 2\theta$ (by applying to triangle OAB the theorem that the exterior angle of a triangle equals the sum of the interior opposite angles). By symmetry, $\angle COH = 2\theta$, so $\angle AOD = 2\theta$. Thus the triangles ADO and BPO are congruent., and consequently $BP = AD$. Since we are given that $AD = BC$, it follows that $BP = BC$, and hence that triangle BPC is equilateral. Now $\angle BPH = 3\theta$ (again because the exterior angle of a triangle equals the sum of the interior opposite angles), so $6\theta = 60°$ and hence $\theta = 10°$. Therefore $\angle BAC = 20°$.

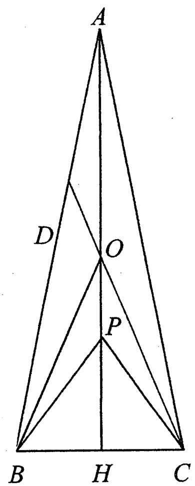


Figure 1

Note that this proof does not use the hint that was provided when the problem was proposed in the October 1997 issue of *Function*.

Solutions were also submitted by Julius Guest, Keith Anker, and the proposer.

PROBLEM 21.5.3 (K R S Sastry, Bangalore, India)

Let $\overline{AD}$ be an altitude of triangle ABC. If the reciprocals of the lengths AD, AB and AC are the side lengths of a right-angled triangle, find the maximum value of the degree measure of angle BAC.

SOLUTION

Since the reciprocals of the lengths AD, AB and AC are the side lengths of a right-angled triangle, and AD is the smallest of the three lengths, the reciprocal of AD must correspond to the hypotenuse. Thus we have:

$$\frac{1}{AD^2}=\frac{1}{AB^2}+\frac{1}{AC^2} \tag{1}$$

Now, triangles ABD and ACD are right-angled, so:

$$AB^2 = AD^2 + BD^2 \text{ and } AC^2 = AD^2 + CD^2 \tag{2}$$

Substituting these results into equation (1), we obtain:

$$\frac{1}{AD^2}=\frac{1}{AD^2+BD^2}+\frac{1}{AD^2+CD^2} \tag{3}$$

After multiplying equation (3) through to clear all the fractions, and then simplifying as far as possible, we eventually obtain:

$$BD.CD = AD^2 \tag{4}$$

If the altitude AD lies within the triangle ABC, as shown in Figure 2, then $BC = BD + DC$. Therefore:

$$\begin{aligned} BC^2 &= (BD+DC)^2 \\ &= BD^2 + 2BD.CD + CD^2 \\ &= AB^2 - AD^2 + 2AD^2 + AC^2 - AD^2 \quad \text{by equations (2) and (4)} \\ &= AB^2 + AC^2 \end{aligned}$$

Hence the angle BAC measures $90°$. If, on the other hand, AD lies outside the triangle ABC, then ABC must be an obtuse-angled triangle (with the obtuse angle at either B or C), and so angle BAC must be less than $90°$ in this case.

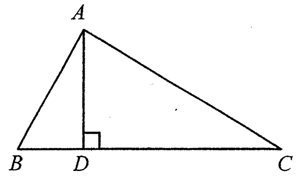


Figure 2

We received solutions to this problem from Carlos Victor, Keith Anker, and the proposer.

PROBLEM 21.5.4

A list of finite sequences of numbers is constructed as follows. The first sequence is just the number 1. Each sequence apart from the first "describes" the previous sequence in the following sense: the first number in the previous sequence is listed, preceded by the number of times it occurs consecutively, then the next number is listed, preceded by the number of times it occurs consecutively, and so on.

Thus:

First sequence: 1

Second sequence: 1, 1 (since the first sequence can be described as "one 1")

Third sequence: 2, 1 (since the second sequence is "two 1s")

Fourth sequence: 1, 2, 1, 1 (since the third sequence is "one 2, one 1")

Fifth sequence: 1, 1, 1, 2, 2, 1 (since the fourth sequence is "one 1, one 2, two 1s")

etc.

Which numbers can occur as terms in these sequences?

