Function

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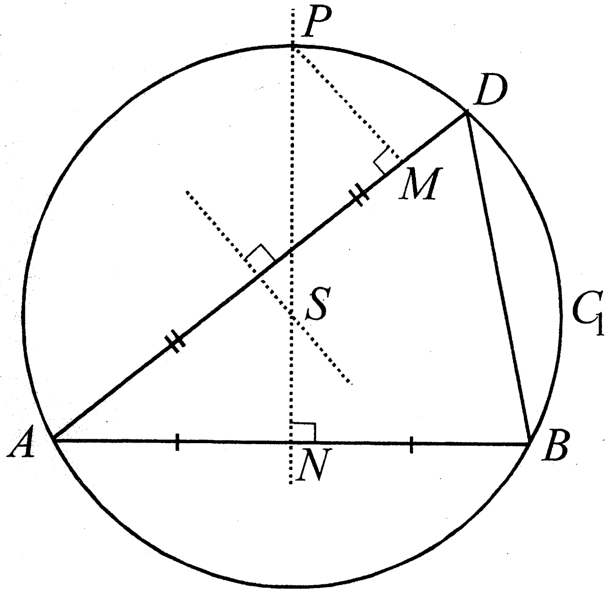

Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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EDITORIAL

Welcome to our readers!

This issue of *Function* has a rather geometric flavour with three articles discussing geometric constructions, so we recommend you have a compass and a straightedge (unmarked ruler) ready.

The Ancient Greeks were interested in objects than can be constructed by means of a straightedge and a compass. The figure on the front cover is from the article by our editor Ken Evans in which he presents such a problem solved by Archimedes: the construction of a point halfway along a three-point journey.

The second feature article is by our reader D F Charles, who shows how we can construct—again, using only a straightedge and a compass—any number that involves the five basic operations: addition and subtraction, multiplication and division, and finding the square root. In addition, the author also presents a more complex example: the construction of a circle through two given points and tangent to a given line.

The Ancient Greeks were also interested in solving in the same manner the problems of duplicating a cube, squaring a circle, and trisecting an angle. Our *History of Mathematics* editor gives an account of how only in relatively modern times the almost unknown mathematician Pierre Laurent Wantzel showed—by algebraic means—that it is impossible to solve these problems using only a straightedge and a compass.

For the *Computers and Computing* column you will need three pegs and some rings to solve the classic Tower of Hanoi and to develop an understanding of recursion and recursive algorithms.

Finally, the *Problem Corner* editor includes solutions to the problems in the April issue and new problems for the readers. He will also publish the best solutions received by December 1.

Happy reading!

* * * * *

A Construction of Archimedes

Ken Evans

Archimedes (c.287 BC–212 BC) lived in Syracuse in Sicily until he was killed there during a Roman invasion. His extraordinary contributions to mathematics have led to him being regarded as one of the three greatest mathematicians ever to have lived (the other two being Newton and Gauss).

In this article, the following problem — just one of those solved by Archimedes is presented.

If you go from point A to point B via point D, devise a construction to find M, the point which is halfway along the journey (Figure 1),i.e. construct the point M such that

$$AM = MD + DB$$

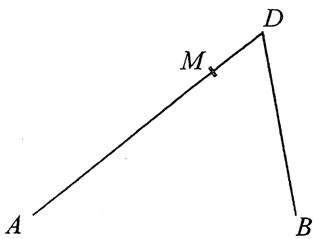

Figure 1

In his construction of M, Archimedes followed the convention of Greek geometry (formalized by Plato) that only a compass and ungraduated ruler are to be used.

Construction

First construct the circumcircle, C_1, of ΔADB (The centre, S, of C_1 is the point of concurrence of the perpendicular bisectors of $\overline{AB}$, $\overline{BD}$, $\overline{DA}$.) Let P be the point of intersection of the perpendicular bisector, $\overleftrightarrow{SN}$, of $\overline{AB}$ and the arc ADB.

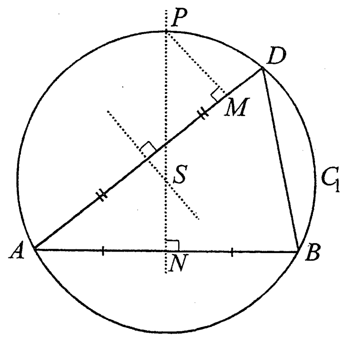

Figure 2

Case (1) If $D = P$, then D is the required point M.

Case (2) If $D \neq P$, construct the perpendicular from P to the longer of the sides $\overline{AD}$, $\overline{DB}$ ($\overline{AD}$ in Figure 2) intersecting $\overline{AD}$ at M. M is the required point.

Proof

Case (1) If $D = P$, then D belongs to the perpendicular bisector of $\overline{AB}$. Hence $DA = DB$, so M is the point D.

Case (2) If $D \neq P$ an additional construction is needed.

With centre P and radius length PA, draw a circle C_2. Note that B belongs to C_2 since $PA = PB$. Draw say $\overline{AD}$ to intersect C_2 at E. Draw $\overline{BE}$. Denote the magnitudes of the angles by the letters shown in Figure 3.

Now	$\beta = 2\alpha$	(central angle of C_2)
Also	$\beta = \gamma$	(angles in a segment of C_1)
So	$\gamma = 2\alpha$.	
But	$\gamma = \alpha + \theta$ (exterior angle of ΔBED).	
Hence	$\alpha + \theta = 2\alpha$, and so $\theta = \alpha$.	

Therefore in ΔDEB, $DE = DB$. (1)

In C_2, $\overline{PM} \perp \overline{AE}$, and since P is the centre of C_2, M is the midpoint of $\overline{AE}$. That is,

$$\begin{aligned} AM &= ME \\ &= MD + DE \qquad (2) \\ &= MD + DB \end{aligned}$$

which completes the proof.

