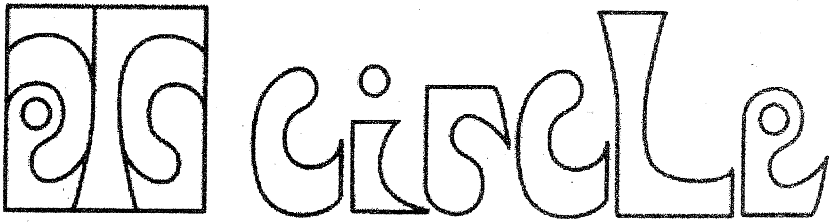
Function

A School Mathematics Journal

Volume 24 Part 5 October 2000



Department of Mathematics & Statistics – Monash University

Reg. by Aust. Post Publ. No. PP338685/0015

Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

* * * * *

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors, *Function*
Department of Mathematics & Statistics
PO BOX 28M
Monash University VIC 3800, AUSTRALIA
Fax: +61 3 9905 4403
e-mail: function@maths.monash.edu.au

Function is published five times a year, appearing in February, April, June, August, and October. Price for five issues (including postage): $27.50* ; single issues $7.50. Payments should be sent to: The Business Manager, Function, Department of Mathematics & Statistics, PO Box 28M, Monash University VIC 3800, AUSTRALIA; cheques and money orders should be made payable to Monash University.

For more information about *Function* see the journal home page at `http://www.maths.monash.edu.au/~cristina/function.html`.

---

* $14 for *bona fide* secondary or tertiary students.

EDITORIAL

Welcome to our readers!

Most readers will be familiar with the classical Greek problem of "squaring the circle": given an arbitrary circle, is it possible to construct a square of the same area using straight edge and compass only. Mathematicians have attempted to solve this problem from as early as 200 BC, but it was only in 1882 that it was finally proved impossible.

The front cover is a humorous look at the converse problem: given a square, is it possible to find a circle with the same area? The answer is Yes: cut the square into pieces (cutting along the indicated smooth curves) and re-arrange the shapes to give the word "circle"!

The issue starts with a reminder that most mathematicians do not work alone but collaborate with one another. The legendary Hungarian-born mathematician Paul Erdös has collaborated to such an extent that mathematicians have devised a new quantity, an Erdös Number, to describe it.

Bert Bolton and Graeme Hunt describe some surprising extensions to results concerning the well-known Fibonacci sequence. Of particular interest are the limiting ratios of successive terms of related Fibonacci sequences.

Marko Razpet looks at various extensions to the curve known as a cycloid, which in its simplest form is the path traced out by a fixed point on the rim of a rolling wheel. Have a look at the intricate curves that can be produced!

In the *History of Mathematics* column, Michael Deakin writes about the life and achievements of the mathematician Ada Lovelace, the daughter of the poet Byron. He also discusses various related philosophical issues, such as why purely mechanical procedures can produce valid results.

Probability and randomness are a fascinating area of mathematics. But as many readers know, the calculation of probabilities can be very difficult. One solution is to use a computer to simulate random phenomena; our *Computers and Computing* column gives some illustrations of how to do this.

Finally, *Problem Corner* contains another batch of solutions to past problems and new problems to exercise your minds over the summer break!

* * * * *

WHAT IS YOUR ERDÖS NUMBER?

Malcolm Clark, Monash University

Many people have the impression that mathematical research is a solitary pursuit. They imagine a mathematician locked away in some dingy room or lonely cabin, oblivious to everyday concerns, and focused on a single problem, scribbling weird symbols on scraps of paper, and thinking long and hard before emerging with a triumphant "Eureka!" and a proof.

The dramatic announcement in 1993 by Andrew Wiles that he had proved Fermat's Last Theorem seemed to support this popular view, for Wiles had worked almost alone and independently of other mathematicians for 8 years on this problem. Yet he still relied on much previous work by other mathematicians who had tackled the same problem.

Doing mathematics is really a social process. Mathematicians do not work in isolation: the abundance of meetings, seminars, conferences and other gatherings confirms this. Electronic communication has sped the process of collaboration.

This world-wide collaboration between mathematicians is no better demonstrated than by the legendary Hungarian-born mathematician Paul Erdös. Over a working life of at least 60 years, he wrote almost 1500 research papers on a huge variety of topics, and collaborated with hundreds of fellow mathematicians.

The extent of Erdös's collaboration is such that, in typical fashion, mathematicians have invented a new quantity to define it, an Erdös Number.

Erdös himself is assigned the Erdös Number 0. All those who have published a paper (research article) jointly with him are given Erdös Number 1. Those who have published a paper with someone who has published a paper with Erdös are given Erdös Number 2; those who have published a paper with someone who has published a paper with someone who has published a paper with Erdös receive number 3. And so it goes. Any person not yet assigned an Erdös Number who has written a joint mathematical paper with a person having an Erdös Number n earns the Erdös Number $n + 1$. Anyone left out of this assignment has the Erdös Number infinity.

This score-keeping system started years ago, but Jerrold Grossman at Oakland University has become the compiler and guardian of this list of collaborators. He maintains an interesting web site, giving information about the project, Erdös himself, and related topics at

`http://www.acs.oakland.edu/~grossman/edoshp.html`

As at January 2000, there were 507 mathematicians with Erdös Number 1 (including 200 with more than one joint paper with the master), and 5897 with Erdös Number 2. A person's Erdös Number measures the closeness of their collaboration with Erdös, not necessarily their innate mathematical ability; Einstein had the Erdös Number 2, and Andrew Wiles has an Erdös Number of at most 4.

Another way of looking at mathematical collaboration is by means of a graph. Not the sort of graph like $y = x^2$ or a bar chart, but as an array of *vertices* connected by *edges*. Each mathematician is represented by a vertex, and any two mathematicians who have collaborated on a paper have their vertices joined by an edge. Erdös himself is somewhere in the centre of the graph. The result is an enormous tangle of lines that snares almost all mathematicians, with branches reaching out into computer science, biological sciences, economics and even the social sciences. And any person's Erdös Number is given by the shortest number of edges linking that person to Erdös in this enormous collaboration graph.

So what's *your* Erdös Number? If you have never co-authored a mathematical article, then regrettably it must, by definition, be infinity. But maybe you know somebody who has, and in principle, their Erdös Number *could* be finite. Grossman's web site gives some tips on finding out your Erdös Number, saying "You never know: it could be lower than you think!"

* * * * *

In the quest for simplification, mathematics stands to computer science as diamond mining to coal mining. The former is a search for gems. ... The latter is permanently involved with bulldozing large masses of ore—extremely useful bulk material.

—Jacob T Schwartz in *Discrete Thoughts: Essays on Mathematics, Science and Philososphy*, Boston: Birkhäuser, 1986

* * * * *

A good proof is one that makes us wiser.

—Yu I Manin in *A Course in Mathematical Logic* New York: Springer Verlag, 1977

EXTENSIONS TO THE FIBONACCI SEQUENCE

Bert Bolton and Graeme Hunt, University of Melbourne

The mathematician whose name occurs in the title is the Italian Leonardo Fibonacci (c1170–c1240). His name is pronounced Fee-bon-ah-chee, and the c's that occur in the brackets after his name stand for *circa*, a Latin word meaning "about". They show the uncertainty in both the date of his birth and that of his death. [An easy way to find good information about Fibonacci and other influential mathematicians is to look them up in *The Dictionary of Scientific Biography* (*DSB*). This work runs to many volumes and is usually found only in large libraries, but the *Concise DSB* is often found in local libraries.]

