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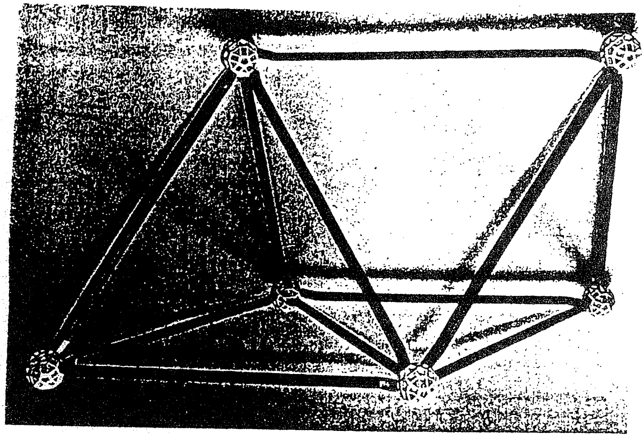

School of Mathematical Sciences – Monash University

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. Founded in 1977 by Professor G B Preston, *Function* is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* may include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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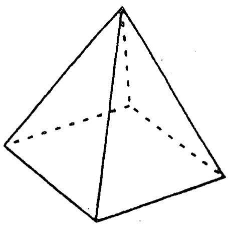
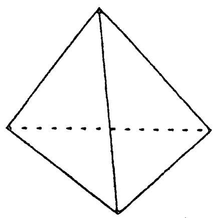
* $14 for *bona fide* secondary or tertiary students.

THE FRONT COVER

Our cover diagram for this issue shows a shape known as a "puptent". The photograph is the work of Dr Burkhard Polster, of the School of Mathematical Sciences at Monash University. Burkhard also constructed the model that forms the subject of the photo, and the same holds true for the other models whose photographs appear later in this article. The models were all made using a Zome geometry kit.

If we begin with a square pyramid, in which the sides of the base are all of length 1, and the sloping sides are also of length 1, and adjoin to it a tetrahedron (triangular pyramid) all of whose sides are also of length 1, then we have a "puptent". This word is not in any dictionary we have consulted, but it is to be found in an account by Edward J Barbeau in his delightful book *Mathematical Fallacies, Flaws and Flimflam.* Barbeau tells an interesting story. To follow it, look at the figure below.





To left is the square pyramid, to right the tetrahedron. If we paste the tetrahedron to the square pyramid in such a way that one of the equilateral triangles comprising the tetrahedron coincides exactly with one of the triangular faces of the square pyramid, then how many faces will the resulting polyhedron possess?

This question was posed in a scholastic aptitude test set in 1980 by the Educational Testing Service (New Jersey, USA). The test was multiple-choice in nature, and five answers were suggested: 5, 6, 7, 8, 9. Reasoning that there were 5 + 4 = 9 faces before the paste and that the paste removed 2 of these, the examiners counted the answer 7 as correct.

There is a certain logic to this argument, but it is wrong (although it would be true if some of the data were appropriately modified). The point is that the two triangles whose edges meet on the front side of the paste become coplanar, as do the two at the back. We thus have only 5 faces in all! In order to prove this assertion, imagine two identical square pyramids side by side and a join inserted between their two apices, as in the top photo opposite. Clearly all the edges of the double pyramid are of length 1, and it is very easy to show that the join between the apices is also of length 1. So the space between the two square pyramids is exactly filled by a tetrahedron whose edges are all of length 1. But very clearly, the two front sides of the two square pyramids are coplanar (as are those at the back) and so the faces of this intermediate tetrahedron must lie respectively in these two planes also.

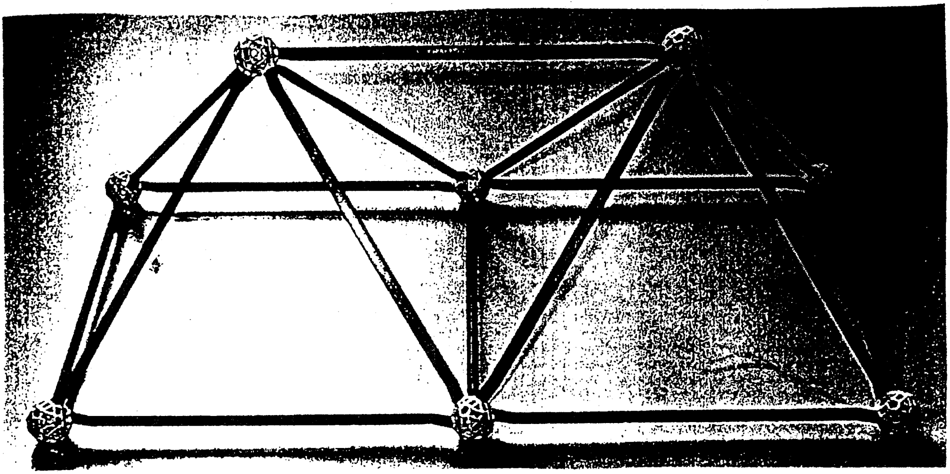
Barbeau imagines the first square pyramid as a tent, but supposes one sloping face replaced by a canopied entrance, formed by supporting a strut parallel to the horizontal base by means of two sloping bearers. This is what he calls the puptent (and presumably the reason for the name). However, there is much more that can be said. The lower photo opposite shows the result of joining two puptents together at the "base". The two square pyramids now form a regular octahedron, and attached to each of its two of its adjacent faces is a regular tetrahedron. This new element is rigid, whereas the puptent itself is not (because a square can deform without altering the lengths of its sides).

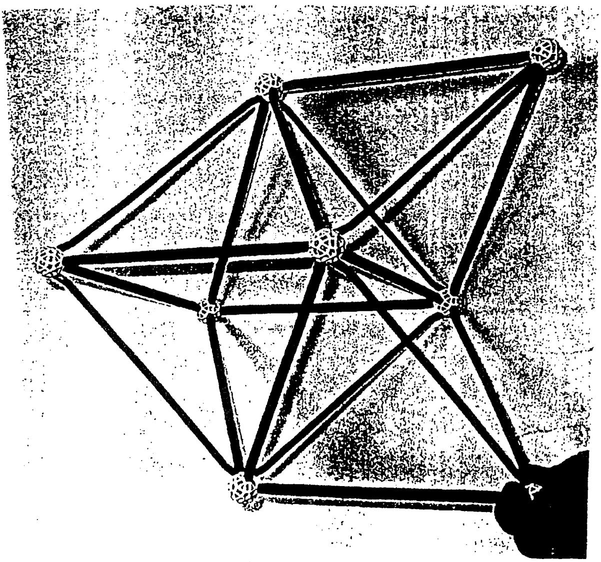
(The late Sir Louis Matheson, Monash University's first Vice-Chancellor and a civil engineer by profession, was fond of saying that the whole of civil engineering could be encapsulated in four principles, of which the second was: "Buildings are made of triangles"[1]. An octahedron is made of triangles; a square pyramid is not.)