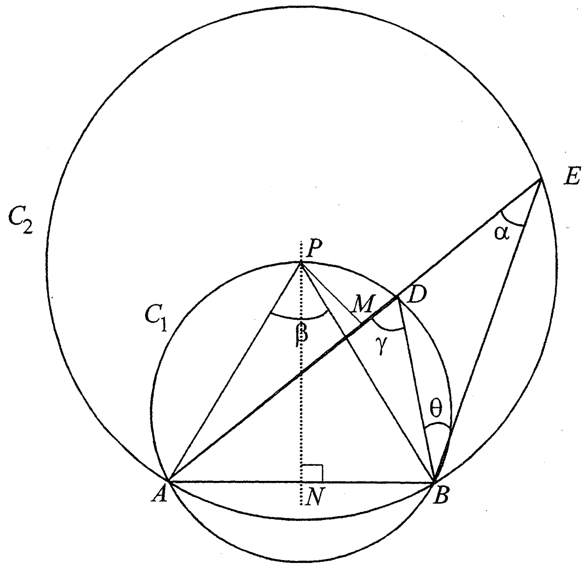

Figure 3

Corollary

If a line segment is drawn from M to the mid-point, N, of $\overline{AB}$ (Figure 4), ΔABD is divided into two parts of equal perimeter. This is easily demonstrated.

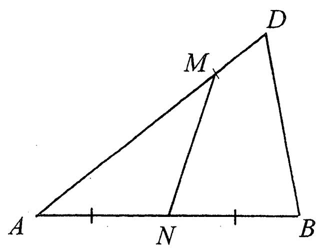

Figure 4

From (2) $AM = MD + DB$

Therefore $AM + AN + NM = MD + DB + BN + NM$

i.e. perimeter of ΔAMN = perimeter of quad $MDBN$

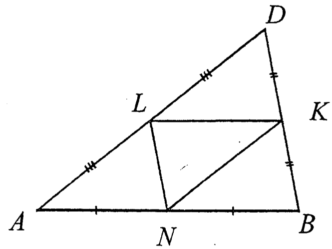

Figure 5

For this reason, the segment $\overline{MN}$ is called, somewhat graphically, a *cleaver* of ΔADB. Triangle ABD has three cleavers and, remarkably, it can be shown that the three cleavers are concurrent by showing that the cleavers are the angle bisectors of triangle KLN (Figure 5).

Finally the reader is left the task of constructing the halfway point, M, in the cases (originally excluded) where D belongs to $\overleftrightarrow{AB}$.

Reference:

Ross Horsberger *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Chapter 1, The Mathematical Association of America, 1995.

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It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never-satisfied man is so strange if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others.

–Karl Friedrich Gauss in Letter to Bolyai, 1808.

RULER AND COMPASS CONSTRUCTIONS

D F Charles

Although the Ancient Greeks undoubtedly possessed conventional rulers with scales on them, they were interested in the problem of what geometrical objects could be constructed by means of a simple unscaled ruler, here called a straightedge, and a compass. Many familiar objects can be so constructed, but others eluded all attempts. Three problems in particular became notorious in this regard. They were the **Duplication of the Cube**, the **Squaring of the Circle** and the **Trisection of an Angle**.

By the first of these is meant the construction, from a line-segment of length 1, another of length $\sqrt[3]{2}$. The second was the construction of a square equal in area to a given circle, or else (what can be shown to be equivalent) a line-segment whose length is the circumference of that circle. [It is interesting to note that a circular arc of this length may be so constructed, but not a straight line-segment, so this problem could also be dubbed *the straightening of the circumference*.] In the case of the third problem, it is possible to trisect any given straight line-segment, but it is not possible to so trisect an arc of a circle. Note that this last comment applies to *general* angles and arcs. Certain angles (e.g. 90°) may be trisected, but others (e.g. 60°) may not.

In fact none of these problems can be solved by the means that the Ancient Greeks sought to use. This has now been shown, but the proofs had to wait till relatively modern times.

If we begin from a line-segment whose length we take to be 1, then the construction of other lengths can be represented as the construction of other *numbers*. Call the initial length the *unit*.

We can construct any integral multiple of the unit, and it is also possible to construct any rational fraction of it. Indeed we can construct any number that involves the five basic operations:

- Addition and Subtraction
- Multiplication and Division
- Finding the Square-Root.

These are called the *five basic construction operations* and they will now be considered in turn.

Addition and subtraction of two given lengths may readily be carried out. Represent each length by an interval and use the straightedge to extend one or other of these and next use the compass to mark off an interval equal in length to the other interval and so placed that the ends adjoin. See Figure 1.

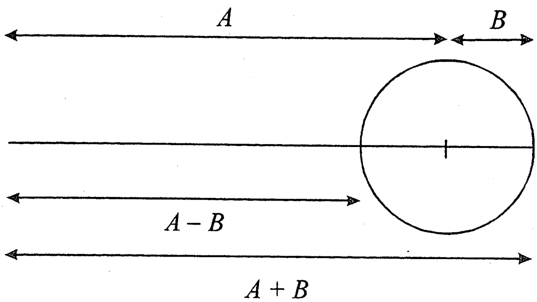

Figure 1

Multiplication and division may be accomplished using similar triangles. A right angle can be constructed using straightedge and compass alone, and so we may form two right-angled triangles as shown in Figure 2a (multiplication) and Figure 2b (division).

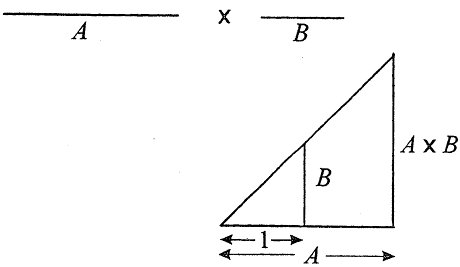

Figure 2a

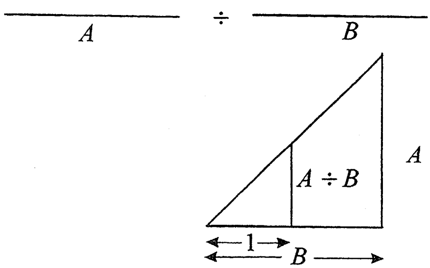

Figure 2b

The construction of a length $\sqrt{x}$, if a length x is given, is more complicated. There are three separate cases. The first has $x < 2$, in the second $2 < x < 4$, and the last possibility is $x > 4$. [The special cases in which $x = 2$ or $x = 4$ are easily dealt with separately.] The full details will not be given here, but only the case $2 < x < 4$ will be discussed, as an illustration. See Figure 3.