The mathematical sequence named after Fibonacci is given by the pattern of numbers

$$F_1 : 1, 1, 2, 3, 5, 8, 13, \ldots \tag{1}$$

where the subscript in F_1 is the value of the second term. The two initial numbers are given and from then on each number is the sum of the two to its left. The ratio of two successive numbers, with the right-hand one in the numerator and the left-hand one in the denominator, yields the pattern

$$1, 2, 1.5, 1.666\ldots, 1.6, 1.625, 1.6153\ldots,$$

which suggests that every ratio is not only a member of this sequence but also a member of one or the other of two sub-sequences, one of which seems to converge from above and the other from below. Readers who like computing may like to write a program and check that the two sub-sequences do each converge to 1.6180… . This value may be confirmed by an algebraic argument. We will show that if such a limit exists, then its value may be determined.

Represent the mth term of the sequence by a_m, so that the next terms are a_{m+1} and a_{m+2}, and

$$a_{m+2} = a_{m+1} + a_m . \tag{2}$$

Dividing by a_{m+1} gives

$$\frac{a_{m+2}}{a_{m+1}} = 1 + \frac{a_m}{a_{m+1}} \tag{3}$$

Assuming that there is a limit to the ratio $\frac{a_m}{a_{m+1}}$ as m gets very large, let this limit be called x. Then equation (3) becomes

$$x = 1 + \frac{1}{x}$$

which may be solved as a quadratic equation with roots $x = \frac{1 \pm \sqrt{5}}{2}$. Because x must be positive, we find

$$x = \frac{1+\sqrt{5}}{2} = 1.6180\ldots \,. \tag{4}$$

This is the required value, applying to both of the subsequences, the increasing one and the decreasing one. The rectangle that has the ratio of its longer side to its shorter exactly equal to this value is called the *golden rectangle*, which is also said to be the best rectangle to look at! [For more on the *golden rectangle* and its *rectangular spiral* see Bert Bolton's article in *Function*, *Vol* 21, *Part* 1; see also *Function*, *Vol* 16, *Part* 5.]

There are other similar sequences related to the Fibonacci Sequence. First consider the sequence in which the second member is 2 (replacing 1), but with equation (2) holding as before. This new sequence is

$$F_2 : 1, 2, 3, 5, 8, 13, \ldots ,$$

which is the same sequence as F_1 apart from its first term.

But now try putting a 3 in the second position to get

$$F_3 : 1, 3, 4, 7, 11, 18, 29, \ldots \tag{5}$$

and this time the ratios of successive pairs become

3, 1.3333…, 1.75, 1.5714…, 1.6363…, 1.6111…, … .

Again we notice that there are two subsequences of ratios both apparently converging to the value given by equation (4). Algebraically, the sequence is still determined by equation (2), and so the argument given earlier still holds for this new sequence. Furthermore new sequences, all different, may be written as follows:

$$F_4 : 1, 4, 5, 9, 14, 23, ...$$
$$F_5 : 1, 5, 6, 11, 17, 28, ...$$
etc.

All such sequences will give the same limit as found by equation (4).

Some new insight is gained when the values of the terms of the sequences are plotted on the x and y axes of a graph. In Figure 1, some of the terms (the odd-numbered ones) of F_1 (1, 2, 5, …) are marked on the x–axis and the corresponding terms of F_2 (1, 3, 8, …) (which are also the even-numbered terms of F_1) on the y–axis. Corresponding components on the two axes are joined by straight lines as shown, suggesting strongly that if these lines make angles θ with the x–axis, then these angles converge to a limiting value.

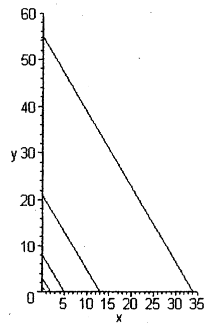


Figure 1

In Table 1, the components of both sequences are labelled with the subscript m to indicate the mth term in the sequence. Thus F_{1m} represents the mth term or entry in the sequence F_1. We note that $F_{2m} = F_{1\,m+1}$ and that F_{1m} is the same as the number previously called a_m. We now have:

m	1	2	3	3	5	6	etc
$F_{1\,m}$	1	1	2	3	5	8	etc
$F_{2\,m}$	1	2	3	5	8	13	etc

Table 1

The value of tan θ is just the ratio of the two terms at the mth position. Each value of tan θ is a value of $\frac{F_{2m}}{F_{1m}}$. These values have already been shown to converge to the value 1.6180... , as defined by equation (4). This limiting value now has an interpretation of tan θ where

$$\theta = 58.2825...°.$$

The next sequence considered was F_3 with components F_{3m} as shown in Table 2.

m	1	2	3	4	5	6	etc
F_{1m}	1	1	2	3	5	8	etc
F_{3m}	1	3	4	7	11	18	etc
$\frac{F_{3m}}{F_{1m}}$	1	3	2	2.333...	2.2	2.25	etc

Table 2

These numbers making up the sequence F_3 are plotted in Figure 2. Along the horizontal axis are the odd-numbered terms (1, 4, 11, ...) and along the vertical axis are the even-numbered terms (3, 7, 18, ...). The lines joining them reveal a new feature; the values of tan θ are larger than those in Figure 1. Further values of F_{nm} were calculated up to $n = 7$ and the values of $\frac{F_{nm}}{F_{1m}}$ obtained to get their limiting

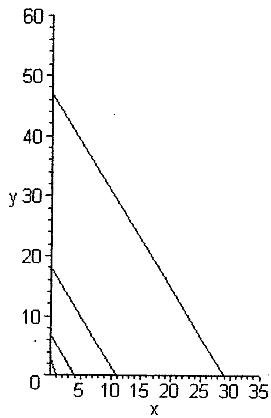


Figure 2

values to 4 significant figures. The results are recorded in Table 3.

n	2	3	4	5	6	7
Limit	1.618	2.236	2.854	3.472	4.090	4.708

Table 3

Figure 3 shows a plot of the values found in Table 3, which indicates that a straight line can be drawn through them. We will now prove that this is indeed the case. In fact we will do this more generally.