The rigidity of the double puptent is one of its properties. Another is the fact that such shapes can pack together to form a space-filling structure. Cubes are regular polyhedra, and they can pack together, to fill space with a "cubic lattice". They are the only regular polyhedra that can. If we try to pack together any of the other regular polyhedra (tetrahedra, octahedra, dodecahedra or icosahedra) we leave spaces, or interstices, between them. However if we use octahedra, we can so arrange them that these interstices can be exactly filled by regular tetrahedra. The overall count is that for every octahedron, there are two tetrahedra, so that it is the double puptent that is the basic building block.

This is the basis of one of M C Escher's famous prints. Copyright considerations prevent our reproducing it here, but you should have little

[1] The others were: 1. Water flows downhill, 3. Use beams on their edges not their flats, 4. You can't push a string.





trouble finding it. The particular print in question is called *Flatworms*, and it is reproduced in (for example) Bruno Ernst's *The Magic Mirror of M C Escher*, published by Ballantyne Books (1976). This particular reproduction is accompanied by a nice descriptive essay (see pp 96, 97).

The space-filling property was what fascinated Escher, who spent much of his energy on the depiction of tesselations of the plane (i.e. subdivision of the plane into regular shapes). This is a three-dimensional counterpart of these others, which are probably better known.

The combination of the space-filling property and the rigidity commended itself to another lateral thinker, R Buckminster Fuller. By joining together enough double puptents, we can build a strut that is made entirely of rods of equal length, and such that the volumes enclosed between them fill the entire volume of the strut, which is theoretically very strong, being made up entirely of triangles.

Fuller called his strut the *Octet truss*, and you can see pictures of it in many books on his work. See, for example, *The Dymaxion World of Buckminster Fuller*, by R Buckminster Fuller and Robert Marks (Anchor Books, 1960), pp 170-175. Again, copyright considerations prevent our reproducing the illustrations.

Fuller attributed the strength of the Octet truss to the fact that each vertex was the join of 12 separate rods. None of our photographs show enough of the structure to demonstrate this property. However, consider the central point on the "base" of the double tent in the top illustration on the previous page. There, three "horizontal" edges meet, and two sloping ones. If two more square pyramids were adjoined along the two forward "bottom" edges, we would add one more "horizontal" and two more sloping edges. If now we adjoined a further 4 square pyramids, reflections in the "horizontal" plane of the "base", then there would be no more "horizontal" edges, but a further 4 sloping edges. This gives a total of 12.

A further feature of the Octet truss is that, if the ends of the constituent rods are carefully and specifically shaped, then the structure holds itself together, without the need for joining elements at the vertices. However, our incomplete examples do not have this character, and the vertices are prominently marked by structural joins.

Perhaps the octet truss would be more popular were it not for the fact that (unlike the cubic lattice) it is fiendishly difficult to draw!

THACKERAY AND THE BELGIAN PROGRESSION

S N Ethier, University of Utah

[The following paper is an edited version of a technical analysis published in *The Mathematical Scientist, Vol* **24** (1999), pp 1-23. It appears here with the kind permission of Professor Ethier and the publishers of *The Mathematical Scientist.*]

In his 1850 Christmas book, *The Kickleburys on the Rhine*, the celebrated English novelist William Makepeace Thackeray (1811-1863) described in detail an incident involving 'a company of adventurers from Belgium' who 'called themselves in their pride the Contrebanque de Noirbourg' and who 'boldly challenged the bank' of the Noirbourg [Homburg] casino using 'an infallible system for playing [the card-game] *rouge et noir*'. According to the story, after consistently winning day after day, the Contrebanquists lost back their accumulated fortune in only about three hours. The narrative, characterized by one critic as 'one of Thackeray's best excursions into the mock-heroic', is evidently regarded by scholars of English literature as lacking a factual basis.

However, historian Russell T Barnhart has found evidence in an obscure nineteenth-century French book on gambling to suggest that it was based on an actual event. This work reported that Édouard Suau de Varennes (a minor French novelist), together with a syndicate of ten financial backers, used his 'progression ascendente' to win 440,000 francs at the Homburg casino over a period of three months, only to lose it all back in just three deals of *rouge et noir*. There are some striking similarities between the two accounts, making it plausible that Thackeray's fictional version was based on the same real incident. This conjecture was confirmed by Barnhart's recent discovery in the archives of the French Ministry of Foreign Affairs of a document titled 'Rapport sur Suau de Varennes' and dated 21 August 1848. The anonymous author of this document stated that in 1844 Suau formed the 'Société de Contre-banque de Hombourg' with the goal of breaking the bank at the Homburg casino. Note that this report predates Thackeray's book by two years.

What was this 'infallible gambling system'? Although none of the authors described it, it is believed to have been a system known as *le* *montant belge*, or the Belgian progression. According to Barnhart, this was confirmed to him in 1961 by General Pierre Polovtsoff, former President of the International Sporting Club in Monte Carlo and author of the 1937 book *Monte Carlo Casino.*

The Belgian progression is applicable to any game of repeated trials with 1:1 payoffs. The system depends on a positive integer parameter K referred to by Barnhart as the key number. In the original formulation $K = 5$. The gambler's status at any time is completely specified by a (finite) sequence of numbers on a score sheet, the sum of whose terms indicates how far behind the gambler is at that time, and by the bet size at the next trial, both of which are updated after each trial. The rules (which will become clearer with the example that follows) are:

- The sequence is initially empty and the initial bet is one unit.
- After a loss the amount just lost is added to the sequence as a single term on the right. The bet size remains the same unless the amount just lost appears K times in the new sequence; in the latter case the bet size is increased by one unit.
- After a win sufficiently many terms on the left of the sequence to sum to the amount just won are deleted. If this cannot be done exactly, the smallest number of terms on the left of the sequence that sum to an amount greater than the amount just won are deleted, and the difference is added to the sequence as a single term on the left. The bet size remains the same unless the amount just won exceeds the sum of the terms in the new sequence by more than 1; in the latter case the bet size is reduced to an amount sufficient to exceed the sum of the terms in the new sequence by exactly 1.
- Once the gambler achieves a net profit of one unit, he/she may quit or start anew.

We regard the one-unit net profit as the gambler's goal. The rule on bet reduction prevents him/her from overshooting that goal. Table 1 opposite illustrates these rules by means of an example.

The gambler outlays a stake of one unit at the outset, in Play No. 1, in accordance with the first of the above rules. As it happens, the gambler loses on this play. So a figure of 1 is entered into the score sheet (on the right) and a further bet of one unit is ventured. This is in accordance with the second of the rules. Plays 2, 3 and 4 are all losses for the gambler, so on each a figure of 1 is entered on the right in the relevant row of the score sheet and a further bet of 1 unit is wagered. After Play 4, the gambler has got a row of four 1s on the score sheet, and has achieved a total loss of 4 units.