1. Begin by constructing the length $(x-2)/2$. Make this length one of the legs of a right-angled triangle.

2. Because $x<4$, $(x-2)/2<1$, and so we can make the hypotenuse of the right-angled triangle equal to 1.

3. The third side of this triangle is then $\sqrt{x-x^2/4}$.

4. From the right-angle so formed, measure out a length $x/2$ back along the base of the right-angled triangle, past where the hypotenuse meets it, and so form a second right-angled triangle.

5. The hypotenuse of this new triangle will be $\sqrt{x}$.

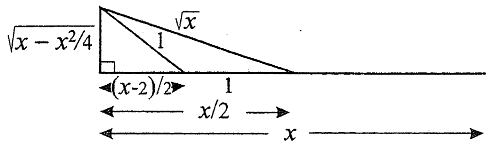

Figure 3

Now consider a more complicated example.

Let two points P and Q be given and let a line L be also given, and such that P does not lie on L. We seek to construct a circle passing through both P and Q and tangent to L.

Let us first investigate the matter algebraically. Set up co-ordinates with the line L as the x-axis. Perpendicular to this draw the y–axis in such a way as to pass through the point P. The length OP will be taken as our unit, and so P has the co-ordinates (0, 1). Let Q be the point (m, n) in these co-ordinates. The required circle can be constructed if we can find the centre and the radius. See Figure 4 .

From the figure we see that if (p, r) is the centre of the circle, then r will also be the required radius. We may also note that the centre lies on a line L^* which passes through the mid-point M of PQ and is perpendicular to PQ. M is the point $\left(\frac{m}{2},\frac{n+1}{2}\right)$ so L^* has equation

$$y-\left(\frac{n+1}{2}\right)=\frac{-m}{n-1}\left(x-\frac{m}{2}\right)$$

and because the centre lies on this line then its co-ordinates $(p,\ r)$ satisfy this equation. This tells us that

$$r-\left(\frac{n+1}{2}\right)=\frac{-m}{n-1}\left(p-\frac{m}{2}\right) \qquad \text{(A)}$$

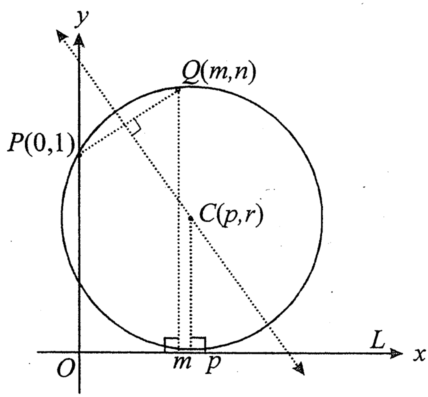

Figure 4

But we also know that the point P lies on the circle and the circle has the equation $(x-p)^2+(y-r)^2=r^2$. So, substituting the co-ordinates of P into this equation, we find

$$(0-p)^2+(1-r)^2=r^2$$

so that

$$r=\frac{1}{2}\left(p^2+1\right). \qquad \text{(B)}$$

If we now substitute for r from Equation (B) into Equation (A), we find

$$\frac{p^2+1}{2}-\left(\frac{n-1}{2}\right)=\frac{m\left(p-\frac{m}{2}\right)}{n-1}$$

This is a quadratic equation in p, whose solutions are

$$p=\frac{-m\pm\sqrt{n}\sqrt{(n-1)^2+m^2}}{n-1}$$

We see that there are two different circles that satisfy the conditions, and this is as we would expect from geometric considerations. Here concentrate on the solution for which $p > 0$.

Note that $\sqrt{(n-1)^2+m^2}$ is simply the length, l say, of the line-segment PQ, so that we now have

$$p=\frac{-m+l\sqrt{n}}{n-1}$$

and so we have a recipe for constructing p.

It goes like this.

1. With ruler and compass, we may drop a perpendicular from Q to L. The length of this perpendicular is n.

2. Using the process outlined earlier, construct $\sqrt{n}$.

3. Multiply $\sqrt{n}$ by l.

4. The distance from O to the point constructed in Step 1 is m. Subtract m from the result of Step 3.

5. Divide this latest result by $n-1$.

We may now measure off a length p from O along L and erect a perpendicular from the point found. On this perpendicular, measure off a length r, which comes directly from equation (B). This gives both the precise centre and also the radius of the required circle.

It should be noted that the method outlined here, while it can always be made to work if a straightedge and compass construction exists, need not represent the most elegant way to carry out that construction. However, we readily see that if an algebraic expression involves no operations other than the five allowed, then the number represented by the expression may always be constructed.

* * * * *

The Amen Bug

As midnight of 31 December turned into the first day of 1999, computer crashes, which could be precursor of the much-forecast Y2K disaster, occurred in different parts of the world. The programming error, nicknamed "the Amen bug", affected computer users from a Singapore taxi fleet to computers at immigration desks at Swedish airports.

The error was quickly tracked down to a program which interpreted "99" in the year field of the data base as an "end of file" instruction.

Why the "Amen" bug? Amen is a word used at the end of prayers. It comes directly from the Greek word $\alpha\mu\eta\nu$, meaning "so be it". In classical Greek, letters also represented numbers, and significance was attached to the "numerical value" of words. Since α denoted 1, μ denoted 40, η denoted 8, and ν denoted 50, the numerical value of $\alpha\mu\eta\nu$ was $1+40+8+50=99$. And so "99" was linked to "amen" in medieval times. Much later, computer programmers adopted 99 as an "end of file" instruction.

— From *Mathematical Digest, N⁰ 115*

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A mathematician is a blind man in a dark room looking for a black cat which isn't there.