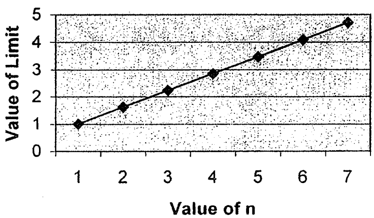


Figure 3

To prove the linearity, consider Table 4, which is a generalisation of Table 2.

m	0	1	2	3	4	5	6	7
$F_{1\,m}$	1	1	2	3	5	8	13	21
$F_{n\,m}$	1	n	$n+1$	$2n+1$	$3n+2$	$5n+3$	$8n+5$	$13n+8$

Table 4

In the bottom row, the coefficients of n are all entries from the row above, and if we now form the required ratio, we get, for example

$$\frac{F_{n5}}{F_{15}} = \frac{5n+3}{8} = \frac{nF_{14}+F_{13}}{F_{15}}$$

More generally, it can be shown that

$$\begin{aligned}\frac{F_{n\,m}}{F_{1\,m}} &= \frac{nF_{1\,m-1}+F_{1\,m-2}}{F_{1\,m}} \\ &= \frac{n\,F_{1\,m-1}}{F_{mm}} + \frac{F_{1\,m-2}}{F_{1m}}\end{aligned} \tag{6}$$

In equation (6), the last term can be written as

$$\left(\frac{F_{1\,m-2}}{F_{1\,m-1}}\right)\left(\frac{F_{1\,m-1}}{F_{1\,m}}\right)$$

and in the limit as $m \to \infty$, each of these ratios is the reciprocal of x as given by equation (4). So, for very large values of m, we have the ratio R_n (let us call it) given by

$$R_n \approx \frac{n}{x} + \frac{1}{x^2}$$

As x is a given number, this represents a straight line in the limit as $m \to \infty$, as claimed. We have

$$Limit = \frac{n}{x} + \frac{1}{x^2}$$

or, numerically,

$$\text{Limit} = 0.618n + 0.382.$$

This result further confirms the linear relationship depicted in the graph in Figure 3.

* * * * *

THE CYCLOID AND ITS RELATIVES

Marko Razpet, Institute of Mathematics, Physics and Mechanics, Ljubljana

Suppose P is a point on the rim of a wheel of radius a, and this wheel rolls along a straight line which we will take as the x-axis. Suppose the point P is initially on this axis, and that the wheel rolls through an angle t (measured in *radians*). Then the x and y co-ordinates of P are given by the equations:

$$x = a(t - \sin t)$$
$$y = a(1 - \cos t)$$

and the curve is called a *cycloid*.

[You may be most familiar with the description of a curve in terms of an equation of the form $y = f(x)$, in which a value of y is calculated from some given value of x. When a curve is described by means of two equations, as here, each dependent on some third variable, we say it is given in *parametric form*. The cycloid is one of the standard examples of a curve specified in parametric form.]

Figure 1 shows the cycloid and its relation to the circular wheel that generates it.

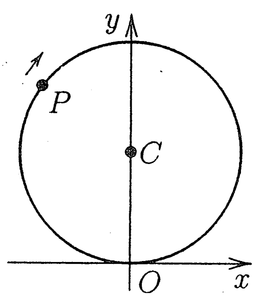


Figure 1. The Cycloid and its generating circle.

The cycloid has many interesting properties, but it is only one among a set of more general curves, which we will now examine. Consider the equations

$$x = a(\kappa t - \sin mt)$$
$$y = a(1 - \cos t)$$

where the first equation has been modified by the insertion of two new constants, κ and m. The different values of κ and m give rise to different curves of which the cycloid itself is one (with $\kappa = m = 1$). Because the second equation is not altered when we pass from the cycloid to its generalisations, the curves all share the property of the cycloid itself of lying within a strip bounded by the lines $y = 0$ and $y = 2a$.

Let us now look at some special cases.

The Case $m = 1$

Start with the special case $m = 1$. This gives a set of curves, whose properties depend on the value of the other constant, κ. When $y = 0$, we have $\cos t = 1$, so that $t = 2n\pi$, where n is an arbitrary integer. Similarly, when $y = 2a$, $t = (2n+1)\pi$. If we put these values into the equations for x and y, we find that $y = 0$ when $x = 2n\kappa\pi a$ and $y = 2a$ when $x = (2n+1)\kappa\pi a$. All the curves exhibit these properties.

Figure 2 shows some of the curves from the full set. The curve for which $\kappa = 1$ is the cycloid, and that for which $\kappa = 0$ is readily seen to be a circle. These curves are drawn with solid lines in the figure. The others shown are those for which $\kappa = 3/4$ (dots) and $\kappa = 5/4$ (dashes).

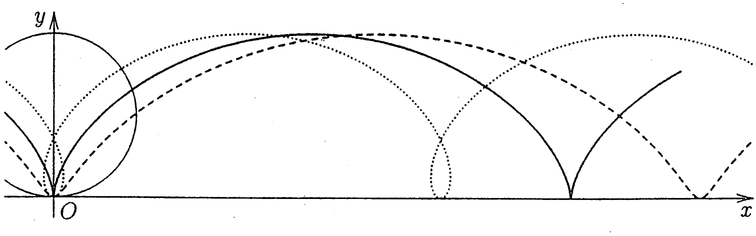


Figure 2. Curves $x = a(\kappa t - \sin t)$, $y = a(1 - \cos t)$

If we allow the possibility of negative κ, this leads to nothing really new. For if we replace κ by $-\kappa$ throughout and also replace x by $x+a\kappa\pi$ and y by $2a-y$ also throughout, the formulae are unaltered. Thus the curves for negative values of κ are the same as those for positive values except for a translation and a reflection. This property is illustrated in Figure 3.

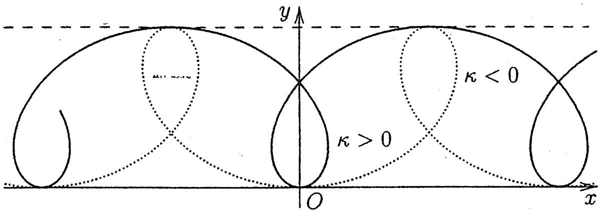


Figure 3. The curves $x = a(\kappa t - \sin t)$, $y = a(1-\cos t)$ for $\kappa = \pm 1/2$

The Case *m* =2

When $m = 2$, we get more interesting curves. The special case $\kappa = 0$ gives a curve with the parametric equations

$$x = -a \sin 2t$$
$$y = a(1-\cos t)$$

and this replaces the circle we found in the previous section. This new curve is a member of a set of curves known as *Lissajous Figures*. [The circle is also a Lissajous Figure.] This curve has the equation

$$4(y-a)^4 = a^2\{4(y-a)^2 - x^2\},$$

which is derived by eliminating t. Figure 4 (on the next page) shows a graph of this curve.

We may also derive the curve in this new family for which $\kappa = 1$. Just as in the previous family we had a circle and a cycloid as special members of the set, so here he have the curve shown in Figure 4 and a new curve shown in Figure 5 (also on the next page).

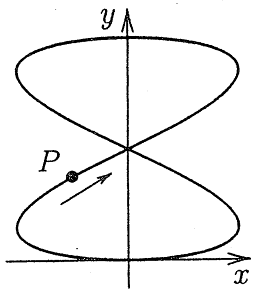


Figure 4. The curve $x = -a\sin 2t,\ \ y = a(1-\cos t)$

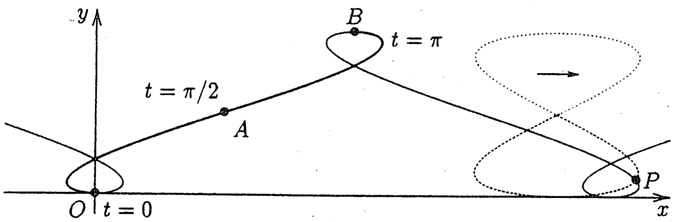


Figure 5. The curve $x = a(1-\sin 2t),\ \ y = a(1-\cos t)$

Other values of κ give other curves, many of them most attractive to the eye. Some examples are shown in Figure 6 (on the next page). This shows the cases arising when κ is positive and relatively small. When κ is a large positive number, the curve is relatively uninteresting, but they become nicer to look at as κ decreases. All these curves meet the line $y = 0$ at the points where $x = 2n\kappa\pi a$. They also meet the line $y = 2a$ at the points where $x = (2n+1)\kappa\pi a$. The lines $x = (2n+1)\kappa\pi a/2$ are axes of symmetry for all these curves.