SOLUTION by Keith Anker

Only 1, 2 and 3 can occur. The numbers 1 and 2 occur in the sequences given, and 3 occurs in the sixth sequence. If any larger number were to occur, then there would be four identical numbers, x, x, x, x, appearing consecutively in the preceding sequence. Now, either the first or the second x is a "count". In the former case, we have the description "x x's followed by another x x's", which would have correctly been written "$2x$ x's". In the latter case, we have "y x's followed by another x x's" (where y denotes the term preceding the first x), which would have correctly been written "$(y + x)$ x's". Therefore, no number larger than 3 can occur.

PROBLEM 21.5.5 (from *Mathematical Spectrum*)

Find all prime numbers p for which $2p - 1$ and $2p + 1$ are both prime.

SOLUTION

If $p = 2$ then $2p - 1 = 3$ and $2p + 1 = 5$, which are both prime. If $p = 3$ then $2p - 1 = 5$ and $2p + 1 = 7$, which again are both prime.

Now suppose p is a prime other than 2 or 3. One of the consecutive numbers $2p - 1$, $2p$, $2p + 1$ must be divisible by 3. Since p is prime and $p \neq 3$, $2p$ is not divisible by 3. Therefore at least one of $2p - 1$ and $2p + 1$ must be divisible by 3. But the only prime that is divisible by 3 is 3 itself, so either $2p - 1 = 3$ or $2p + 1 = 3$. The first equation gives $p = 2$, contrary to assumption, while the second equation gives $p = 1$, which is not prime. Therefore 2 and 3 are the only prime numbers that satisfy the condition.

We received solutions to this problem from Lachlan Harris,Carlos Victor, and Keith Anker.

PROBLEM 21.5.6

Find the mistake in the following reasoning:

"Let $I = \int \frac{1}{x} dx$. Integrate by parts:

$$I = \int 1 \times \frac{1}{x} dx = x \times \frac{1}{x} - \int x \left(-\frac{1}{x^2} \right) dx = 1 + \int \frac{1}{x} dx = 1 + I$$

Now subtract I from both sides to obtain $0 = 1$."

SOLUTION

We can resolve the paradox if we first clarify the exact meaning of the expression $\int \frac{1}{x}\,dx$. The expression stands for *an* antiderivative of $\frac{1}{x}$. (It doesn't matter *which one* it stands for, but it must stand for a particular one.) Each of the other antiderivatives of $\frac{1}{x}$ can be obtained by adding a constant to this one. Now, when the expression $\int \frac{1}{x}\,dx$ appears for a second time in the calculation, we cannot assume it is the same antiderivative as appeared the first time. In order to allow for this, we must write the second occurrence of the antiderivative as $\int \frac{1}{x}\,dx + c$, where c is a constant (as yet unspecified). When we reach the end of the calculation, we discover that c does in fact have a specific value: -1.

We received a solution from Carlos Victor, and an imaginative solution in dramatised form from Keith Anker (which limitations of space unfortunately prevent us from reproducing).

A correction and a clarification to two problems from the February 1998 issue

In Problem 22.1.2, the equation to be proved should read $DC = DA + DB$.

In Problem 22.1.6, it is stated in the hint that $\cot x$ can be expressed in terms of $\tan x$ in two different ways. Strictly speaking, one of the two ways is not an expression for $\cot x$ in terms of $\tan x$, but rather an equation relating the cot function to the tan function.

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 1 June will be acknowledged in the August issue, and the best solutions will be published.

Lachlan Harris, who was in Grade 6 last year when he produced his solutions to Problems 21.5.1 and 21.5.5, also sent us several entertaining problems from a book called *The Chicken from Minsk*. We give two of them below. Lachlan drew an amusing cartoon to accompany the first problem, which regrettably we are unable to reproduce.

PROBLEM 22.2.1 (from *The Chicken from Minsk*; submitted by Lachlan Harris)

Old Man Mazay rows for his vodka

Old Man Mazay (the alcoholic) is rowing down a river. The current is 2 miles/hour. Just as Mazay is passing under a bridge, he takes a drink, but instead of returning the bottle to the stern of the boat, he drops it into the river! Mazay continues rowing downstream for half an hour, until he realises he is thirsty once again. Mazay rows at 3 miles/hour, but aided by the current, he goes at 5 miles/hour. How long will Mazay take to retrieve the bottle, and how far from the bridge will he be at that time?