— Charles Darwin

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HISTORY OF MATHEMATICS

Trisecting an Angle

Michael A B Deakin

Elsewhere in this issue, D F Charles discusses the problem of constructing geometric objects with the aid only of a compass and a straightedge (unmarked ruler). Many elegant constructions can be performed with the aid of just these two simple instruments, but three problems (all posed in ancient times) continued to elude the best efforts of all who tried to solve them. These were "the squaring of the circle", "the duplication of the cube" and "the trisection of the angle".

As Charles discusses, the first of these problems amounts to the construction of a length of either π or $\sqrt{\pi}$ times a given length; the second amounts to the construction of a length $\sqrt[3]{2}$ times a given length. The third is rather more complicated to describe in such brief numerical form and it will be discussed in more detail as this article proceeds. I will also talk further about the second problem, but not about the first, which turns out to be rather different.

From a *geometric* point of view however, the third problem is easily described— if an angle (3θ we will call it for convenience) is given to us, we are to use the compass and straightedge to construct the angle θ with one third the measure of the given angle.

Now it was known in ancient times that any given angle could be bisected, i.e. cut exactly in two, by such means. In fact, it is quite easy. If the angle is BAC, formed by the intersection of two lines AB and AC at A, then we use the compass to draw a circle with centre A and with an arbitrary radius r. This will intersect AB in D, say, and AC in E, say. Now with centres E, F and the same radius r, draw two further circles to intersect at G, say. The line AG now bisects the angle BAC.

Now notice three things about this construction. First, it involves only a finite number of steps. Second, it produces an exact result. Third, it can be applied to any angle whatsoever. And of course, we have an obvious fourth thing also: it involves only the straightedge and compass.

Now, trisection of an angle is quite possible if we relax any one of these conditions. Even if we merely allow our straightedge to be marked, then a simple construction gives an exact result. This construction was the subject of a short article in *Function, Volume* 2, *Part* 5, p. 20. Alternatively we may use curves other than straight lines and circles. For example, our cover picture for *Volume* 20, *Part* 1 showed the *unifolium*, a curve that trisects angles, and the use of conic sections for this purpose has also been very long known. (See for example Victor Katz's book *A History of Mathematics* – in the first edition, p. 120.)

Then too there are a number of constructions that give excellent *approximations* over a very wide range of angles. For some examples, see *Function, Volume* 3, *Part* 5, p. 24. And it is also possible to get an exact trisection by straightedge and compass alone if we allow an infinite number of steps. One easy way to do this is to notice that the bisection construction allows us to take quarters, eighths, sixteenths, etc. of any given angle. So if we begin with our angle 3θ, continue to take quarters and then add, we find

$$3\theta\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots\right) = \theta,$$

so that an exact trisection has been performed. Indeed we can stop after any finite number of steps to get approximations that can be as accurate as we like, although none of these will be *exact*.

We may also remark that *some* angles *can* be exactly trisected in a finite number of steps using straightedge and compass alone. For example both the 90° and the 60° angles can be constructed, and if we bisect the latter, we reach an angle of 30°, so that the 90° angle can be trisected. By bisecting both these angles we find that we can construct a 15° angle and so trisect the 45° angle. This last (and important) example will be further discussed below.

But a method of trisection that satisfies *all* the requirements never was found and mathematicians came to suspect that the task might actually be impossible. This is now known to be the case, and we know this result because of the work of a highly gifted but very little-known mathematician, Pierre Laurent Wantzel.

The best available biography of this obscure figure is that to be found on the web at:

www-groups.docs.st-and.ac.uk/~history/Mathematicians/Wantzel

It relies on what I suspect are the only primary sources in existence, the principal one being an obituary notice by his friend and research collaborator Saint-Venant (whose full name was Adhémar Jean Claude Barré de Saint-Venant — Barré de Saint-Venant being the surname). [Saint-Venant incidentally turned up briefly in an earlier *Function* article; see *Volume* 22, *Part* 1, p. 5.]

Wantzel was a highly gifted mathematician, engineer and also linguist. He was as well almost certainly a drug addict, which may explain his short life. He died in 1848, about two weeks before his 34th birthday. In this short lifetime however, he published over twenty pieces of research, some of it joint work with Saint-Venant, and his final paper appeared after his death, having been found among the papers he left behind.

As Saint-Venant said:

> "Ordinarily he worked evenings, not lying down till late; then he read, and took only a few hours of troubled sleep, making alternately wrong use of coffee and opium, and taking his meals at irregular hours until he was married. He put unlimited trust in his constitution, very strong by nature, which he taunted at pleasure by all sorts of abuse. He brought sadness to those who mourn his premature death."

[One might remark perhaps the way Saint-Venant seems to have put coffee and opium on the same footing; Saint-Venant appears to have been a pious man and so may well have disapproved of both, but there is also the consideration that both medical knowledge and social attitudes have changed since 1848.]

The research paper that has assured Wantzel an enduring place in the history of Mathematics is actually quite a short one. It appeared in 1837 and it showed the impossibility both of a trisection construction and also of the duplication of the cube.

In what follows, I shall describe the gist of Wantzel's argument. Nowadays, although we tend to follow the same outline as he did, his insights have been more carefully codified as part of the branch of Algebra known as Field Theory. The best account of the modern theory I have seen is that in I N Herstein's *Topics in Algebra* (in the second edition, see pp. 230-231). However, this still needs a lot of specialist vocabulary and preliminary work that will be well beyond most readers of *Function*.

Much of this specialist background was established in order to formalise and to extend Wantzel's work (an endeavour begun by Wantzel himself), but for the purposes of this paper, I will try to give the flavour of what Wantzel did on the trisection and duplication problems.

Essentially, he took the *geometric* problems and turned them into *algebraic* ones. This he did by (in essence) setting up a co-ordinate system. If we begin from a given unit length, we can take its direction as that of the x–axis, with one end of the unit as the origin and the other as the point (1, 0). Now the right angle is constructible (as noted above), so we may also draw a y–axis. And now, because we can add, subtract, multiply and divide (as outlined in Charles's article), we can build up a grid in which any given rational point may be constructed. That is to say, we can use our tools to construct the point (p, q), whenever p and q are rational numbers.

This I will call the starting position: whatever rational points we need, we may assume to be already present. This much of the work is presumed done already. So what follows next I will call **Step 1**.