We may also investigate the cases for which κ is negative. Again if its absolute value is large, the curves look relatively uninteresting, but when κ lies between -1 and 0, the curves are both complicated and ornamental. Some examples are shown in Figure 7.

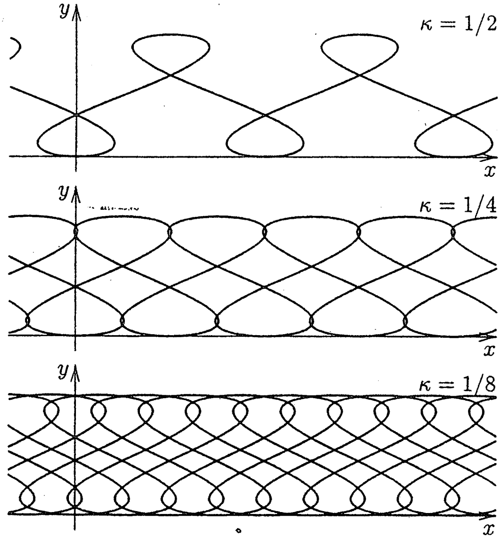


Figure 6. The curves $x = a(\kappa t - \sin t)$, $y = a(1 - \cos t)$ for various positive values of κ.

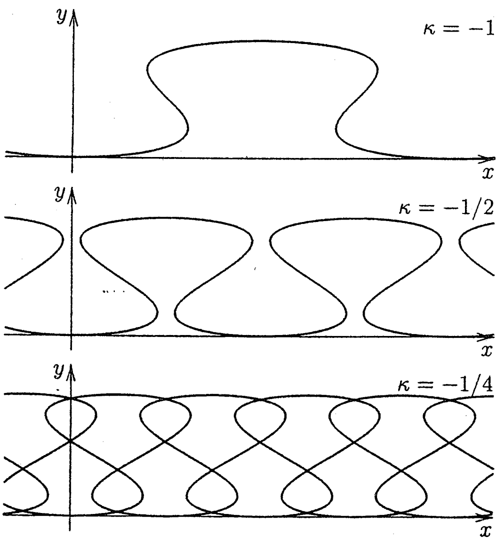


Figure 7. The curves $x = a(\kappa t - \sin t)$, $y = a(1 - \cos t)$ for various negative values of κ.

NEWS

Blessed Francesco Faà di Bruno

In *Function*, *Vol* 10, *Part* 5, the story was told of Francesco Faà di Bruno, a mathematician and Roman Catholic Priest, whose good works led to his becoming a candidate for canonisation. There are four steps in this process. First, the Vatican must be sufficiently interested in the case to open an official file on the individual concerned. When this happens, the person under review becomes known as a "Servant of God". If the initial investigation is favorable, they give a limited endorsement to the cult of the candidate, who then becomes known as "The Venerable ...". The third step is more serious and involves a ceremonial announcement endorsing the candidate's life as manifesting good works and a lifestyle to be regarded as exemplary. This step is known technically as Beatification, and it bestows on the candidate, the title of "Blessed". Australia's Mother Mary McKillop reached this stage a few years back. The final stage is called *canonisation*, and it bestows the title "Saint" on the candidate.

Francesco Faà di Bruno (1825–1888) is said to have lived a life of exemplary piety and has a devoted band of admirers, especially in his native Turin. He was a mathematician, and in that capacity is best remembered for his generalisation of the familiar chain rule of elementary calculus, to the case of the *n*th derivative. That is to say, he gave a precise formula (a very complicated one) for $\left(\frac{d}{dx}\right)^n f(g(x))$. It is today called Faà di Bruno's Formula. [As an exercise, try working out the first few cases: $n = 2, 3, \ldots$.] He is also remembered for a number of inventions, notably an early form of typewriter for the use of the blind, and also for his good works, especially his concern for "fallen women" as the Vatican put it.

He was accorded the title of "Venerable" in 1971, and we learned of this rather late in the day and so reported it in *Function*, *Vol* 10, *Part* 5 (1986). We were behind the times then and the same is true of this present news-item; he has been beatified. This event took place on September 25, 1988, a little over a hundred years after his death. You may look up brief further details of his life on the website

`http://www-history.mcs.st-and.ac.uk/history/Mathematicians/Faa_di_Bruno.html` which is part of an ambitious project to make large numbers of (usually brief) biographies of mathematicians available on the internet.

This one gives a typically brief, and mostly accurate, account of Faà di Bruno, although it makes a common mistake in that it claims that he has been made a saint (much as our own media did in the case of Mother Mary McKillop). This has not yet happened, as the above explanation makes clear. However, there are links to two more authoritative sources, one of them the relevant entry in *The Catholic Encyclopedia*, the other a large specialised source, that you may browse if you read Italian.

These sources both tend to be more concerned with his piety and good works than with his Mathematics. However, the larger of the two (the second) also covers this second aspect of Faà di Bruno's life.

* * * * *

What indeed is mathematics? ... A neat little answer ... is preserved in the writings of a church father of the 3rd century A.D.; Anatolius of Alexandria, bishop of Laodicea, reports that a certain (unnamed) "jokester", using words of Homer which had been intended for something entirely different, put it thus:

> Small at her birth, but rising every hour,
> While scarce the skies her horrid [mighty] head can bound,
> She stalks on earth and shakes the world around.
> (Illiad, IV, 442–445, Pope's translation)

For, explains Anatolius, mathematics begins with a point and a line, and forthwith it takes in heaven itself and all things within its compass. If the bishop were among us today, he might have worded the same explanation thus: For mathematics, as a means of articulation and theoretization of physics, spans the universe, all the way from the smallest elementary particle to the largest galaxy at the rim of the cosmos.

—Salomon Bochner in *The Role of Mathematics in the Rise of Science*, Princeton University Press, 1966

* * * * *

HISTORY OF MATHEMATICS

Mathematical Sprites and Fairies

Michael A B Deakin

In this column, I finally and reluctantly address an issue that I would rather not. I do so however, because I have had more requests for this topic than for any other, and these requests keep coming. So here goes; I will write about Ada Augusta King, Countess of Lovelace (1815–1852), often known as Ada Lovelace. My article, however, will be in two parts. The first will give a brief summary of her life and achievements; the second will treat a number of philosophical issues raised by the first. I shall refer to my subject as Ada throughout, not to belittle her, but rather to avoid the problems that arise in trying to decide her surname.

1. Ada's Life and Achievements

Ada was the only legitimate child of the poet Byron, who legally separated from her mother shortly after Ada was born. [Salacious gossip attributes various illegitimate children to Byron, with greater or lesser degrees of plausibility in the various cases.] Grown to womanhood she married Baron William King, who later became the Earl of Lovelace. After the fashion of the day, she adopted King's surname on her marriage, and so her name came to be as I have given it. However, she is also known under various other names, with some reason behind the more sensible of them.