PROBLEM 22.2.2 (from *The Chicken from Minsk*; submitted by Lachlan Harris)

Masha's mathematical turtles

Masha has trained her four turtles to always follow each other. She arranges them at the corners of a square, as shown in the diagram [Figure 3], with each turtle facing its clockwise neighbour. The turtles move at one constant speed, V. What will happen to the square [with a vertex at each turtle]? When will the turtles meet?

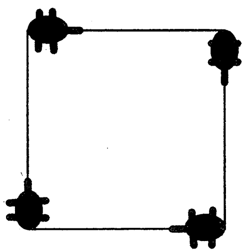


Figure 3

PROBLEM 22.2.3 (based on a problem in *New Scientist*)

Three churches A, B and C are equally spaced from one another, ie., they lie at the vertices of an equilateral triangle. George is standing at a point which is 8 km from A, 5 km from B and 3 km from C.

(a) Show that George must be outside the triangle.
(b) How far apart are the churches?

PROBLEM 22.2.4 (Garnet J Greenbury, Brisbane, Qld)

Let k be the positive solution of the equation $k^2 + k - 1 = 0$. Prove that:

$$k^n = (-1)^{n+1}(a_n k - a_{n-1}) \quad (n \geq 2)$$

where a_n is the nth term of the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$.

The next problem is a variation of Problem 20.5.5, which appeared in the October 1996 issue of *Function*.

PROBLEM 22.2.5 (from *Crux Mathematicorum with Mathematical Mayhem*)

Lucy and Anna play a game where they try to form a ten-digit number. Lucy begins by writing any digit other than zero in the first place, then Anna selects a different digit and writes it down in the second place, and they take turns, adding one digit at a time to the number. In each turn, the digit selected must be different from all previous digits chosen, and the number formed by the first n digits must be divisible by n. For example, 3, 2, 1 can be the first three moves of a game, since 3 is divisible by 1, 32 is divisible by 2 and 321 is divisible by 3. If a player cannot make a legitimate move, she loses. If the game lasts ten moves, a draw is declared.

(a) Show that the game can end up in a draw. (For an easy answer, look up the solution to Problem 20.5.5!)

(b) Show that Lucy has a winning strategy and describe it.

PROBLEM 22.2.6 (Mathematical Team Contest "Baltic Way – 92")

Find all integers satisfying the equation $2^x(4 - x) = 2x + 4$.

* * * * *

OLYMPIAD NEWS

The 1998 Mathematical Olympiad

The contest was held in Australian schools on February 11 and 12. On either day about 115 students in years 9 to 12 sat a paper consisting of four problems, for which they were given four hours. As a result, twenty-six students were invited to represent Australia at the Tenth Asian Pacific Mathematics Olympiad, (APMO), a major international competition for students from about twenty countries on the Pacific Rim and Argentina, South Africa and Trinidad & Tobago.

First Day

Wednesday, 11th February, 1998

Time allowed: 4 hours
NO *calculators are to be used*
Each question is worth seven points

1. Determine all real solutions of the equation:

$$(x + 1998)(x + 1999)(x + 2000)(x + 2001) + 1 = 0$$

2. Find all pairs (r, s) of non-negative real numbers that satisfy the following two conditions:

(i) $2^{r^4 + s^2} + 2^{r^2 + s^4} = 8$

(ii) $r + s = 2$

3. Let ABC be a triangle, let D be a point on AB and E be a point on AC such that DE and BC are parallel and DE is a tangent to the incircle of the triangle ABC. Prove that

$$8\,DE \le AB + BC + CA$$

4. Determine all functions f with real values that satisfy the following conditions:

(i) f is defined for all non-zero real numbers;

(ii) $f(-x) = f(x)$ for all non-zero real numbers x;

(iii) $f\left(\dfrac{1}{x+y}\right) = f\left(\dfrac{1}{x}\right) + f\left(\dfrac{1}{y}\right) + 2(xy - 1000)$ for all non-zero real numbers x and y such that $x + y \neq 0$.