At **Step 1** we take a circle with some rational point (p,q) as its centre and some rational number r (say) as its radius and we intersect this either with some other such circle or else with a straight line passing through some two rational points. [There is no need to consider the intersection of two such straight lines, as this will merely give another point of the pre-existing grid, and so will come up with nothing new.]

But intersections as introduced above will involve the solution of a quadratic equation that Wantzel wrote as

$$x_1^2 + Ax_1 + B = 0,$$

where A and B are rational numbers. Any intersection of a circle with a circle or with a straight line, as allowed by **Step 1**, results in such an equation.

[We may also remark that the solution of any quadratic with rational coefficients may be achieved with straightedge and compass. The way to do this was given in *Function, Volume* 3, *Part* 5, p. 24. Very briefly, to solve the equation just given, construct the line-segment joining (0, 1) to $(-A,B)$ and make it the diameter of a circle. Then the roots of the equation are the intersections of this circle with the x–axis. Readers may like to prove this detail for themselves.]

We may thus construct all points (and thus all lengths) that are either rational or else the roots of a quadratic equation such as that given above with rational coefficients. Our "grid" of points is now extended to cover all the rational points and also all those points whose co-ordinates are rational multiples of roots of quadratic equations with rational coefficients. Call these new points **Type 1 points**. The numbers giving their co-ordinates will be called **Type 1 numbers.**

Now move on to **Step 2**. This is the same as **Step 1**, except that now we have a quadratic whose coefficients are not restricted to be rational, but can also be **Type 1 numbers**. Otherwise, we proceed as before; set up a quadratic equation

$$x_2^2 + A_1 x_2 + B_1 = 0,$$

and so set up **Type 2 points** as the new points on an even further refined grid.

From our starting grid (the set of all rational points) we can in one construction reach all **Type 1 points**; in two constructions, we can reach all **Type 2 points.** And so on. We can continue the process to **Type 3 points**, **Type 4 points**, etc. At each stage we have a new type of number: **Type 2 numbers**, etc.

Now notice that the construction of a **Type 2 Point** is the same as the solution of two quadratic equations, and this is readily shown to be the same as the solution of a *quartic* (degree 4) equation, a quartic equation whose coefficients will all be rational; the construction of a **Type 3 point** is the same as the solution of an equation of degree 8, and again the coefficients will all be rational; and so on. In general, the construction of a **Type *n* point** is the same as the solution of an equation of degree 2^n, and again the coefficients of *this* equation will all be rational.

It follows that any point or length that can be described as the solution of an equation of degree 2^n can be constructed as long as all its coefficients are rational. In fact, these points are the *only points* that can be constructed, but this observation does not entirely settle the matter. There is a complication.

To appreciate the nature of the problem, let us look in more detail at the trisection question. What Wantzel did was to look at the possibility of trisecting a particular angle: the 60° angle. If we cannot trisect this then clearly no *general* method is possible.

So in this special case we have, in our earlier notation, $3\theta = 60°$ which is to say $\theta = 20°$. Wantzel now reasoned that if we can construct a length equal to cos 20°, then we can construct an angle of 20°, and so trisect the 60° angle. And the argument works in reverse also: if we can do *these* things, then we can construct a length equal to cos 20°. So it all boils down finding the equation satisfied by cos 20°.

We may find this quite easily because in general

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

and in particular cos $60^\circ = \frac{1}{2}$ so that cos 20° is a solution of the equation

$$x^3 - \frac{3x}{4} - \frac{1}{8} = 0. \tag{1}$$

This is a cubic equation, and so has not got an order of 2^n for any integral value of n. However this simple observation, as Wantzel recognised, is not enough entirely to settle the matter. To see why, let's get back to our example of the 15° angle.

Proceeding exactly as above, but using 45° instead of 60°, we reach an equation very like equation (1), namely

$$x^3 - \frac{3x}{4} - \frac{1}{4\sqrt{2}} = 0. \tag{2}$$

But now come at the problem another way, and think of 15° as the half of 30° which is itself the half of 60°. To bisect an angle, we proceed algebraically by noting that in general

$$\cos 2\theta \;=\; 2\cos^2\theta - 1$$

and that in the particular case of $\theta = 60^\circ$, this reduces to making cos 30° equal to $\frac{\sqrt{3}}{2}$, which we know to be the correct value. So now if $\theta = 15^\circ$, we have cos 15° as a root of

$$2x^2 - 1 = \frac{\sqrt{3}}{2} \tag{3}$$

This gives

$$\cos 15^\circ = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}} = \frac{\sqrt{6}+\sqrt{2}}{4} \tag{4}$$

[You may care to check this last piece of working as an exercise.]

The point to notice is that the right-hand side of equation (4) is a **Type 2 number**, because the right-hand side of equation (3) is itself a **Type 1 number.** If we are to get back to an equation with rational coefficients, then we need to square both sides of equation (3), and so produce aquartic equation. Similarly, the removal of the square root from equation (2) produces asextic (degree 6) equation. Now 6 is not a power of 2, but here it turns out that thesextic expression can be factored into two simpler expressions, one of them a quartic and the other a quadratic, and *both with rational coefficients*. In fact we find from equation (2):

$$\left(x^4 - x^2 - \frac{1}{2}\right)\left(x^2 - \frac{1}{2}\right) = 0,$$

so the sextic is actually reducible to two simpler equations, both of the type that Wantzel allowed.

Thus Wantzel argued that if equation (1), a cubic with rational coefficients, is to allow the construction of cos 20°, then the only way this could happen is that its left-hand side be similarly factorisable, with this time the factors being on the one hand a quadratic and on the other a simple (degree 1 or "linear") factor, and *both these factors must involve only rational numbers*.

But in this case, if we set the simple linear factor equal to zero, we must get a rational root, and so we must have:

If cos 20° *is constructible, then equation* (1) *has a rational root.*

But it is a relatively simple matter to show that equation (1) does *not* have a rational root. The details are here omitted, but readers may find them (in slightly altered notation) on page 246 of W Dunham's *The Mathematical Universe* (New York: Wiley, 1994).

