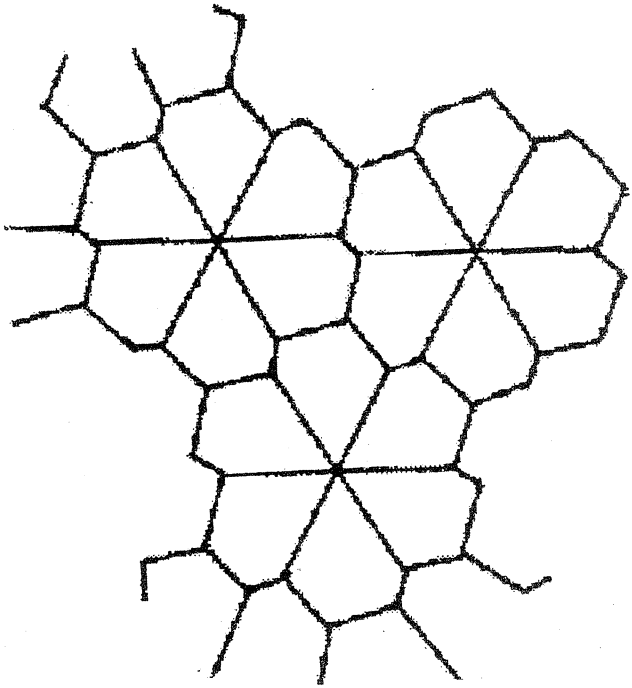
Function

A School Mathematics Journal

Volume 28 Part 4 **August 2004**



School of Mathematical Sciences – Monash University

Reg. by Aust. Post Publ. No. PP338685/0015

Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston, and is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Function is published five times a year, appearing in February, April, June, August, and October. Price for five issues (including postage and GST): $33* ; single issues $7. Payments should be sent to: The Business Manager, Function, School of Mathematical Sciences, PO Box 28M, Monash University VIC 3800, AUSTRALIA; cheques and money orders should be made payable to Monash University.

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* $17 for *bona fide* secondary or tertiary students.

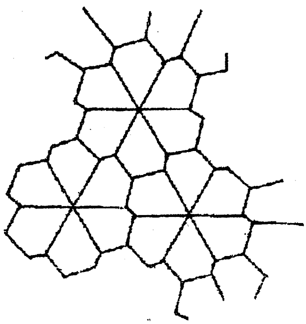
THE FRONT COVER

Our front cover for this issue displays one of the diagrams from our feature article on tilings. This is an important study within Mathematics and moreover a topical one. In our previous issue, the book review took account of several tilings to be seen in the public architecture of Melbourne.

In particular, the 15th and last of Jill Vincent's list of notable mathematical sights concerned the "Tesselating Pentagon Pavement" at the University of Melbourne.

Our article provides a proof that it is not possible to use a *regular* pentagon to tile, or tessellate, a plane. However, if we drop the condition of regularity, then there a number of such tilings known. (According to our article, there are 14 that have been found so far, but there may be yet others that no one has yet discovered!)

The diagram is reproduced again below for the reader's convenience.



Notice that the pentagons combine to form a rose-shaped structure, each "head" of which contains six congruent pentagons. These six pentagons make up a structure with the symmetry properties of a regular hexagon. In fact the boundary could be seen as a "deformed hexagon". It follows that the tiling shown here is a periodic (repeating) tiling. Contrast this with the Penrose tiling, also mentioned in the article, which is not periodic.

TILINGS

Josefina Alvarez, University of New Mexico
and
Cristina Varsavsky, Monash University

In bathrooms, kitchens and beyond, house tilings are manifestations of a craft that has adorned buildings from ancient Rome to the Islamic world, from Victorian England to colonial Mexico.