It is very clear that she wanted not only to learn but also to excel in Mathematics. Her current reputation is based on the belief that she succeeded in this ambition and contributed significantly to the field (unless one also counts the notoriety of her being her father's daughter). I believe that this reputation is greatly overblown, and that her mathematical talents were minimal. Which is why I have been so reluctant for so long to devote a column to her. She learned her Mathematics from textbooks and with the help of a variety of tutors, of whom Mary Somerville (1780–1872), Augustus De Morgan (1806–1871) and Charles Babbage (1791–1871) are the best known.

She is the subject of at least three book-length biographies, of which by far the best is Dorothy Stein's *Ada: A Life and a Legacy* (MIT Press, 1985). This was the second to appear and is the only one (at least of these three) to be written by a biographer who knew enough Mathematics to do the job well. However, much of what we read about Ada comes from other sources and is unreliable and uncritical. Stein's book details the origins of the present exaggerated claims made on Ada's behalf, and goes on to a more realistic assessment of her true significance. Her own work is a model of biographical writing and is based on very careful analysis of original documents such as letters and manuscripts.

What is very clear is that Ada had a passion for Mathematics, and that her womanhood was a difficulty in her pursuit of this goal. Mathematics was seen as most unwomanly: even more so than other very definitely unwomanly things in Victorian England! As Stein remarks (pp. *xi-xii*), "Ada swears unabashedly, gambles, and takes a lover, but feels constrained to buy her books and geometry models anonymously".

But Stein also notes severe limitations in her understanding of the mathematical enterprise. One is that she never really came to terms with the simple procedure of substituting one expression into another. So we find in one of her letters to De Morgan:

> "It had not struck me that, calling $(x+\theta)=v$, the form $\dfrac{(x+\theta)^n - x^n}{\theta}$ becomes $\dfrac{v^n - x^n}{v - x}$. And by the bye, I may remark that the curious *transformations* many formulae can undergo, the suspected & to a beginner apparently *impossible identity* of forms exceedingly *dissimilar* at first sight, is I think one of the chief difficulties in the early part of mathematical studies. I am often reminded of certain sprites and fairies one reads of, who are at one's elbow in *one* shape now, the next minute in a form most dissimilar; and uncommonly deceptive, troublesome and tantalizing are the mathematical sprites and fairies sometimes; like the types I have found for them in the world of fiction."

That a beginner might have such difficulties is understandable, but as the context makes clear, Ada is here learning Calculus, and these difficulties should by now be behind her. However, they persisted. Even later, when she was working with Babbage, they were evident. Stein comments:

> "For the subject [Mathematics] itself she had little natural talent; its techniques, despite hard work, continued to elude her; its symbols remained the doings of 'mathematical sprites & fairies'."

This weakness is all the more remarkable, when it is considered that her present claim to fame rests entirely on an alleged aptitude for computing, which relies to an enormous extent on formal manipulations of symbols, and a trust that this process is valid.

Ada's mathematical reputation rests on a single paper, to which I now turn. It is widely held that Charles Babbage contributed in a significant way to the early development of the computer. Whether this is true or not is a moot point, but it is clear that one of his designs, for the so-called *Analytical Engine*, anticipated many of the ideas nowadays seen as central to computer design. [Babbage, however, saw his *Analytical Engine* as being machined in metal, and the extreme engineering precision required was right at the edge of what could then be achieved; he also experienced continual difficulties with the funding of his project.]

Babbage never published anything on his *Analytical Engine*, but he lectured on it and so it happened that one of his lectures was the subject of a report, written in French published by the Italian engineer and politician Luigi Menebrea. Ada set out to translate this work into English, and did so, including not only the translation itself, but also copious notes whose total length greatly exceeded that of the work itself. She was encouraged in this enterprise by Babbage, and struggled to complete the task to his satisfaction despite quarrels and disagreements.

It is a measure of her will to succeed and of the importance she attached to the work that she referred to the paper as "this first child of mine": this at a time when she had already given birth to three (human) children! The work did appear and its author was identified as "A. A. L.", (so that the name Ada Augusta Lovelace has some historical basis).

This paper may be read in a number of places, of which the most convenient is probably the anthology *Charles Babbage and his Calculating Engines*, by Philip and Emily Morrison (Dover, 1961). There you may find a rather curious passage:

> "Let us now examine the following expression:

$$2\cdot\frac{2^2.4^2.6^2.8^2.10^2...(2n)^2}{1^2.3^2.5^2.7^2.9^2...(2n-1)^2.(2n+1)^2},$$

which we know becomes equal to the ratio of the circumference to the diameter, when n is infinite. We may require the machine not only to perform the calculation of this fractional expression, but further to give indication as soon as the value becomes identical with that of the ratio of the circumference to the diameter when n is infinite, a case in which the computation would be impossible. Observe that we should thus require of the machine to interpret a result not of itself evident, and this is not amongst its attributes, since it is no thinking being. Nevertheless, when the cos of n = 1/0 has been foreseen, a card may immediately order the substitution of the value of π (π being the ratio of the circumference to the diameter), without going through the series of calculations indicated."

The formula quoted is Wallis's Product (see *Function*, *Vol* 22, *Part* 4), and the sensible point is made that the machine itself is incapable of taking the limit as $n \to \infty$, as that formula requires. But *we*, knowing this answer, can tell the machine what we know, so that we may have it check the progress of its calculation as n gets larger. The use of the term 1/0 should not trouble us unduly; it is simply an old symbol for ∞. But what of "the cos of $n = \infty$?" This is a plain nonsense, and Ada should have known as much. But she didn't. Stein tracked down the origin of this queer turn of phrase. Back in the original French, Menbrea had "Cependant, lorsque le cos. de n = 1/0 a été prévu ...", and here it is clear that we are dealing with a misprint. It *should* read "Cependant, lorsque le cas de n = 1/0 a été prévu ...", which means "Nevertheless, when the case of n = 1/0 has been foreseen ...", and which makes perfect sense.

Ada should have recognised the mistake and have corrected it. But she didn't.

And I am sorry to be so negative about her, but it seems to me that Stein has made out her case most convincingly.

2. The Wonder of the Mathematical Sprites and Fairies

But now let's leave Ada herself, and concentrate on the question that her naivete raises for us. Why does substitution work? How is it that purely mechanical procedures produce valid results?

Now this is not itself a mathematical question. The techniques of Mathematics itself are unable to answer it. But it does help to address it if we look at mathematical argument and what it does.

Essentially a chain of mathematical reasoning is no different from any other chain of reasoning. What distinguishes the mathematical from, say, the verbal argument is that its greater complexity demands aids to memory and flags detailing where the argument has got to. These are usually written symbols, but they can also find embodiment in hardware or in stored programs, etc.

But the laws of Logic that make a mathematical argument valid or invalid are those same laws of Logic that apply in everyday discourse. If we say "The cat is on the mat" and if we happen to know that the mat is in the living room, then we may deduce that the cat is also in the living room.

Nonetheless, mathematical results can be surprising, even though supported by an impeccable chain of argument. For example, if a, b, c are the sides of a triangle and A, B, C are the angles subtended respectively by those sides, then the statement "C is a right angle" implies that $c^2 = a^2 + b^2$ (Pythagoras' Theorem) and we also know that the statement $c^2 = a^2 + b^2$ implies that C is a right angle (Converse of Pythagoras' Theorem). Both these inferences are capable of strict proof: they proceed by means of chains of valid deduction, and so in this sense the two statements are equivalent.