Trial no.	Bet size	Outcome	Sequence on Score Sheet	Loss so far	(X, Y, Z)
					(1, 1, 0)
1	1	loss	1	1	(2, 1, 1)
2	1	loss	1 1	2	(3, 1, 2)
3	1	loss	1 1 1	3	(4, 1, 3)
4	1	loss	1 1 1 1	4	(5, 1, 4)
5	1	win	1 1 1	3	(4, 1, 3)
6	1	loss	1 1 1 1	4	(5, 1, 4)
7	1	loss	1 1 1 1 1	5	(6, 2, 0)
8	2	loss	1 1 1 1 1 2	7	(8, 2, 1)
9	2	loss	1 1 1 1 1 2 2	9	(10, 2, 2)
10	2	win	1 1 1 2 2	7	(8, 2, 2)
11	2	loss	1 1 1 2 2 2	9	(10, 2, 3)
12	2	loss	1 1 1 2 2 2 2	11	(12, 2, 4)
13	2	loss	1 1 1 2 2 2 2 2	13	(14, 3, 0)
14	3	loss	1 1 1 2 2 2 2 2 3	16	(17, 3, 1)
15	3	win	2 2 2 2 2 3	13	(14, 3, 1)
16	3	win	1 2 2 2 3	10	(11, 3, 1)
17	3	win	2 2 3	7	(8, 3, 1)
18	3	win	1 3	4	(5, 3, 1)
19	3	win	1	1	(2, 2, 0)
20	2	loss	1 2	3	(4, 2, 1)
21	2	win	1	1	(2, 2, 0)
22	2	win	-1	-1	(0, 0, 0)

Table 1

On Play 5, the gambler wins. The amount of the win is 1 unit, and so, in accordance with the third of the rules, the leftmost 1 is deleted from the score sheet. The play continues as before until the loss on Play 7. An amount of 1 unit has now been lost 5 times, that is to say *K* times. So now, in accordance with the second of the rules, the bet size is increased to 2. 2 units are bet on the losing Plays 8 and 9, and on the winning Play 10. With this win (of 2 units), 2 of the 1's are deleted from the left of the score sheet, in accordance with the third rule. On each of Plays 11, 12 and 13, the gambler loses 2 units, and so reaches a point where 5 2's have been entered onto the score sheet.

Thus the bet size is increased to 3 units for Play 14 which is also a loss. However the gambler wins (an amount of 3 units) on Play 15, and so three of the 1's are deleted from the left of the score sheet. Play 16 wins a further 3 units, and so a total of 1 + 2 = 3 is deleted from the left of the score sheet. This removes one of the 2's and alters the other to a 1. This is in line with the 'fine print' in the third rule. Play 17 results in a further win and so the 1 and the 2 are both deleted from the score sheet. The further win on Play 18 deletes one of the 2's on the left of the score sheet and replaces another by a 1.

A further win on Play 19 then deletes the 1 on the left of the score sheet and sees the 3 reduced to a 1. At this point, the gambler is precisely 1 unit down, and can end up ahead by wagering 2 units and winning. Thus, in accordance with the 'fine print' of the third rule, the bet size is reduced to 2 units for Play 20, which however results in a loss. 2 more units are bet on Play 21, and this results in a win, so once more a winning bet of 2 units will put the gambler ahead.

This is what happens on Play 22, which leaves the gambler ahead by precisely 1 unit and so, in accordance with the fourth of the rules, this sequence of play ends.

The fourth column gives the gambler's overall loss as the play proceeds. The final column refers to the analysis below. The figure in the fourth column is merely the total of the numbers on the relevant row of the score sheet, i.e. the third column. Once the total becomes negative, the sequence ends. Thus in theory, a gambler can continue each sequence until this point is reached, pocket the unit which has been won, begin again and play again until a further unit is won, and so on. On first appearances, the gambler should always come out ahead.

Look again at Table 1. Out of 22 Plays, the gambler has actually lost 13, and won only 9. The winning streak from (essentially) Play 15 on is not as pronounced as the losing streak that all but lasted from Play 1 till Play 14. What the system does is to adjust the bet size to compensate for the losing streak and to capitalize on the winning one.

However, this rosy picture which allows the gambler always to win in the end, is over-simplistic. Several real-life considerations intervene to complicate it.

In the first instance, we must consider the probability of winning on any particular play. If the game were perfectly fair, then this probability would be 1/2 . However, games played in casinos are never perfectly fair; the casinos expect to return a profit and so load the probabilities in their own favor. Thus consider the probability of the gambler's winning on any particular play to be p, where $0 < p < 1$, as a general rule, but realistically $p < 1/2$.

The second complication is that the amounts that can be wagered and the amounts of the payouts are both bounded. Neither can increase without limit. Thus a point may be reached where the system calls for a bet beyond the gambler's capacity to pay. Alternatively the casino may limit the size of the allowable bets, and this barrier may prevent the gambler from following the system. Given a sufficient capital and a fairly generous policy on the part of the casino, the gambler has always a small probability of taking the casino for all that it has. This is what provides the drama of Thackeray's novel.

The Belgian progression, like many other systems, yields a small win with high probability or a large loss with low probability. That the balance tilts in favor of the casino is a consequence of the 20th-century mathematical result known as The Optional Stopping Theorem. It says that, regardless of the betting system chosen, if $p < 1/2$, the gambler's expected profit cannot be positive. Expected profit can be thought of as long-term average profit when the session is repeated indefinitely. More precisely, it is the weighted average of all possible profit values, weighted by the probabilities with which they occur. For example, if the possible profit values are 1 with probability 999/1000 and –1000 with probability 1/1000, then the expected profit is

$$(1)(999/1000) + (-1000)(1/1000) = -1/1000.$$

Thus, without even knowing the details of the Belgian progression, we know that it cannot possibly be successful on a consistent basis. Of course, in the short run anything can happen.

A detailed analysis of the Belgian progression is provided in the full paper from which this simplified version is taken. There is one important idea in the paper that is not hard to understand. Instead of keeping track of the sequence on the gambler's score sheet and the bet size at the next trial, it suffices to keep track of just three numbers, namely

X = the number of units the gambler needs to reach his/her goal following the most recent trial,

Y = the gambler's bet size at the next trial, and

Z = the number of times Y appears in the gambler's sequence (on his/her score sheet) following the most recent trial.

Before the first trial, $(X, Y, Z) = (1, 1, 0)$. After each trial, and depending on whether a win or loss occurred, (X, Y, Z) can be updated using only the most recent value of (X, Y, Z). (This makes (X, Y, Z) what is known as a Markov chain.) See the example in Table 1 to get a better understanding of how this works.

This simple idea is the basis for the following computer simulation of the Belgian progression, written in True BASIC.

```
LET p = .49
FOR s = 1 to 1000
   RANDOMIZE
   LET t = 0
   LET x = 1
   LET y = 1
   LET z = 0
   PRINT " "," "," ",x,y,z
   DO while x > 0
     LET t = t + 1
     LET xx = x
     LET yy = y
     IF rnd < p then
       LET x = xx-yy
       LET y = min(yy,xx-yy)
       IF xx-yy*z <= yy then LET z = z-1
       IF z = -1 then LET z = 0
       PRINT s,"W",x,y,z
     ELSE
```

```
        LET x = xx+yy
        IF z = 4 then LET y = yy+1
        LET z = z+1
        IF z = 5 then LET z=0
        PRINT s,"L",x,y,z
      END IF
    LOOP
    PRINT t
    PRINT
  NEXT s
  END
```

Try running this program, or a suitable modification of it, to see the Belgian progression in action.