Second Day

Thursday, 12th February, 1998

Time allowed: 4 hours
NO *calculators are to be used*
Each question is worth seven points

5. Consider all 2×2 arrays with each entry being either 0 or 1. We say that a pair of such 2×2 arrays A and B is compatible if there exists a 3×3 array within which both A and B appear as 2×2 arrays. For example, the pair

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is compatible since both arrays occur within $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Determine all pairs of 2×2 arrays that are not compatible.

6. Prove that for any positive integer n,

$$(1998n)! \leq \left(\frac{3995n+1}{2} \cdot \frac{3993n+1}{2} \cdot \frac{3991n+1}{2} \cdots \frac{n+1}{2} \right)^n$$

7. Let ABC be a triangle whose area is 1998 cm^2. Let G be the centroid of the triangle ABC. Each line through G cuts the triangle ABC into two regions having areas A_1 and A_2, say. Determine (with proof) the largest possible value for $A_1 - A_2$.

8. A team of archaeologists was able to establish that forty thousand years ago there had been a flourishing civilisation in the crocodile infested jungle of Udakak. As many as 12 cities were excavated. After some effort, theUdakak script was deciphered, and the rich Udakak literature was studied by historians. They discovered that:

(i) Udakak could not have had more than 28 cities;

(ii) The Udakakan cities had peculiar trade arrangements:

(a) wherever a city A had no trade relations with another city B, there were exactly two cities with which both A and B had trade relations;

(b) whenever a city A had trade relation with another city B, no city had trace relations with both A and B.

For budgetary reasons, the archaeologists wanted to know how much excavation work was still lying ahead of them. Show how the archaeologists can work out the exact number of Udakakan cities.

The Tenth Asian Pacific Mathematics Olympiad

The Asian Pacific Mathematics Olympiad (APMO), and annual competition, was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the number of participating countries has grown to twenty. This year 27 Australian students sat the APMO on March 10.

Time allowed: 4 hours
NO *calculators are to be used*
Each question is worth seven points

1. Let F be the set of all n-tuples $(A_1, A_2, \ldots, A_n)$ where each A_i, $i = 1, 2, \ldots, n$ is a subset of $\{1, 2, 3, \ldots, 1998\}$. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{A_1, A_2, \ldots, A_n} |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

2. Show that there are no positive integers a and b such that

$$(36a + b)(a + 36b)$$

is a power of 2.

3. Let a, b, c be positive real numbers. Prove that

$$(1 + \frac{a}{b})(1 + \frac{b}{c})(1 + \frac{c}{a}) \geq 2\left(1 + \frac{a + b + c}{\sqrt[3]{abc}}\right)$$

4. Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

5. Determine the largest of all integers n with the property that n is divisible by all positive integers that are less than $\sqrt[3]{n}$.

The performance of students at the APMO as well as at the Australian Mathematical Olympiad (AMO) in February was used in selecting two members and nine candidates for the team which is to represent Australia at this year's International Mathematical Olympiad (IMO). Also, thirteen highly gifted students, having at least one more year of secondary education ahead of them, were singled out for further training.

These 25 students will participate in the ten–day Team Selection School of the Australian Mathematical Olympiad Committee. Following a tradition, the school will be held in Sydney. Participants will have to undergo a day–and–evening–filling program consisting of test and examinations, problem sessions and lectures given by mathematicians. Finally, the 1998 Australian IMO Team will be selected.

Taipei (Republic of China-Taiwan) is the venue of the XXXIX IMO scheduled for July. There the Australian team, consisting of six members, will have to contend with six problems during nine hours spread equally over two days in succession. The following students have qualified for team membership and team candidature respectively:

Team members:
Stephen Farrar (Year 12), James Ruse Agricultural High School, NSW;
Justin Ghan (Year 12), Pembroke School, SA.
Team candidates:
Andrew Cheeseman (Year 11), Mentone Grammar School, VIC;
Geoffrey Chu (Year 10), Scotch College, VIC;
Steven Irrgang (Year 12), Caringbah High School, NSW;
Justin Koonin (Year 12), Sydney Grammar School, NSW;
Hiroshi Miyazaki (Year 12), Scotch College, VIC;
David Varodayan (Year 12), Sydney Grammar School, NSW;
Thomas Xia (Year 10), James Ruse Agricultural High School, NSW;
Geordie Zhang (Year 11), Melbourne Grammar School, VIC.

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