In general, the word 'tiling' refers to any pattern that covers a flat surface, like a painting on a canvas, using non-overlapping repetitions of one or more shapes, so that the design does not leave any empty spaces. Contemporary tilings can be found in African-American quilts, Indonesian batiks, molas (traditional blouses sewn by the Cuna women from the San Blas Islands off the coast of Panama), and Aboriginal paintings. Tiling was a favourite means of expression for the Dutch artist M C Escher (1898-1972). He had this to say about tiling: "... I have embarked on this geometric problem again and again over the years, trying to throw light on different aspects each time. I cannot imagine what my life would be like if this problem had never occurred to me; one might say that I am head over heels in love with it, and I still don't know why"[1]. In these and other words, Escher repeatedly expressed his love for this art-form, acknowledging at the same time the influence on his work of the mosaics he admired and sketched at Moorish buildings in Southern Spain. In spite of this influence, Escher's art went far beyond anything seen before. He produced enigmatic tilings, with strange creatures and mutating landscapes that suggest a craft free from any worldly limitation. Despite this appearance, tiling is a very precise art, where not much can be left to chance. Even the simplest tilings fall under the sway of mathematical principles. We can push and turn and wiggle, but if the maths is not right, it isn't going to tile.

To see how Mathematics can limit the fancy of the best tile installer, we first try our hand at tiling with copies of just one regular polygon. Not only do we want to use copies of just one regular polygon, but we also want to place them vertex to vertex, that is to say, with the vertices of one copy only touching the vertices of another copy. These tilings are called regular. For instance, in Figure 1, tilings (a), (b) and (c)

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[1] In F Cajori, *A History of Mathematics*, 5th ed, AMS Chelsea Publishing, AMS 2000

are regular, while tilings (d), (e) and (f) are not. In fact, the squares in tiling (d) lie in parallels that can "slide past" one another. Tilings (e) and (f) use more than one kind of regular polygon each, regular octagons and squares for (e), regular hexagons and equilateral triangles for (f).

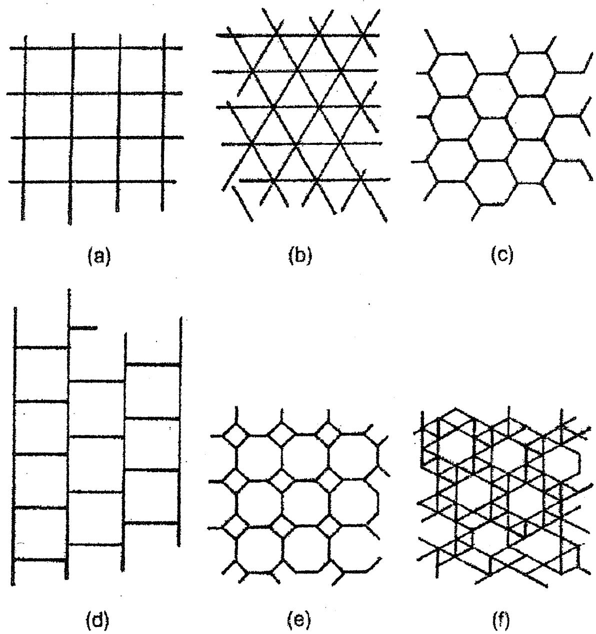

Figure 1

As the tilings (a), (b) and (c) in Figure 1 show, squares, equilateral triangles and regular hexagons do make up regular tilings, a fact that was known to Pythagoras's followers in the fifth century BC. But Mother Mathematics says that no other regular polygon can make the same claim. Why? If we look at the regular tilings in Figure 1, we can see that the magnitudes of the angles meeting at each vertex add up to exactly 360°. What happens with, say, regular octagons? As Figure 2 (overleaf) illustrates, two regular octagons fall short of completing 360°, while three regular octagons produce some overlapping. Tiling (e) shows that the perfect tiling companion of two regular octagons is a smaller square wedged between them.

So our claim about which regular polygons can tile a plane regularly is really a claim about the size of their angles. To say that a regular polygon will produce a regular tiling of a plane is the same as saying that the size of its angle (in degrees) divides exactly into 360. In other words, that the ratio $\frac{360}{\text{angle measure}}$ is equal to one of the numbers $1, 2, 3, \ldots$.