As may be conjectured from the simulation, and as is proved in the technical paper, if $p < 1/2$, then there is positive probability that the one-unit goal will never be achieved, and this result holds even in the case where no house limit is imposed (and the gambler has unlimited wealth)!

Notes

1. Thackeray's account was written under the pseudonym M A Titmarsh. That it is based on a real event is made even more plausible by other thinly veiled bits of evidence: e.g., the Homburg casino was run by the brothers François and Louis Blanc [White]; their fictional counterparts are called Lenoir [Black].

2. The Belgian progression bears some resemblance to another system, known as 'Oscar's System', the subject of an earlier analysis by the author.

3. The Optional Stopping Theorem is covered in many books. Perhaps the best account to begin with is that found on p 261 of *A First Course in Stochastic Processes, 2nd Edition*, Ed S Karlin and H M Taylor (Academic Press, 1975).

4. The author thanks Russell T Barnhart for generously making his research available prior to its publication.

THE BOUNDS OF THE UNIVERSE

A BRIEF HISTORY OF ASTRONOMY: PART 2

K C Westfold

[This is the third in our series of astronomical expositions by the late Professor Westfold. It concludes the second, published in our previous issue. Eds]

The modern science of stellar Astronomy dates from the work of William Herschel (1738-1822). He was a professional musician with a keen amateur interest in Astronomy, going so far as to construct his own telescopes. With these he tried unsuccessfully to measure stellar parallax; however these attempts led him to the study of double stars, and more generally the nature of the various objects in the sky. He determined on a systematic survey of the sky, noting all the remarkable features.

In all he made four such surveys with Newtonian telescopes of increasing power. In the course of his second survey he discovered what at first he thought was a comet, but which later turned out to be the planet Uranus. For this discovery he was granted £200 pa by King George III, and the title of Astronomer Royal. This enabled him to give up his musical career and devote his whole time to Astronomy.

He pioneered the methods of statistical Astronomy; by his system of gauging – counting the number of stars in his telescopic field for each line of sight – he was able to estimate the number of stars in the Milky Way. He thus found that the stars were concentrated about a comparatively thin disc in the plane of the Milky Way. As this appeared to divide the sky into two halves he deduced that the Sun must be in this disc. He also believed it to be near the centre because there was little difference in brightness around the Milky Way.

He catalogued double stars, variable stars and nebulae. As the power of his telescopes increased he was able to resolve some of the nebulae into discrete stars, although others remained unresolved. This work led him to establish a classification of celestial objects into star clusters, nebulae (bright and dark) and external island universes like our own galaxy. This last class consisted of the apparently smaller nebulae which were more abundant away from the plane of the Milky Way.

He ordered the stars in his catalogues in terms of their relative brightness (later to be expressed as "magnitudes"). Assuming an average uniform brightness and no loss of light in traversing space, he had a statistical criterion for the relative distances of the stars he counted. We know now that neither assumption is correct and that the actual law of absorption of light in space is one of the greatest difficulties in Cosmology.

Since Herschel's time there have been further discoveries. We should note first the spiral form of the nebula M51. This is the most common form of the external galaxies. Then there has been the replacement of the human observer by the photographic plate. This has made possible many great developments in modern Astronomy. With long exposure times, objects invisible to the naked eye reveal themselves, and the astronomer can examine the plate intensively without time constraints! The first such use of the photographic plate occurred in about 1839-1840, and it was carried out by the photographic pioneer Daguerre.

The application of spectroscopy to the stars was pioneered by Secchi (1818-1878) and Huggins (1824-1910), although the Sun's spectrum had previously been studied. Today the spectroscope ranks second (only behind the telescope) in its importance to Astronomy. Secchi and Huggins found the lines of hydrogen, iron, sodium, calcium and other elements in their stellar spectra. This showed that these terrestrial elements were also present in the stars. Secchi classified the various stars into different spectral types and so began the work that has culminated in the Harvard system of classification used today.

Huggins also applied Doppler's principle, which says that the light from a moving source will show increased or decreased wavelength, according as the source is moving away from us or towards us. He thus determined the velocities of various celestial objects. The Doppler technique determines the component of velocity along the line of sight. Before Huggins' day only the component across the line of sight could be determined (in some cases).

In the twentieth century, improved photographic techniques were used to make more accurate observations of the positions, motions and types of stars. Statistical analysis of such data was advanced by Kapteyn, through his plan of studying only selected typical areas of sky. The view he formed of our galactic system confirmed Herschel's general idea of a disc-like distribution. It was not until a new criterion was found for determining distances had been found that it became possible to deduce that the Sun does not occupy a preferred position at the centre of the galaxy. Just as the Earth was dethroned from the position at the centre of the Solar System, so in turn was the Solar System itself dethroned from a central position in our galaxy!

The somewhat egocentric view that had placed us at the centre of the Universe was seen as unlikely on probabilistic grounds. When it was further realized that the concentration of obscuring dust in the plane of the Milky Way prevented light from more than a few thousand light-years from reaching us, this new position was further confirmed. (The total diameter of our galaxy was at this time estimated at about 50,000 light-years.) The presence of this dust vitiates the second of Herschel's two assumptions (see above) and explains why the Milky Way seems to be more or less uniformly bright all around.

In 1912, Miss Leavitt of Harvard, studying a class of star known as the Cepheid variables, discovered a relation between their absolute luminosities and the periods of their fluctuations in brightness. Thus observations of their periods could be used to determine their absolute luminosity. The absolute luminosity is greater than the apparent luminosity (what we actually observe) because of the effects of the inverse square law, so that from the apparent and the absolute luminosities we are able to deduce the distance.

The older method of estimating distances had relied on parallax and was only applicable to those stars sufficiently near to us to allow its use. However, now that a more powerful method was available, Shapley was able to show that the Sun was situated well away from the centre of our galaxy, near the edge of the system. He estimated that the centre of our galaxy was some 60,000 light-years away. Later work has however considerably reduced this figure.

There was no yardstick with which to measure the distances to other galaxies until 1924, when Hubble discovered Cepheid variables in nearby galaxies and so was able to estimate their distances, distances of the order of 1,000,000 light-years.

The flattened form of our galaxy established by the researches I have just described suggests that it is in a state of rotation. Spectroscopic measurements of the Doppler shifts of nearby galaxies with flattened, spiral structures established that these indeed rotate. This led to the speculation that our galaxy also had spiral arms. The general theory of galactic rotation was advanced by B Lindblad of Stockholm in 1926, and subsequent research by J H Oort and others has shown that our galaxy *is* in a state of differential rotation, decreasing in angular velocity outwards from the centre.

The linear velocity of the stars in the vicinity of the Sun is about 300 km/sec and the distance from the centre about 25,000 light-years. This corresponds to a period of rotation of about 2×10^8 years. The mass of the galaxy is about 2×10^{11} solar masses, of which about half is probably gas and dust. When we consider that we can see only about one-tenth of the way to the galactic centre, the deduction of these results is seen as a remarkable achievement.