Yet the two give rise to two quite different responses in our minds. One refers to the measure of an angle; the other to the equality of two areas. Logically, the statements are equivalent; psychologically, they are not. It is very much as if the statement about the angle has metamorphosed into a quite different statement about areas, much as sprites and fairies may assume different guises. Ada's is not a silly question therefore, but it is not a mathematical one as such. Most of us, as we learn Mathematics, learn not to worry about such questions, at least not to worry in such a way as to impede our actual mathematical endeavours.

Philosophers, looking at such questions, tend to classify statements into various categories. Thus *analytic* statements are those for which the subject (the thing we are talking about) necessarily possesses the properties asserted of it by virtue of its very nature. "All squares have four sides" is an analytic statement, because it is part of the definition of a square that it have four sides. *Synthetic* statements, by contrast, do not follow simply from the definition of the subject. "The cat is on the mat" is synthetic, because nothing intrinsic to the nature of the cat entails its being on the mat.

Then there is a further classification into *a priori* and *a posteriori* statements. *A priori* statements are those that are knowable independent of experience; they are necessarily true, and can never be otherwise. Our earlier example of Pythagoras' Theorem is an example of an *a priori* statement. *A posterori* statements, by contrast could be false. "The cat is on the mat" is an *a posteriori* statement; it could well be false. The cat could (under other conditions) very well be elsewhere, but right now it is on the mat.

It was the philosopher Immanuel Kant (1724–1804) who proposed this twofold classification and who pointed out that the two modes of classification are not exactly the same. There is a tendency to imagine that analytic statements are *a priori* while synthetic statements are *a posteriori*. This is because the first group are logically forced on us, while the second depend on experiment and observation.

Kant recognised that analytic *a posteriori* statements could not exist. We cannot possibly imagine a situation where something is necessarily true as a matter of simple definition and nonetheless requires experimental checking; the check would always be unnecessary and irrelevant. But he did allow the case of synthetic *a priori* statements, and he saw mathematical statements as being examples of this class. Thus, Pythagoras' Theorem does not follow immediately from the definition of a right angle, but it is nonetheless the case that all right angles must possess this property.

I would say that the difference lies in our own ability to see or not to see the connection. A synthetic *a priori* statement is, on this account, one that we have to *work* to appreciate. (And perhaps here we can sympathise with Ada's difficulties over the "apparently *impossible identity* of forms exceedingly *dissimilar* at first sight, [which] is I think one of the chief difficulties in the early part of mathematical studies".)

For Kant, the synthetic *a priori* classification included even such apparently simple statements as $7 + 5 = 12$, because the definition of (say) 12 makes no reference either to 7 or to 5. We would define 12 as "the next number after 11" or else perhaps as $11 + 1$. Nonetheless, it is not a difficult matter to show that this definition readily entails the property asserted. By contrast, it turned out to be very difficult to prove that Fermat's Last Theorem was entailed by the arithmetical properties of the natural numbers!

When mathematicians approach such problems, they tend to think of there being a mathematical world in which such things as right-angles and natural numbers actually exist. This is not a "real world" in the sense that the world of everyday experience is real; it is often called an "ideal world", and the underlying philosophy is called Idealism, or Platonism (after the philosopher Plato). By no means all mathematicians agree with this view of matters, and some argue strongly against it, but it underlies a lot of the day-to-day language of Mathematics.

Those who argue against such a view hold that Mathematics is the task that builds logical consequences on previously accepted results and does so in such a way that no contradiction can ensue. Those who hold to it emphasise the process of abstraction, of arriving at universal results from specific instances. So, for example, if I (by counting them) find that

$$7 \text{ bottletops} + 5 \text{ bottletops} = 12 \text{ bottletops},$$

then I intuit, without need of further experimentation, that $7 + 5 = 12$. The property is then applied universally, not just to other objects, but is seen as intrinsic to the numbers themselves.

That such mental processes can actually provide us with information about our world (both real and ideal) is a great mystery, but we depend on it in everything we do. It was well put by the theoretical psychologist Warren MacCulloch: "What is a number, that a man may know it, and a man, that he may know a number?" The gendered language should not worry us; it was the fashion of the day (1961), but the questions remain and are as important and as puzzling today as they have always been.

* * * * *

> Neither you nor I nor anybody else knows what makes a mathematician tick. It is not a question of cleverness. I know many mathematicians who are far abler than I am, but they have not been so lucky. An illustration may be given by considering two miners. One may be an expert geologist, but he does not find the golden nuggets that the ignorant miner does.

—L J Mordell in H. Eves *Mathematical Circles Adieu*,
Boston: Prindle, Weber and Schmidt, 1977.

* * * * *

COMPUTERS AND COMPUTING

Simulating Random Phenomena

Cristina Varsavsky

Probability is a fascinating area of mathematics. Over the years we have included in *Function* a number of articles in this area—*The Buffon Paperclip*, *Are your Tattslotto Numbers Overdue?*, *Heads and Tails with Pi* and *On Biased Coins and Loaded Dice* are some of the articles that come to my mind, apart from the recent *History of Mathematics* columns.

Probability deals with random phenomena. We encounter randomness in our every day life, but we rarely get to see enough repetitions of the same random phenomenon to be able to see the regularity in the long run that appears in it. So we usually use mathematics to calculate probabilities of particular outcomes. But this is not always easy and it could be quite challenging even for the more experienced.

However, in many cases there is a simple way of tackling probability problems. Since computers are very good at doing complex, repetitive, and lengthy calculations, why not use them to simulate many repetitions of the same phenomenon? Let us see how this works. Take the following example:

> A coin is tossed 4 consecutive times, what is the probability of a run of at least two tails in a row?

Using the computer to simulate repeated tosses of a coin is straight forward: since there are only two possible outcomes—head or tail— both equally likely and the outcome of each toss is independent from the previous one, we simply generate a random sequence of 1's and 0's where the 1's represent (say) heads and the 0's represent tails. I used *Excel* to obtain the following sequence:

0101 1000 1101 1100 1111 0011 0000 0101 0011 1001

This corresponds to repeating 10 times the phenomenon of tossing a coin four times in a row. Of the 10 times, the 2^{nd}, 4^{th}, 6^{th}, 7^{th} , 9^{th} and 10^{th} group of four has at least two consecutives 0's in it, ie. it corresponds to at least two tails in row. So our estimated probability is

$$\frac{6}{10} = 0.6$$

Of course 10 repetitions is not enough to give you a good estimate, but now that we know how to simulate the coin tossing we could leave to the computer to do the hard work of repeating an counting for us. Here is, in pseudocode, a program that you can adapt to your preferred programming language:

1. $n \leftarrow$ number of repetitions
2. $i \leftarrow 1$
3. $counter \leftarrow 0$
4. While $i \leq n$ do
   - 4.1 $m = 0$
   - 4.2 $k = 0$
   - 4.3 Repeat until $m = 3$
     - 4.3.1 $m \leftarrow m + 1$
     - 4.3.2 $r \leftarrow$ random number 0 or 1
     - 4.3.3 If $r = 0$ then
       - 4.3.3.1 $k \leftarrow k + 1$
       - else
       - 4.3.3.2 if $k \geq 2$ then $counter \leftarrow counter + 1$
       - 4.3.3.3 $k \leftarrow 0$
   - 4.4 If $k \geq 2$ then $counter \leftarrow counter + 1$
   - 4.5 $i \leftarrow i + 1$
5. $probability \leftarrow counter / n$
6. Output *probability*

I wrote this in QuickBasic. In this language, a random number between 0 and 1 is generated with `RND`; but this has to be first initialised with `RANDOMIZE` (for example, `RANDOMIZE TIMER`). To obtain a random number 0 or 1, we use

```
INT(RND + 0.5)
```

These are the different estimates of probabilities obtained with this program:

1000 repetitions	→	0.509
2000 repetitions	→	0.495
5000 repetitions	→	0.4974
10000 repetitions	→	0.4997

20000 repetitions → 0.49785

Therefore the probability of having a run of at least two tails when an unbiased coin is tossed 4 times seems to be around 0.49. Perhaps you would like to see whether you can prove this using mathematics.