So finally we are done. Wantzel had shown conclusively that the age-old problem was impossible.

Because a very similar (actually somewhat simpler) analysis may be applied to the equation

$$x^3 - 2 = 0$$

we also see that the duplication of the cube is likewise impossible. This too Wantzel noticed. So his short (7-page) paper settled fully two of the oldest questions in Mathematics!

It is a pity that such an important contribution has been almost completely forgotten. The result, of course, has endured, but Wantzel's is now an almost unknown name.

Perhaps the most telling instance of the neglect befalling him is this. One of the basic tools that any historian of Mathematics uses is a reference work known as *Poggendorff's Handwörterbuch*. Volume 2 of this does give a brief notice to Wantzel, but it misspells his name, queries the date of his death, and in its (very incomplete) list of his published works completely overlooks his most enduring work: the work I have just finished describing!

* * * * *

Diophantus' epitaph

This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.

—In Ivor Thomas, Greek Mathematics, in J. R. Newman (ed) *The World of Mathematics*, New York: Simon and Schuster, 1956

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COMPUTERS AND COMPUTING

About Towers and Recursion

Cristina Varsavsky

The Towers of Hanoi is a famous puzzle. It involves a set of rings of different sizes and three pegs. Initially, the rings are all in one peg arranged from largest at the bottom to smallest at the top. The object is to move all rings to another peg following two rules:

1. Only one ring can be transferred at a time;
2. A larger ring may never be on top of a smaller ring.

The recreational mathematics writer W Rouse-Ball attributes this puzzle to the following legend.

In the great temple at Benares ... beneath the dome which marks the centre of the world rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed 64 disks of pure gold, the largest disk resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the disks from one diamond needle to the other ... When the 64 disks shall have been thus transferred [i.e, according to rules 1 and 2 above] from the needle on which at the creation God placed them to one of the other needles, the tower, temple, and Brahmins alike will crumble into dust and then with a thunderclap the world will vanish.

Since the Towers of Hanoi is a knowledge-lean problem it is used by psychologists to experiment with children's ability to solve problems, although usually only in the simpler cases of 2 or 3 rings. University students are often introduced to this puzzle as an example of a *recursive algorithm.*

In this column we have repeatedly dealt with *algorithms*, that is, definite procedures to solve problems defined by a finite number of steps. But what is a *recursive* algorithm?

Recursion is a process of defining an object in terms of itself. We can use recursion to define sequences, functions and sets.

Here is an example of a sequence defined by recursion:

$$a_0 = 3 \qquad a_n = 2\,a_{n-1} \quad \text{for} \quad n \geq 1$$

The first term is 3, and the subsequent terms are obtained by multiplying the previous one by 2 , generating the sequence

$$3,\ 6,\ 12,\ 24,\ 48, 96,\ \ldots$$

The famous *Fibonacci sequence* is also defined by recursion:

$$f_0 = 1,\ \ f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2$$

to give

$$1, 1, 2, 3, 5, 8, 13, 21,\ \ldots$$

An example of recursive definition of a function is the factorial function $f(n) = n!$. We specify the initial value of this function as $F(0) = 1$ and give a rule for finding $f(n)$ from $f(n-1)$:

$$f(n) = n\,f(n-1)$$

For example, $3! = 3 \cdot 2!$, but $2! = 2 \cdot 1!$ and $1! = 1 \cdot 0! = 1 \cdot 1 = 1$. Therefore

$$3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$$

Recursive definitions are often used to define sets. Such a definition starts by giving an initial collection of elements, and then defining the rules to be used in the construction of new elements from other elements already known.

For example, the set defined by

1. $4 \in A$
2. If a and $b \in A$ then $a + b \in A$

is the set of all positive integers multiples of 4.

In all examples above we defined objects in terms of themselves to produce recursive definitions of sequences, functions and sets. Just in the same way, a recursive algorithm is one that solves a problem by reducing it to a problem with a smaller input.

For example, we could use the following algorithm to compute the nth power of 2:

Algorithm A
1. Input n
2. $power = 1$
3. For $k = 1$ to n
   3.1 $power = 2 \cdot power$

The nth power of 2 is calculated by iterating through a loop; it first computes 2^1 then 2^2, 2^3, 2^4, etc. until it finally computes 2^n. This is an *iterative* algorithm. The same task could be achieved with the following algorithm

Algorithm B
1. Input n
2. **procedure** $power(n)$
   2.1 **If** $n = 0$ **then**
      2.1.1 $power = 1$
      else
      2.1.2 $power = 2 \cdot power(n-1)$

This algorithm includes a call to itself in line 2.1.2; it is therefore a *recursive* algorithm.

Let us see how the algorithm goes about computing 2^2, ie. $power(2)$. Here $n = 2$ and because n is not 0, execution is directed to the **else** clause. Now, the computation of $power(2)$ must be suspended until $power(1)$ is known. Any known information about the computation of $power(2)$ is stored in the computer memory to be retrieved when the computation can be completed. The algorithm is invoked again with the input value $n = 1$. Again, the **else** clause is executed to compute $power(0)$ while the computation of $power(1)$ is put on hold. This time the **then** clause is executed returning $power(0)=1$ and this value is returned to the previous invocation. Next $power(1)$ is computed using the information stored in the memory, returning $power(1) = 2$ to the call by $power(2)$ and removing from the memory any information about $power(1)$. Finally, using the returned value of $power(1)$, $power(2)$ is computed as 4 and the memory is cleared.

Although the recursive approach appears to be more complex and has the disadvantage of consuming too much memory, it provides a natural way to think about many problems, some of which would have very complex non-recursive solution.

Now, let us go back to the Towers of Hanoi puzzle and its recursive solution. Here the basic non-trivial problem is the one with 2 rings and 3 pegs. I am sure many readers know how to solve this particular case. Figure 1 shows the procedure to move the two rings from peg 1 to peg 2, using peg 3 as an auxiliary peg.