Radio Astronomy has an advantage over visual observation in this respect for radio waves are not affected by the dust clouds in the galactic plane. This means that, using radio observations, we can effectively "see" to the edge of our galaxy. This has enabled important inferences to be made concerning the galactic structure, including good evidence for the existence of spiral arms.

The work of Hubble and others on external galaxies has shown that our galaxy is about average in size and mass. Hubble classified galaxies into three types: elliptical, spiral and barred-spiral. From his spectroscopic measurements and distance estimates, he made the remarkable discovery that all the external galaxies are receding from us with their velocities of recession being proportional to their distances from us. The furthest of them are receding at velocities close to that of light.

Such considerations have led to the concept of "the expanding universe".

Thus, as Astronomy developed, its horizons extended. At first, it was concerned with the Sun, Moon and nearer planets. The most distant planet is Pluto, and light from Pluto takes about $5\frac{1}{2}$ hours to reach us. The nearest star is $4\frac{1}{2}$ light-years away, and the diameter of our galaxy is about 100,000 light-years. The nearest spiral galaxy outside our own is the "Great Nebula in Andromeda", which is about one million light-years away, so that the universe now studied by astronomers is orders of magnitude larger than even this awesome figure!

[Professor Westfold's article was written over twenty years ago. Since that time, there has been yet another advance with the development of orbiting telescopes and "space-based Astronomy". With the advent of the Hubble telescope (named after the astronomer) and its like, we can now look even further and even more clearly into the remote corners of the Universe. Eds]

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HISTORY OF MATHEMATICS

Omar Khayyám: One Person or Two?

In my previous column I detailed the life of James Cockle, who achieved distinction both as a mathematician and as a jurist. This column will treat a similar case. The name of Omar Khayyám is best known to the general public as that of a poet who lived about 1000 years ago in what was known till quite recently as Persia (now Iran). Mathematicians know the name as that of a mathematician and astronomer. Most accounts of his life take it for granted that these two persons are one and the same. Thus the authoritative *Dictionary of Scientific Biography* describes Omar (as I shall call him) as contributing to the Mathematics and the Astronomy of his day and also as the writer of works of Philosophy and of Poetry. The same can be said of the life posted on the web at

http://www-history.mcs.st-and.ac.uk/~history/Mathematicians/Khayyam.html

However, this identification has recently been called into question.

Let us proceed by speaking first of the poet, whose work is what the general public is likely to associate with the name. Omar the poet is associated with a collection of short (4-line) poems collectively called the *Rubái'yát*. This word merely means "quatrains", i.e. 4-line verses. Quite how many of the quatrains that have been attributed to Omar are actually by him is unclear; some are almost certainly the work of others. Most readers will know the *Rubâi'yât* through one or other of four different versions put out in English by the nineteenth century poet Edward Fitzgerald, who produced a free English rendering, and furthermore so ordered the quatrains that they coalesced into a coherent whole.

Perhaps the best-known quatrain is the first, which, in Fitzgerald's first (1859) edition, goes:

> Awake! for Morning in the Bowl of Night
> Has flung the Stone that puts the Stars to Flight:
>   And Lo! the Hunter of the East has caught
> The Sultan's Turret in a Noose of Light.

This quatrain was altered in the fourth edition, issued 20 years later, but it is the earlier version that is most familiar. However, it is the fourth edition that has provided the text we recall today of all the others, for example:

> A Book of Verses underneath the Bough,
> A Jug of Wine, a Loaf of Bread — and Thou

and

> The Moving Finger writes; and, having writ
> Moves on: nor all your Piety nor Wit
>   Shall lure it back to cancel half a Line,
> Nor all your Tears wash out a Word of it.

and many many more. As *The Oxford Companion to English Literature* (6th, Drabble, Edition) remarks, the Fitzgerald translation provides some of "the most frequently quoted lines in English poetry."

Omar was a Muslim, although hardly a devout one. Many of the quatrains express a tough agnosticism, and the very many devoted to the subject of wine (forbidden to Muslims) not only celebrate the product of the vine, but even seem to have been written by a man with a serious drinking problem!

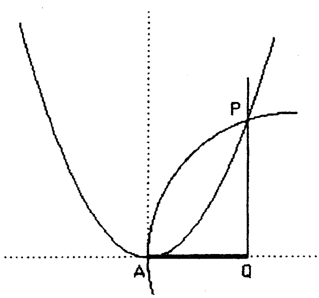
Omar the mathematician wrote a number of works on Mathematics and Music (then seen as a branch of Mathematics) before he reached his 25th birthday. He had been born in 1048 in Nishapur (Persia), but in

1070 he travelled to Samarkand in what is now Uzbekistan. There he worked under the patronage of a prominent citizen, Abu Tahir, and produced his most famous book *Treatise on Demonstration Problems of Algebra*. He also sought to prove Euclid's parallel postulate.

He later (and back in Persia) became active in the reform of the calendar. Along with seven collaborators, he produced a reform to the old (Julian) calendar, along the lines of the later Western Gregorian calendar, but in fact even more accurate.

As a mathematician, he investigated problems of Arithmetic, Algebra, Astronomy and Geometry, but perhaps he is most remembered today for his pioneering studies of cubic equations. He was aware that quadratic equations could be solved by means of ruler and compass constructions, and stated (correctly, but without proof) that this was not possible with cubics. He realised that cubic equations could possess more than a single solution, but did not consider the possibility of three such solutions. (He did not anticipate the possibility of complex solutions.) Cubics, he said, again correctly, required conic sections for their graphical solution.

Here is how Omar solved the cubic $x^3 + a^2x = b$.



First draw the parabola $x^2 = ay$. Label the vertex of the parabola A. Draw the tangent to the parabola at A. On this tangent draw a semi-circle AC of diameter b/a^2. The parabola and the semi-circle intersect at A and also at another point P. From P drop a perpendicular PQ to AC. Then AQ has a length satisfying the cubic.

You could check the correctness of this solution as an exercise.

As I said at the outset, it is widely accepted that the poet and the mathematician are one and the same person. However, this has recently been queried by Roshdi Rashed, an eminent scholar of Islamic Mathematics, whom we met in my column for April this year. Rashed, while explicitly leaving the question open, nevertheless clearly favours a "two Omars" theory.

His reasons for this are set out in a book *Omar Khayyam the Mathematician*, which he co-authored with B Vahabzadeh, and which appeared in print about two years ago. I have not been able to see this work and so I rely on discussions of its contents that I received as emails from others. Luis Puig of the University of València summarises Rashed's reasons by listing various early authors who discuss Omar's Mathematics without mention of the *Rubái'yát*, and others who discuss the poetry, but not the Mathematics. He also pointed to alleged inconsistencies between the two outlooks, as expressed on religious matters.