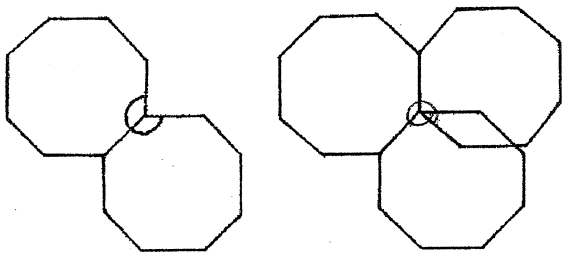

Figure 2

Now, if we want to show that only equilateral triangles, squares and regular hexagons make up regular tilings, we need to show that the angle-measure of any other regular polygon will not divide exactly into 360. How do we do this? A mix of geometry and algebra will do the trick very nicely. We first use geometry to come up with a formula for the size of the angle of any regular polygon. Let us see how. Figure 3(a) (opposite) shows a regular polygon with n sides and angle measure a. The generic number of sides n can be 3, 4, 5, ... We haven't completed the picture of our polygon because we do not want to fall into thinking about a particular polygon. Whatever we do has to work for any regular polygon. In Figure 3(b), we have outlined one of the n isosceles triangles whose apices meet at the centre of our polygon.

What can we say about the magnitudes of the angles of this isosceles triangle? They are equal to $\frac{a}{2}$, $\frac{a}{2}$ and $\frac{360°}{n}$, as Figure 3(b) suggests. We also know that the sum of these three angle-measures has to be $180°$. Or

$$\frac{a}{2} + \frac{a}{2} + \frac{360°}{n} = 180°.$$

That is to say:

$$a = 180° - \frac{360°}{n}$$

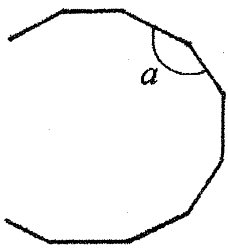

Figure 3(a)

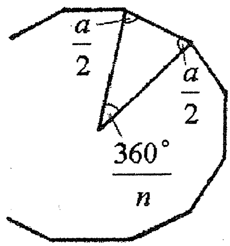

Figure 3(b)

This is the formula we will use for the angle of our regular polygon. Of course, it produces the values we have already found for the angle of an equilateral triangle, $n = 3$, a square, $n = 4$ and a regular hexagon, $n = 6$. Now, let's remember that our regular polygon will make a regular tiling only when the measure of the angle a divides exactly into 360. This means that we are looking for those regular polygons for which the ratio $\frac{360}{a}$ equals one of the numbers 1, 2, 3, ... If we substitute in this condition our newly acquired formula for a, the condition becomes

$$\frac{360}{180 - \frac{360}{n}} \text{ must be equal to 1 or 2 or 3 or ... .}$$

After some simplification, this condition reads

$$\frac{2n}{n-2} \text{ must be equal to 1 or 2 or 3 or ... .}$$

So, our geometric problem of tiling has become the following algebraic problem: to show that $2n$ is divisible by $n-2$ only when $n = 3, 4, 6$. We can quickly see that the condition is true for $n = 3, 4, 6$; we can also check that is not true when $n = 5$ (pentagon) or when $n = 7$ (heptagon). But checking individual values for n will not do the job. We need to do some further algebra to prove the assertion. By division, we can write

$$\frac{2n}{n-2} = 2 + \frac{4}{n-2}$$

Now we only need to show that $\frac{4}{n-2}$ cannot be a counting number, for $n \geq 8$. But this is easy, because when $n = 8$ we have $\frac{4}{8-2} = \frac{2}{3}$ and, as n increases, the ratio $\frac{4}{n-2}$ decreases. So there is no way that $\frac{4}{n-2}$ could ever become a counting number, for any $n \geq 8$, and the same must then be true for $2 + \frac{4}{n-2}$. And we are done. We now know for sure that the only regular tilings of the plane using one regular polygon are the first three tilings depicted in Figure 1.

It has been said that mathematicians do not know where to stop, meaning that we always find yet another wrinkle to explore. Here is my new wrinkle: Tiling (d) in Figure 1 shows that we can build a sliding tiling using copies of a square. Is this true for an equilateral triangle? How about a regular hexagon? How about other regular polygons?

Let us consider first the case of equilateral triangles. We can start with two equilateral triangles sitting as in Figure 4(a).