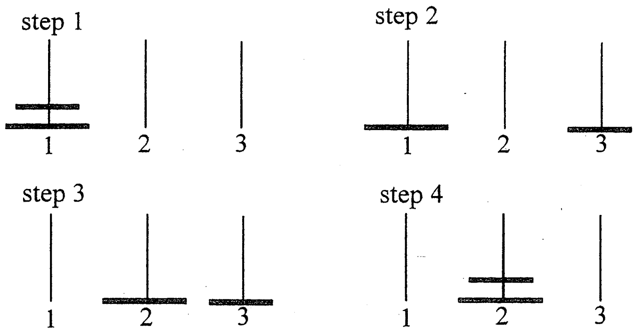

Figure 1

As shown in Figure 2, this same procedure could be applied to the tower with any number of rings n, provided we know how to solve the problem with $n-1$ rings.

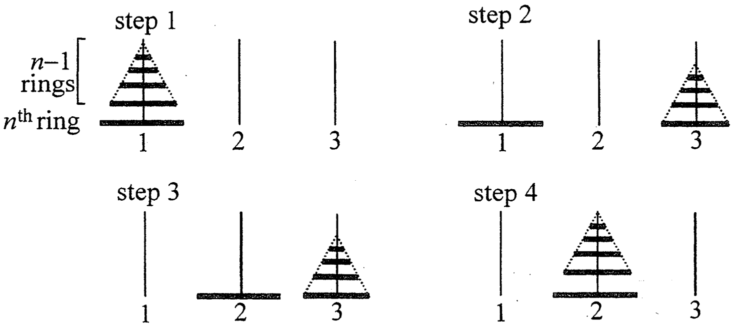

Figure 2

First, we move the $n-1$ rings to peg 3, then we move the nth ring from peg 1 to peg 2, and finally, we move the $n-1$ rings from peg 3 to peg 2.

This approach may be translated into algorithmic language where a procedure calls itself. Since at each stage the auxiliary role will be taken by a different peg, we need to be able to identify them each time. This is easily done by observing that if p and q are the numbers of two different pegs, then the number of the 3rd peg is $6-p-q$. Here is the recursive algorithm:

Algorithm Towers of Hanoi

1. **Input** n.
2. **procedure** *hanoi* (n, p, q)
   - 2.1 **If** $n=1$ **then**
     - 2.1.1 Move ring from peg p to peg q
     - **else**
     - 2.1.2 $hanoi(n-1, p, 6-p-q)$
     - 2.1.3 Move ring from peg p to peg q
     - 2.1.4 $hanoi(n-1, 6-p-q, q)$

Does this algorithm solve the *Towers of Hanoi* problem? Let us check that this works for $n=2$, to move the two rings from peg 1 to peg 2, i.e. we invoke the procedure $hanoi(2,1,2)$ ($p=1$ and $q=2$). Since n is not equal to 1, the **else** clause is executed, calling the procedure $hanoi(1,1,3)$ (line 2.1.2). Now $n=1$, so the top disk is moved from peg 1 to peg 3. Continuing with the second statement in the **else** clause 2.1.3, the largest disk is moved from peg 1 to peg 2; finally the ring in peg 3 is moved to peg 2.

Readers make care to trace the algorithm for $n=3$ to convince themselves that it solves the puzzle and also to develop a better understanding of how recursion works. Over to you!

* * * * *

The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics.

—Johannes Kepler

PROBLEM CORNER

SOLUTIONS

PROBLEM 23.2.1 (from Crux Mathematicorum with Mathematical Mayhem)

Suppose that a, b, c are the sides of a triangle with semi-perimeter s and *area* A. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{s}{A}.$$

SOLUTION

Let C be the angle opposite to side c of the triangle. We have $A = \frac{1}{2}ab \sin C \leq \frac{1}{2}ab$, from which it follows that $\frac{1}{a} \leq \frac{b}{2A}$. The equality sign holds if and only if C is a right angle.

Similarly $\frac{1}{b} \leq \frac{c}{2A}$, $\frac{1}{c} \leq \frac{a}{2A}$, so that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{b+c+a}{2A} = \frac{s}{A}$.

But equality cannot hold simultaneously (a triangle has at most one right angle) so we in fact have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < \frac{s}{A}.$$

Solutions were also received from Carlos Victor (Rio de Janeiro, Brazil) and Julius Guest (East Bentleigh).

PROBLEM 23.2.2 (from Mathematical Mayhem)

The quartic $5x^4 - ax^3 + bx^2 + cd - d = 0$ has roots 2, 3, $\frac{N}{271}$ and $\frac{11111}{N}$. Determine the value of $(a+b+c+d)$.

SOLUTION (Carlos Victor, Rio de Janeiro, Brazil)

Using the relation between the roots and the coefficients of the polynomial we find:

(i) $\frac{a}{5} = 2+3+\frac{N}{271}+\frac{11111}{N}$

(ii) $\frac{b}{5} = 6+\frac{5N}{271}+\frac{5(11111)}{N}+\frac{11111}{271}$

(iii) $\frac{c}{5} = -\left(\frac{6N}{271}+\frac{6(11111)}{N}+\frac{5(11111)}{271}\right)$

(iv) $\frac{d}{5} = -\frac{6(11111)}{271}$

Adding (i) - (iv) we find

$$\frac{(a+b+c+d)}{5} = -399$$

and hence $(a+b+c+d) = -1995$

Another solution was obtained by Julius Guest (East Bentleigh).

PROBEM 23.2.3 (from Mathematical Mayhem)

Show that for all positive integers, a, b, c, d the polynomial $x^{4a}+x^{4b+1}+x^{4c+2}+x^{4d+3}$ is divisible by $1+x+x^2+x^3$.

SOLUTION (Carlos Victor, Rio de Janeiro, Brazil)

Let $P(x) = x^{4a}+x^{4b+1}+x^{4c+2}+x^{4d+3}$ and let $Q(x) = 1+x+x^2+x^3$.

If α is a solution of $Q(x) = 0$, then $\alpha^4 = 1$; we now have

$$\begin{aligned} P(\alpha) &= (\alpha^4)^a+(\alpha^4)^b\alpha+(\alpha^4)^c\alpha^2+(\alpha^4)^d\ \alpha^3 \\ &= 1+\alpha+\alpha^2+\alpha^3 = 0. \end{aligned}$$

Hence all roots of $Q(x)$ are also roots of $P(x)$, and so $P(x)$ is divisible by $Q(x)$.