However, Jeff Oaks of the University of Indianapolis argues that the two *are* identified by the 13th century historian and biographer al-Qifti (whose testimony is dismissed by Rashed) and that some degree of caution was advisable when discussing religion, as these were dangerous times! He also passes on the opinions of J P Hogendijk, another eminent authority. Hogendijk mentions other Islamic scholars who also combined poetry with Mathematics, and also pointed out that some of the quatrains have clear astronomical overtones.

To my mind, one of these is very strong evidence for the "one Omar" theory. In Fitzgerald's translation (4th edition), Quatrain 57 reads:

> Ah, but my Computations, People say,
> Reduced the Year to better reckoning? — Nay,
> 'Twas only striking from the Calendar
> Unborn Tomorrow, and dead Yesterday.

This would seem to refer directly to Omar's work on calendric reform.

So perhaps the older opinion still rules!

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"I will sette as I doe often in woorke use, a paire of paralleles, or [twin] lines of one lengthe, thus = , bicause noe 2. thynges, can be moare equalle."

Robert Recorde, 1557

COMPUTERS AND COMPUTING

Solving Non-Linear Equations: Part1, Overview

J C Lattanzio, Monash University

In this paper and those to follow, I shall be concerned with the numerical solution of non-linear equations in a single unknown. If the unknown is denoted by x, then a *linear* equation is one of the form $ax+b=c$, where a, b, c are given constants. Any other form involving x is *non-linear*. For example, the quadratic equation

$$ax^2+bx+c=0$$

is non-linear, as are all such equations involving polynomials in the unknown.

The roots of the quadratic are determined by means of a well-known formula, but for most non-linear equations, no such formula exists. I remember when I was at school, I assumed that as I was trained to higher levels, I would learn the exact solutions of more complicated equations. It was quite a shock to learn that for most equations there is no exact solution. In particular, it is known that no such formula is possible where we seek the zeroes of a polynomial of degree greater than 4.

In such a situation, we need to resort to computational methods. In this paper I will give general descriptions of the principal such methods, and in the later papers in this sequence, I will go into the details of each.

We may usefully classify some of the various methods available as

(a) graphical methods
(b) bracketing methods
(c) fixed-point iteration
(d) the Newton-Raphson method
(e) the secant method.

The following papers in this sequence will describe each of these in turn. Here I begin with some general descriptions and remarks. (If you have a computer, you should try writing programs to implement these methods. You can use your favourite language. But even with a calculator you should be able to implement these methods very nicely.)

(a) **Graphical Methods:** If an equation has the form $f(x)=0$, then we can use a graph of $f(x)$ to determine such useful information as the number of roots and their approximate locations. A careful sketch-graph is all that is really required. In today's world the aim of a graphical approach is to guide the choice of what other method to use for the more exact determination of the roots, and to warn us of any pitfalls we may be likely to encounter.

(b) **Bracketing Methods:** The basis of all bracketing methods is Bolzano's Theorem from Calculus, which states that *if* $f(x)$ *is a continuous function for all x such that* $a \leq x \leq b$ *and* $f(a)<0<f(b)$ *or* $f(b)<0<f(a)$ *then there is some number x satisfying* $a \leq x \leq b$ *and* $f(x)=0$. (This may sound very complicated, but it really says simply that if you start with a positive function value somewhere and move continuously to a negative function value, then the function must be zero somewhere in between. Not very difficult really!)

We then seek to use an algorithm progressively to reduce the size of the interval containing x.

(c) **Fixed-Point Iteration:** Fixed-point methods involve the rewriting of the original equation in one or another special form, such as $x=g(x)$, and then applying an algorithm repetitively to improve on some initial estimate x_0 of the value of the required root. Fixed-point methods belong to a class called "open methods" because, unlike the bracketing methods, they do not rely on restricting the search for a root to the interior of a closed interval $a \leq x \leq b$.

(d) **The Newton-Raphson Method:** The Newton-Raphson method is another "open method". It is probably the most popular of all methods, and it works best when the non-linear function $f(x)$ is easily differentiable. It too depends on the progressive refinement of an initial estimate x_0.

(e) **The Secant Method:** This is a modification of the Newton-Raphson method suitable for cases in which $f(x)$ is not easily differentiable. It is yet another "open method" but it proceeds by refining a *pair* of approximations at each step of an iterative process.

Subsequent articles in this sequence will discuss these different methods in turn. Here I confine myself to some general remarks.

The first concerns the convergence of any iterative scheme. As we are most unlikely to arrive at the *exact* value of the root we seek, we need some criterion to tell us when to stop the computation. Four obvious criteria are:

(1) stop after N iterations
(2) stop when the difference between estimates is less than some small quantity
(3) stop when the size of $f(x)$ is sufficiently small
(4) stop when the relative difference between estimates is less than some small quantity.

Criterion (1) by itself is not very good. However, a maximum number of iterations should always be specified, to prevent the computation from proceeding indefinitely. Condition (3) is also poor, because it is possible to have functions $f(x)$ that maintain small values over large domains of x. However it has considerable value as a check. Of the others, Criterion (4) is almost always preferable to Criterion (2) , since it estimates the root to within some specified tolerance. However if we only needed to know the root to n decimal places, regardless of its actual value, then we would use the simpler Criterion (2).

The other general point to make is that of all the methods, the one that converges most rapidly is Newton-Raphson. This is therefore to be preferred provided that $f(x)$ is differentiable and that the evaluation of its derivative is not computationally prohibitive. After this, the secant method would be preferred. This shares the property of relatively rapid convergence, but does not require an explicit formula for the derivative.

However if these methods, for one reason or another, are not available (e.g. because they fail to converge) then a fixed-point iteration would be the next choice. Again, however, convergence is not guaranteed. Finally, we would resort to a bracketing method, because these methods *are* guaranteed to converge, albeit slowly.

NEWS ITEMS

The Poincaré Conjecture

In *Function's* issue for April 2001, we reported on the Clay Challenge problems issued by the Clay Mathematics Institute to mark the dawn of the new millenium. These are seven outstanding problems in the Mathematics of today, the solution to any one of which would represent a considerable mathematical advance. The third of these is the so-called Poincaré Conjecture. Recently there was a flurry of excitement over the claim that this had now been proved.

It is a little difficult even to state the conjecture precisely without recourse to advanced and difficult mathematical concepts. In this it differs from (say) Fermat's Last Theorem, whose statement, at least, could be expressed in relatively simple terms. So what follows is not intended to provide a full statement of the conjecture, but rather a simplified description that will indicate the "flavour" of the Mathematics involved.

Begin with the description posted on the Clay website

wysiwyg://5/http://www.claymath.org/prizeproblems/poincare.htm

This begins with a popular exposition that runs as follows.

"If we stretch a rubber band around the surface of an apple, then we can shrink it down to a point by moving it slowly, without tearing it and without allowing it to leave the surface. On the other hand, if we imagine that the same rubber band has somehow been stretched in the appropriate direction around a doughnut, then there is no way of shrinking it to a point without breaking either the rubber band or the doughnut. We say that the surface of the apple is "simply connected", but that the surface of the doughnut is not. Poincaré, almost a hundred years ago, knew that the two dimensional sphere is essentially characterized by this property of simple connectivity, and asked the corresponding question for the three dimensional sphere (the set of points in four dimensional space at unit distance from the origin). This question turned out to be extraordinarily difficult, and mathematicians have been struggling with it ever since."