Since $3 \times 60^\circ = 180^\circ$, we should be able to fit exactly another four copies of the triangle, two above and two below, as shown in Figure 4(b) below. The result is a regular hexagon broken into two halves, with one half slid along the other. Now we can see what is going on: If we break the regular tiling (b) in Figure 1 along all or some of the horizontal lines, or along all or some of the slanted lines, and if we slide the strips along the fracture lines, we obtain a tiling, like the one shown in Figure 4(c). We could try to reason in the same way with regular hexagons, but as it happens, doing the same thing does not always guarantee the same outcome. As Figure 4 suggests, the angles refuse to cooperate, leaving annoying empty spots.

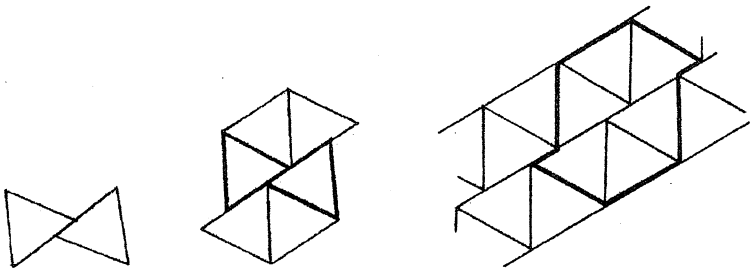

Figure 4(a) Figure 4(b) Figure 4(c)

So, the answer for the triangular wrinkle is "yes", while the answer for the hexagonal wrinkle is "no". What happens with other regular polygons? A moment's reflection will show that a regular polygon will produce one of these sliding tilings only when its angle measure divides exactly into 180°. Or,

$$\frac{180}{180-\frac{360}{n}} \text{ must be equal to 1 or 2 or 3 or } \ldots .$$

Reorganising this expression as before, we can see that the possibility of the sliding tiling using one regular polygon goes hand in hand with the truth of the condition

$$\frac{n}{n-2} \text{ must be equal to 1 or 2 or 3 or } \ldots .$$

That is to say, $1+\dfrac{2}{n-2}$ must be equal to 1 or 2 or 3 or ... .

But $\dfrac{2}{n-2}$ is a counting number only for $n = 3, 4$. In other words, only equilateral triangles and squares can produce this kind of sliding tiling using just one regular polygon.

You can see how the rules and regulations of Mathematics appear very quickly even in the simplest tiling designs. No pentagons in the bathroom floor! No pentagons? Well, thinking of it, we only know that regular pentagons do not work. But what happens if we drop the word 'regular' from the specifications? What if we just want to tile with copies of one convex pentagon, which means a five-sided polygon with all the angles less than $180°$? If we do, a very different and interesting story will unfold, because there are quite a few convex pentagons that will tile a plane. For instance, how would you like to have one of these patterns in your bathroom floor?

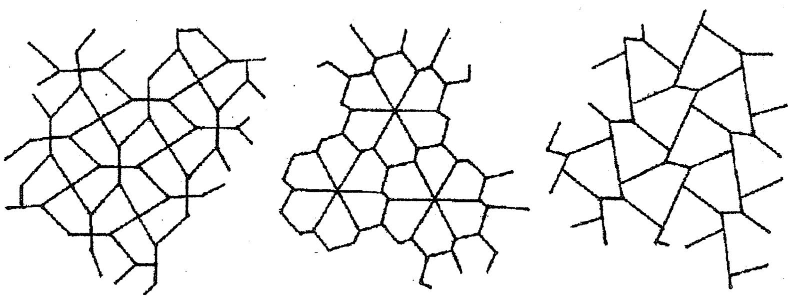

Figure 5

As you can see, non-regular pentagons give quite a lot of choices. They are also more difficult to handle. The problem of tiling with copies of one convex pentagon is open. To date, 14 different tilings are known, but nobody knows whether this list is complete.