A solution was also received from Keith Anker, and an incomplete solution was received from Julius Guest (East Bentleigh).

PROBLEM 23.2.4 (from Crux Mathematicorum with Mathematical Mayhem)

Find the smallest integer in base eight for which the square root (also in base eight) has digits 10 following the 'decimal' point. In base 10 the answer would be 199 with $\sqrt{199} = 14.10673\ldots$

SOLUTION

Let n be a positive integer with the given property. If m is the integer part of $\sqrt{n}$, let $n = m^2 + r$. We will use the notation $(x)_8$ for the base 8 representation of the number x. By assumption,

$$m + (0.10)_8 \le \sqrt{n} < m + (0.11)_8$$

which is equivalent to

$$\left(m + \frac{1}{8}\right)^2 \le n < \left(m + \frac{9}{64}\right)^2$$

The last inequality may be replaced by the following stronger inequality by taking into account that the expressions on both sides of the inequality above cannot be integer.

$$m^2 + \frac{m+1}{4} \le n < m^2 + \frac{9m}{32}$$

We now have that $\frac{m+1}{4} \le r \le \frac{9m}{32}$. To satisfy this inequality we must have $m \ge 8$. If $8 \le m \le 10$ the lower limit of the inequality is greater than 2 but the upper limit is less than 3, so no integer r can be found in this case. If $m = 11$ then $r = 3$, hence the smallest m is 11 and the smallest n is $n = m^2 + p = 124 = (174)_8$.

Solutions along similar lines were received from Keith Anker, Carlos Victor and Ian Preston.

PROBLEM 23.2.5 (from Crux Mathematicorum with Mathematical Mayhem)

An autobiographical number is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells us how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's and so on. For example, 6,210,001,000 is autobiographical. Find the smallest autobiographical number and prove that it is the smallest.

SOLUTION (Keith Anker, Mt Waverley)

Consider the leading digit.

The leading digit (in particular, only digit) cannot be zero; it would say that there are no zeroes!

If the leading digit is "1" then:
there is one zero;
there cannot be just one "1", because we would then need another "1" in second position to say so!
So there are at least two "1"s, and a digit in second place to say how many. It can be "2", and the smallest autobiographical number with leading "1" is 1210.

If the leading digit is "2", that accounts for three digits, and there must be at least one more digit to say how many "2"s. Thus, such a number [is] > 1210. (A possible number is 2020.)

If the leading digit is ≥ 3, then there are at least five digits: "d", d "0"s, and a digit to say how many "d"s.

Solutions were also received from Carlos Victor and Ian Preston.

PROBLEMS

PROBLEM 23.4.1 (from Crux Mathematicorum with Mathematical Mayhem)

Prove that if m, n are natural numbers and $n \geq m^2 \geq 16$, then $2^n \geq n^m$.

PROBLEM 23.4.2 (from Crux Mathematicorum with Mathematical Mayhem)

A certain country contains a (finite) number of towns that are connected by uni-directional roads. It is known, that, for any two such towns, one of them can be reached from the other one. Prove that there is a town such that all the remaining towns can be reached from it.

PROBLEM 23.4.3 (adapted from Crux Mathematicorum with Mathematical Mayhem)

An "n-m party" is a group of n girls and m boys. An "r-clique" is a group of r girls and r boys in which all of the boys know all of the girls, and an "r-anticlique" is a group of r girls and r boys in which none of the boys knows any of the girls. Show that there is a number m_0 such that every 9-m_0 party contains either a 5-clique or a 5-anticlique.

PROBLEM 23.4.4 (Part (a) proposed by Julius Guest, East Bentleigh)

(a) Evaluate $\displaystyle\int_0^1 \frac{x^4}{(1+x^2)^3}\,dx$

(b) Generalise the result for the integral in part (a) to

$$\int_0^1 \frac{x^{2m-2}}{(1+x^2)^m}\,dx, \quad \text{where } m = 1, 2, 3, \ldots$$

PROBLEM 23.4.5 (adapted from 1986 Qualifying Round of the Swedish Mathematics Olympiad

Prove that

$$(1999!)^{\frac{1}{1999}} < (2000!)^{\frac{1}{2000}}$$

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OLYMPIAD NEWS

The XL International Mathematical Olympiad. Australian student achieves second highest score

Between 10 and 22 July, the Romanian capital Bucharest was venue for this year's IMO. Teams, usually having six members from 81 countries had to contend with six problems during nine hours spread equally over two days in succession. As IMO Jury members agreed, this contest was one of the hardest in the forty-year IMO history. The top score, 39 out of 42 available points, was obtained by only three students. Geoffrey Chu (Scotch College, Melbourne) reached the second highest score (38 points), thus finishing fourth out of 450 students. Never before has an Australian student been placed higher at an IMO!

Here are the papers:

Problem 1

Determine all finite sets S of at least three points in the plane which satisfy the following condition:

> for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

Problem 2

Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i \leq j \leq n} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1 \ldots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

Problem 3

Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side.

N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

Determine the smallest possible value of N.

Problem 4

Determine all pairs of positive integers (n, p) such that

(i) p is a prime

(ii) $n \leq 2p$

(iii) n^{p-1} is a divisor of $(p-1)^n + 1$.

Problem 5

Circles Γ_1 and Γ_2 touch circle Γ internally in M and N. The centre of Γ_2 lies on Γ_1. The common chord of Γ_1 and Γ_2 intersects Γ in A and B. MA and MB intersect Γ_1 in C and D, respectively. Prove that Γ_2 is tangent to CD.

Problem 6

Determine all functions $f : R \to R$ which satisfy

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

Australia came fifteenth, with 115 points our of possible 252, and one gold and one silver medal as well as three bronze medals and one honourable mention. Ahead of Australia were the teams from Russia and China (182 points each), Vietnam (177), Romania (173), Bulgaria (170), Belarus (167), Korea (164), Iran