In order to follow even this simplified description, some explanation is needed. The surface of a sphere has the Cartesian equation

$$x^2 + y^2 + z^2 = 1,$$

if we agree to use a scale of measurement where the radius is 1. Although this geometrical object "lives" in the familiar three-dimensional space of our everyday experience, it is itself a two-dimensional object, for two co-ordinates (e.g. latitude and longitude) suffice to determine the position of any point upon it.

Other such two-dimensional objects can also live in our familiar space. Another is the torus, which is the official name applied to the doughnut shape. Yet another is the "Klein bottle", a complicated shape beloved of topologists. And the list goes on. The surface of an apple is not a perfect sphere, but we can imagine it being smoothly massaged into a spherical shape. Topologically, it is equivalent to a sphere.

Poincaré realised that the property of being simply connected was unique to surfaces that could be made into spheres in this way. We now call this the case $n = 2$. Think of an even simpler case: $n = 1$. This will be the case of a loop of string or wire (say) that may be twisted into a variety of shapes. The analogue of our rubber band is a short length of rubber applied to the wire, and allowed to shrink to a point if possible. It is easy to see that this can be done if the wire forms a simple closed loop, but not if it passes through itself.

Another website gives some further background. See

http://mathworld.wolfram.com/news/2002-04-09_poincare.html

This tells us that "the $n = 1$ case is trivial, the $n = 2$ case is classical, [and] $n = 3$ remains open".

Remarkably, the higher cases are now all resolved. One would think that the difficulty would increase with the dimension, but this is not so. In 1961, Smale proved all the cases for which $n > 7$. That same year, Zeeman proved the case $n = 5$, and in 1962, Stallings settled the intermediate case $n = 6$. Smale was later able to modify his proof to cover all cases $n \geq 5$. So, by the mid-60s, only two cases remained. The case $n = 4$ was finally settled by Freedman in 1982, and for this work Freedman was awarded the 1986 Fields Medal (Mathematics' equivalent of a Nobel Prize).

Recently M J Dunwoody of the University of Southampton claimed to have proved the last case, $n = 3$. If this claim holds up, then Dunwoody has earned a prize of $US1 million from the Clay Institute. The rules they apply however insist that the proof must survive two years of expert scrutiny before the prize is awarded.

Dunwoody posted his candidate proof on the internet, and you can see it by following a link from the mathworld site given above. However, it is unlikely that this "proof" will hold water. Dunwoody himself has already changed the title of his posting from "A Proof of the Poincaré Conjecture" to "A Proof of the Poincaré Conjecture?". Furthermore, a step in the proof seems not to be proved, and even Dunwoody himself admits to being unable to fill in the gap.

So it seems that the Poincaré Conjecture, which has survived many previous challenges, has also survived yet another!

"Proof" plays in Melbourne

David Auburn's play *Proof*, noticed in our February issue for this year, had a short run in Melbourne. Advance advertising was scant, but regular subscribers to the Melbourne Theatre Company ensured excellent houses. However, *Function* did not learn of the staging in time to alert readers. Nevertheless, your chief editor managed to get in to see it and came away impressed.

The play concerns what happens following the death of a famous mathematician, identified as Robert. We learn that Robert has fallen victim to a debilitating mental illness. The description of the symptoms make it clear that Auburn is drawing on the real life story of John Nash (see the story in last April's *Function*). Robert has two daughters, Catherine and Claire, of whom the former has inherited some of her father's flair for Mathematics, his great enthusiasm for the subject, but possibly also his tendency to mental instability. Her life has been dedicated to his care, and this has interrupted her formal education, although it has given her unique access to a giant of the discipline.

Claire has some of the same qualities, but they have expressed themselves differently, resulting in a career in finance and a hard-nosed commonsense practicality. Catherine, by contrast, is almost visibly fragile emotionally. The interaction between the sisters is one of the strands in the play's plot. Another is supplied by the fourth character, Hal, a graduate student, who has some knowledge of and respect for Robert's work, but also more than a hint of unscrupulous ambition. He hopes to discover among Robert's papers something he can use to make a name for himself.

As the play progresses, we see an ambiguous relationship develop between Catherine and Hal, and follow the emergence of one particular notebook as the only significant piece of Mathematics to be found among Robert's papers. It contains the proof (or alleged proof) of a significant theorem. The dialogue allows us to deduce that the topic is Number Theory, and that the theorem is one of a cluster loosely related to Fermat's Last Theorem. Beyond this it is not identified, but the details that are provided are convincing.

We need this if we are to believe the intricacies that follow. Hal is a good enough mathematician to recognise that the proof is significant; but he is not sufficiently competent to judge if it is correct. Then there is the question of the proof's authorship. Is it in fact by Robert? If so, what conclusions do we draw as to his mental state? If not, whose is it?

The two questions become intricately intertwined, and the point is explicitly made that two separate canons of proof are involved here. But when the interpersonal issues become part of the plot, further matters arise. Does a mathematical proof compel belief? Is this the same type of belief we need to accept the testimony of a witness? Is it a different thing to believe what someone says, and to believe *in* that someone? And so on.

There is a partial resolution, but it remains partial, and we are left knowing almost nothing of the sequel, but with a lot to think about.

Olympiad News

We normally publish in our August issue a report on the International Mathematics Olympiad and how our Australian team performed. This year the event is being held unusually late, and we have been forced to hold this news over till October.

PROBLEMS AND SOLUTIONS

We begin with the solutions to the problems posed last February.

Solution to Problem 26.1.1

This problem came from Todhunter's *Algebra* (a once popular textbook) and it asked for a proof that

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{a}+\frac{a}{b}\right)\left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{c}\right)$$

under the condition that $abc \neq 0$. (This last condition, not found in the original, was inserted by *Function* to avoid the problem of zero divisors.)

Solutions were received from Keith Anker (who sent two different solutions), J C Barton, J A Deakin, Julius Guest, Awani Kumar (India), Carlos Victor (Brazil) and Colin Wilson. These solutions differed in detail but most began by multiplying throughout by $(abc)^2$.

The result to be proved then becomes

$$a^2(b^2+c^2)^2+b^2(c^2+a^2)^2+c^2(a^2+b^2)^2$$
$$=4a^2b^2c^2+\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right). \qquad (*)$$

Because of the condition prohibiting zero divisors, this equation is almost equivalent to the original. However, Equation (*) may also apply when $abc = 0$, so it is slightly more general. We thus concentrate on Equation (*) in what follows.

From this point on, solutions differed in their detail. We here provide a different proof from any of those submitted, although it is close in spirit to one of Anker's.

The technique to be used is "pseudo-induction", which was the subject of a brief report in *Function*, *Vol* **25**, *Part* **3**. We note that the left- and right-hand sides of Equation (*) are both quartic (i.e. 4th order) in a. Accordingly, if they are equal for five different values of a, then they are identically equal. We may choose these values however we like, but a good strategy is to do so in such a way as to make the resulting algebra as simple as possible.