Now let us try to use the same approach to answer the follow question:

> Tom and Jackie are a young couple who would very much like to have a boy. But they are prepared to have no more than 4 children. How likely are they to have their dreams come true?

To answer this question we have to assume that Tom and Jackie are as likely to have a boy as to have a girl, and that there are no reasons to believe that they will have twins. That is, the phenomenon has two outcomes—boy and girl—each of them with probability of 0.5, and each birth is an independent event. We proceed as before; we assign 1 to "boy" and 0 to "girl". I used *Excel* again to produce a sequence of 1's and 0's:

1010010101110011100010000111110010111 0000

which we group according to what we are looking for, ie groups of 4 or less depending on the 1's:

1–01–001–01–01–1–1–001–1–1–0001–0000–1–1–1–1–1–001–01–1–1–0000

obtaining a total of 20 "successes" in 22 repetitions, which gives a probability of $\frac{20}{22} \approx 0.909$.

A program to automate this process will not look very different from the one before, only the loop in 4. will have some changes:

4. While $i \le n$ do
   - 4.1 $m = 0$
   - 4.2 $k = 0$
   - 4.3 Repeat until $m = 3$
     - 4.3.1 $m \leftarrow m + 1$
     - 4.3.2 $r \leftarrow$ random number 0 or 1
     - 4.3.3 If $r = 1$ then
       - 4.3.3.1 $m = 3$
       - 4.3.3.2 $counter \leftarrow counter + 1$

4.4 If $r=1$ then $counter \leftarrow counter + 1$
4.5 $i \leftarrow i+1$

These are the estimates of probabilities using an increasing number of repetitions:

1000 repetitions	→	0.935
5000 repetitions	→	0.9364
20000 repetitions	→	0.9398

So it appears that the probability that Tom and Jackie will have their dream come true is 0.93.

The technique we showed above with two examples does not have to involve only two equally likely outcomes. For example, let us assume that there are three possible outcomes with probabilities 0.2, 0.3, and 0.5. Then you could generate a number between 0 and 9 (using for example, `INT(RND*10)` in QuickBasic) and assign a range of numbers to each outcome. For example:

numbers 0 and 1 correspond to outcome 1
numbers 2, 3 and 4 correspond to outcome 2
numbers 5 to 9 correspond to outcome 3.

and if the probabilities assigned to each outcome are say 23%, 32% and 45%, then you generate random numbers between 0 and 99 (using for example, `INT(RND*100)`), and proceed with a simulation similar to the one above.

Here are a few further exercises where you can apply your simulation skills:

Exercise 1: A coin is tossed 6 consecutive times. What is that probability of a run of exactly 3 tails or 3 heads?

Exercise 2: A couple wishes to have at least a boy and a girl, and are prepared to have at most 5 children to have their dreams come true. How likely is it for the couple to have their wishes fulfilled?

Exercise 4: Sarah is working on a computer generated multiple choice test consisting of 10 questions with 4 options each, which she is allowed to attempt 4 times. Sarah has not studied for the test, so she has decided to provide a random answer to each question. What is the probability of passing (ie. get 5 correct answers) by attempting the test up to 4 times?

* * * * *

PROBLEM CORNER

PROBLEM 24.3.1

Show that 1 is the sum of all the numbers $\frac{1}{x \cdot y}$, where $1 \le x \le n$, $1 \le y \le n$, $x + y > n$ and x and y are relatively prime.

(For example, when $n = 4$, $\frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 1} + \frac{1}{4 \cdot 3} = 1$).

SOLUTION

For $n \ge 1$, denote by S_n the sum of the fractions, $\frac{1}{x \cdot y}$ where x and y are integers, $1 \le x, y \le n$, $x + y > n$ and $(x, y) = 1$, and where (x, y) denotes the greatest common divisor of x and y. We argue by induction that $S_n = 1$ for each $n \ge 1$.

Base Case: $n = 1$, $S_1 = \frac{1}{1 \cdot 1} = 1$.

Induction Step: Let $n > 1$ and assume that $S_{n-1} = 1$.

We consider the set A_n of terms in the sum S_n which are not in S_{n-1}, and the set B_n of terms in the sum S_{n-1} which are not in S_n. From the definition of these sums it follows that the terms of A_n are all of the form $\frac{1}{x \cdot y}$ with $x = n$ or $y = n$, while the terms of B_n are all of the form $\frac{1}{x \cdot y}$ with $x + y = n$. Consequently we can set up a two to one correspondence

$$\left[\frac{1}{n(n-x)}, \frac{1}{xn}\right] \leftrightarrow \left[\frac{1}{x(n-x)}\right] \qquad (1)$$

between the terms in A_n and the terms in B_n, upon establishing the result that $(n, n-x) = 1$ and $(x, n) = 1 \leftrightarrow (x, n-x) = 1$ (2)

To verify (2), assume first that $(n, n-x)=1$ and $(x,n)=1$ and let $(x, n-x)=d$. Then $d \mid x$ and $d \mid n-x$, hence $d \mid n$, and since $(x,n)=1$, $d=1$. Conversely if $(x, n-x)=1$ and $(n, n-x)=d$, then $d \mid n$ and $d \mid n-x$, hence $d \mid x$. Thus $d=1$. Similarly, if $(x, n-x)=1$ and $(x,n)=d$, then $d \mid x$ and $d \mid n$ and hence $d=1$. This verifies (2).

Now observe that

$$\frac{1}{n(n-x)}+\frac{1}{xn}=\frac{1}{x(n-x)}$$

so that from the correspondence in (1) we can conclude that $S_n - S_{n-1} = S_{n-1} - S_n$ from which we have $S_n = S_{n-1} = 1$. This completes the induction step.

PROBLEM 24.3.2 (proposed by Keith Anker, adapted from a Westpac junior mathematical competition)

Fifteen (equal sized) circular discs, each either red or green, are arranged in an equilateral triangular array with one disc in the top row, two in the second row, etc, down to five in the fifth row. Show that there are three discs of the same colour with centres at the vertices of an equilateral triangle.

SOLUTION (Keith Anker)

We show that every attempt to fill the first four rows will result in an equilateral triangle of a single colour (the triangle condition). The positions in the first four rows are labelled as shown in the diagram below where each letter represents either red (R) or green (G).

I
$J \quad K$
$L \quad M \quad N$
$P \quad Q \quad S \quad T$

Without loss of generality we can assume $I=R$. (Otherwise $I=G$ and the same argument works with the colours reversed). If the triangle condition is not satisfied by ΔIJK then at least one of J and K is G.