Take first the value a = 0, which is allowable because of the remarks above. (Alternatively, take a very small value of a and let it tend towards zero.) In either case, Equation (*) reduces to

$$b^2c^4 + c^2b^4 = b^2(b^2 + c^2)c^2,$$

which is obviously true.

Next choose the two values $a = \pm ib$. This reduces Equation (*) to

$$-b^2(b^2 + c^2)^2 + b^2(c^2 - b^2)^2 = -4b^2c^2.$$

This result is easily proved by expanding the left-hand side using the "difference of two squares" formula.

Finally choose the two values $a = \pm ic$. This works out just like the previous case.

This proof may be simplified by noticing that we may put

$$a^2 = A,\ b^2 = B,\ c^2 = C$$

in Equation (*) and so reduce it to

$$A(B+C)^2 + B(C+A)^2 + C(A+B)^2 = 4ABC + (A+B)(B+C)(C+A). \qquad (\dagger)$$

This equation is quadratic in A, and so we need only verify it for three values, say A = 0, $-B$, $-C$. This is exactly equivalent to the previous proof, but it is simpler in detail, and avoids the explicit use of imaginary quantities. The details are left to the reader.

Other approaches use the cyclic symmetry of (*) or its simplified form (†). That is to say that substitutions in which A replaces B, which in turn replaces C, which in its turn replaces A, leave the expression unaltered. Some of our solvers took this route, using a variety of paths. There are available a number of quite sophisticated arguments from this perspective, but they require specialist knowledge that many readers of *Function* may not have.

However, if A, B, C are the roots of the cubic equation,

$$x^3 - S_1x^2 + S_2x - S_3 = 0,$$

then its coefficients are given by

$$S_1 = A + B + C, \quad S_2 = AB + BC + CA, \quad S_3 = ABC,$$

and all other cyclically symmetric functions may be expressed in terms of these three. A number of elegant proofs may be constructed along these lines. The relevant relations were probably better known in Todhunter's day than they are now.

Solution to Problem 26.1.2

The problem read: "Six men each have some coins; leaving out the first man's share, there are 75; leaving out the second man's share, there are 70; leaving out the third man's share, there are 67; leaving out the fourth man's share, there are 64; leaving out the fifth man's share, there are 54; leaving out the sixth man's share, there are 50. How many coins does each man have?"

The problem is straightforward, and there are many ways to solve it. Its interest derives from its having been proposed by Fibonacci (after whom the famous sequence is named).

Solutions were sent in by Keith Anker, J C Barton, J A Deakin, Julius Guest, Awani Kumar and Carlos Victor. All sent the same solution, which is the simplest available.

Let c_i be the number of coins possessed by the ith man, and let T be the total number of coins. Then there are 6 equations:

$$T - c_1 = 75, \; T - c_2 = 70, T - c_3 = 67, \; T - c_4 = 64, \; T - c_5 = 54, \; T - c_6 = 50.$$

Now add all these six equations to find

$$6T - (c_1 + c_2 + c_3 + c_4 + c_5 + c_6) = 5T = 380,$$

so that $T = 76$. It follows at once that

$$c_1 = 1,\ c_2 = 6,\ c_3 = 9,\ c_4 = 12,\ c_5 = 22,\ c_6 = 26.$$

Readers may care to generalise this problem and to explore it using matrix algebra.

Solution to Problem 26.1.3

This problem was submitted by our regular correspondent Julius Guest. It read: "Ada and Bert each carpeted their living-rooms with the same type of carpet. Ada's living-room is 50 cm longer than it is wide; Bert's is 10 cm longer than Ada's, but also 10 cm narrower. Ada ended up paying \$2.40 more than Bert. What is the price of carpet per square metre?"

Solutions were sent in by Keith Anker, J C Barton, J A Deakin, Awani Kumar, Carlos Victor and the proposer. Again all sent essentially the same solution.

If we let w be the width in cm of Ada's living room, and p be the price (in dollars) per square metre, then Ada's bill comes to $\frac{w(w+50)}{100^2}p$. It also follows that Bert's bill is $\frac{(w-10)(w+60)}{100^2}p$. So now we have:

$$\frac{w(w+50)}{100^2}p - \frac{(w-10)(w+60)}{100^2}p = 2.40.$$

(The factors of 100 convert square centimetres to square metres.)

This equation simplifies to give $p = 40$. The price of the carpet is \$40 per square metre.

It is an interesting feature of the problem that the second unknown, w, cancels out, and we never learn its value.

Solution to Problem 26.1.4

This problem was also submitted by Julius Guest. It read:

"Anderson, Brown, Chester, Driver and Eagle all live in the same street; three are teachers and two are engineers. A detective trying to determine Chester's profession is told:

(1) Neither Anderson nor Brown is an engineer,
(2) Neither nor Driver nor Eagle is an engineer,
(3) Both Driver and Chester are engineers.

However, all these pieces of information turn out to be false. What is Chester's profession?"

Again solutions were sent in by Keith Anker, J C Barton, J A Deakin, Awani Kumar, Carlos Victor and the proposer, and again also these were essentially the same. However, we here print a slight variant on their solution.

The three false pieces of information may all be replaced by their negations, which are thus true. We therefore know:

(1) Either Anderson or Brown *is* an engineer,
(2) Either Driver or Eagle *is* an engineer,
(3) Either Driver or Chester is a teacher.

If now Chester is an engineer, there must then be at least three engineers, and this is contrary to the data.

Therefore Chester is a teacher.

Note that in this problem also we are unable to find out all the details of the situation. The two engineers could be Anderson and Driver, Brown and Driver, Anderson and Eagle or Brown and Eagle, with the others being teachers in each case. All these four possibilities are compatible with the data.

As usual, we close with four new problems.

Problem 26.4.1 (From *Revista de Matematică din Timişoara*)

A function $f(x)$ is said to be convex over its domain of definition if for all x, y in that domain, $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$. Examples are

$f(x) = \ln x$ and $f(x) = \sqrt{x}$, with the domain of definition being $x \geq 0$ in both cases. We seek to extend this definition by defining a notion of "strongly convex functions". A function will be defined to be strongly convex if for all x, y in the domain, and with $x > y$,

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + (x-y).$$

Prove that there are no functions satisfying this definition.

Problem 26.4.2 (From the National Council of Teachers of Mathematics, USA, Calendar, February 2000)

Exactly one of the following statements is false.

(a) Audrey is older than Beatrice
(b) Clement is younger than Beatrice
(c) The sum of the ages of Beatrice and Clement is twice the age of Audrey
(d) Clement is older than Audrey.

Who is the youngest: Audrey, Beatrice or Clement?

Problem 26.4.3 (Submitted by Julius Guest)

Determine the indefinite integral (primitive):

$$\int \left\{5\left(\sin^3 2x\right)^{1/2} / \sin^5 x\right\} dx.$$

Problem 26.4.4 (Submitted by Julius Guest)

Determine the indefinite integral (primitive):

$$\int \left\{x\left(1-x^2\right)\right\}^{1/3} dx.$$

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