Case 1: $K = G$ and $J = R$. If $L = R$ we must have $M = G$ to ensure ΔJLM fails the triangle condition. Now if $N = G$ then ΔKMN satisfies the condition, and if $N = R$ then ΔILN satisfies the condition. To avoid satisfying the condition we must have $L = G$. This choice means that we need to have $S = R$ to avoid ΔKLS satisfying the condition and this in turn implies $P = G$ (for ΔJPS). Continuing we see that we must also have $Q = R$ (ΔLPQ) and $M = G$ (ΔMQS) and $N = G$ (ΔJNQ). But this will make ΔKMN all green so that we cannot have $K = G$ and $J = R$. A similar argument applied to $K = R$, $J = G$ shows this is also not possible without satisfying the triangle condition. We conclude that the only possibility is $K = J = G$.

Case 2: $K = J = G$. Then $M = R$ to avoid to triangle condition for ΔJKM.

Sub-case 2(a) $L = G$. This implies $S = R$ (ΔKLS) and then

$Q = N = G$ (ΔMQS and ΔMNS). However ΔJNQ is all green.

Sub-case 2(b) $L = R$. This implies $N = G$ (ΔILN) and

$M = R$ (ΔKMN). We must then have $Q = G$ (ΔLMQ) and ΔJNQ is then all green.

We see that all attempts result in the triangle condition being satisfied in the construction of the first four rows.

A solution was also received from Carlos Victor.

PROBLEM 24.3.3 (proposed by Julius Guest, East Bentleigh)

Solve the equation $9x^4 + 12x^3 - 3x^2 - 4x + 1 = 0$.

SOLUTION (Carlos Victor)

The equation

$$9x^4 + 12x^3 - 3x^2 - 4x + 1 = 0.$$

can be written

$$(3x^2 + 2x - 1)^2 = x^2$$

so that either

(i) $3x^2 + 2x - 1 = x$

or

(ii) $3x^2 + 2x - 1 = -x$

The quadratics solve to yield

$$x_1 = \frac{-1+\sqrt{13}}{6}, \quad x_2 = \frac{-1-\sqrt{13}}{6}, \quad x_3 = \frac{-3+\sqrt{21}}{6}, \quad x_4 = \frac{-3-\sqrt{21}}{6}$$

Solutions were also received from Julius Guest, David Halprin and J A Deakin.

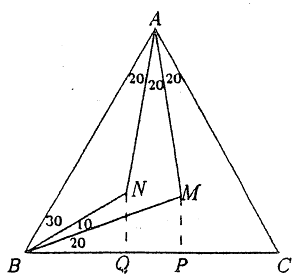
PROBLEM 24.3.4 (from Mathematics and Informatics Quarterly)

Points M and N are drawn inside an equilateral triangle ABC. Given $\angle MAB = \angle MBA = 40°$, $\angle NAB = 20°$ and $\angle NBA = 30°$, prove that MN is parallel to BC.

SOLUTION

Draw the perpendiculars MP and NQ to BC (see diagram). By the sine formula in ΔBAN:

$$\frac{BN}{\sin 20°} = \frac{a}{\sin 50°}$$



where $a = AC = CA = AB$. By the sine formula in ΔBAM:

$$\frac{BM}{\sin 40°} = \frac{a}{\sin 80°}$$

Then

$$NQ = BN \sin 30° = \frac{a \sin 20° \sin 30°}{\sin 50°}$$

and

$$MP = BM \sin 20° = \frac{a \sin 40° \sin 20°}{\sin 80°}$$

But

$$\sin 30° \sin 80° = \frac{1}{2} 2 \sin 40° \cos 40° = \sin 40° \sin 50°.$$

Hence $NQ = MP \Rightarrow MPQN$ is a rectangle $\Rightarrow MN$ is parallel to BC.

A solution using trigonometry was received from Carlos Victor.
Solutions using pure geometry were received from Julius Guest and Garnet J Greenbury.

PROBLEM 24.3.5 (from Mathematics and Informatics Quarterly)

Let x and y be real numbers of the form $\frac{m+n}{\sqrt{m^2+n^2}}$ where m and n are positive integers. Show that if $x < y$ then there is a real number z of the same form such that $x < z < y$.

SOLUTION

If $f(m,n) = \frac{m+n}{\sqrt{m^2+n^2}}$, then f is symmetric (i.e. $f(m,n) = f(n,m)$) and we can assume that $m < n$. Furthermore, notice that $f(m,n) = \frac{m+n}{\sqrt{m^2+n^2}} = \frac{1+(m/n)}{\sqrt{1+(m/n)^2}}$. If we establish that the function $g(x) = \frac{1+x}{\sqrt{1+x^2}}$ is increasing on the interval $(0,1)$, it will follow that

$$f(a,b) = \frac{1+(a/b)}{\sqrt{1+(a/b)^2}} < f(c,d) = \frac{1+(c/d)}{\sqrt{1+(c/d)^2}} \text{ if and only if } \frac{a}{b} < \frac{c}{d}.$$

Therefore, any rational number m/n between a/b and c/d will yield a solution to our problem.

To prove that $g(x) = \frac{1+x}{\sqrt{1+x^2}}$ is increasing on $(0, 1)$, we observe the following equivalences for x and y in $(0, 1)$:

$$\frac{1+x}{\sqrt{1+x^2}} < \frac{1+y}{\sqrt{1+y^2}} \Leftrightarrow \frac{(1+x)^2}{1+x^2} < \frac{(1+y)^2}{1+y^2} \Leftrightarrow \frac{x}{1+x^2} < \frac{y}{1+y^2}$$

$$\Leftrightarrow x + xy^2 - y - x^2y < 0 \Leftrightarrow (x-y)(1-xy) < 0.$$

Since $1 - xy > 0$, it must follow that $x < y$ and therefore, g is increasing on $(0, 1)$.

A solution was also received from Carlos Victor.

PROBLEMS

PROBLEM 24.5.1 (from Parabola)

Show that if m and n are positive integers then $(mn)! \geq (m!)^n (n!)^m$.

PROBLEM 24.5.2 (from Crux Mathematicorum with Mathematical Mayhem)

The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by $a_1 = 2$ and $a_{n+1} =$ the sum of the 10^{th} powers of the digits of a_n, for all $n \geq 1$. Decide whether any number can appear twice in the sequence $\{a_n\}_{n=1}^{\infty}$.

PROBLEM 24.5.3 (from Czechoslovak Olympiad 1993)

Find all natural numbers n for which $7^n - 1$ is a multiple of $6^n - 1$.

PROBLEM 24.5.4 (from Mathematical Spectrum)

There are n sheep in a field, numbered 1 to n, and some integer $m > 1$ is given such that $m^2 \le n$. It is required to separate the sheep into two groups such that (1) no sheep has number m times the number of a sheep in the same group, and (2) no sheep has number the sum of the numbers of two sheep in its group. For which values of m, n is this possible?

PROBLEM 24.5.5 (from Mathematical Spectrum)

A triangle has angles α, β and γ which are whole numbers of degrees, and $\alpha^2 + \beta^2 = \gamma^2$. Find all possibilities for α, β and γ.

* * * * *

INDEX TO VOLUME 24

* * * * *

Published by Department of Mathematics & Statistics, Monash University