What happens if we get back to regular polygons, but this time we allow more than one shape and size to be used? We have seen already examples of this kind of tiling: Tiling (e) in Figure 1 uses at each vertex one copy of a square and two copies of a regular octagon, while Tiling (f) uses at each vertex one copy of a regular hexagon and four copies of an equilateral triangle. Inspired by these tilings, we could demand of our tilings with more than one regular polygon that polygons with the same number of sides have the same size, that vertices meet at vertices and that the same number of polygons of each shape is used at all the vertices. We will call these patterns mixed tilings. Actually, the first three regular tilings of Figure 1 could be considered particular cases of mixed tilings, that use at each vertex four copies of a square, or six copies of an equilateral triangle or three copies of a regular hexagon. Combining geometry and algebra, with some help from a computer, we could find all the possible mixed tilings, but this would need to be another article!

You can see that we could go on forever with this very serious tiling game. What if we allow the polygons to get smaller and smaller? What if we use copies of any triangle or copies of any figure with four sides? What if we want to emulate Escher and try to draw some figurative meaning into the tiles? What if we look for patterns that, in some sense, never repeat? Each of these "what if", and many others that you might imagine, will open up new fascinating possibilities.

As a guide for your explorations, we suggest the excellent presentation in Chapter 20 of *For All Practical Purposes*[2]. There you can read, for instance, about the endeavours of one of Escher's tiling pals, the British mathematician Roger Penrose. Penrose has designed non-repeating tilings that now seem to agree with the internal structure of real materials, such as some composites containing aluminium. On a lighter note, Penrose's designs have also caught the attention of a toilet paper manufacturer, because paper embossed with a non-repeating pattern can be rolled without leaving bulging spots. But Penrose had copyrighted the pattern and the manufacturer got a legal spanking. Beyond the fun side of

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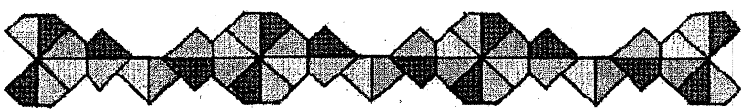
[2] *For All Practical Purposes: Introduction to Contemporary Mathematics* by COMAP, New York: W H Freeman, 2002.

tiling and its more, or less, serious applications, Penrose's interest in tiling relates also to his interest in artificial intelligence and the workings of computers.

For instance, the tiling problem (that is whether a given bunch of shapes will tile a plane) belongs to a class of mathematical problems called non-recursive. The tiling problem is answerable in each particular case, but, Penrose says, "there is no systematic procedure that, once implemented on a machine, could give an answer in any case, without requiring any more thinking." These and many other issues are discussed in Penrose's controversial books *Shadows of the Mind: A Search for the Missing Science of Consciousness*[3] and *The Emperor's New Mind: Concerning Computers, Minds and the Laws of Physics*[4].

Let us get back to tiling. In the books *The Magic Mirror of M C Escher*[5], and *Visions of Symmetry: Notebooks, Periodic Drawings and Related Work of M C Escher*[6], the authors present and explain many of Escher's tiling masterpieces. For more examples on how well Mathematics and tiling play together, you can look into *The World of Patterns*[7]. This CD-ROM includes many tilings classified by their artistic and mathematical traits. It also has an extensive list of references to other works on tiling. An internet search will lead you to several nice computer programs where you can play the tiling game. I also find it very interesting to experiment with shapes cut out of sturdy paper.

From looking at pretty pictures or making them, to doing Mathematics, the choice is yours. For now, we draw the line here.



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[3] Published by Oxford University Press in 1994.
[4] Published by Oxford University Press in 2002.
[5] By B Ernst, published Barnes & Noble in 1994.
[6] By D Schattschneider, published by W H Freeman in 1990.
[7] By B Wichmann, CDROM and booklet published by World Scientific in 2001.

NEWS ITEMS

"Contrived Equations"

A recent article in the *International Journal for Mathematical Education in Science and Technology* raises some interesting and surprising points. It is to be found in their issue for February 2004 (*Volume* **35**, *Part* **1**), pp 135-144. The authors are Miriam Amit, Michael J Fried and Pavel Satianov, all of Ben Gurion University in Israel.

They begin by considering an equation that seems absurd. The idea is to graph the equation:

$$\sqrt{1-x^2}\sqrt{1-y^2} - \sqrt{1-x^2}\sqrt{1-y^2} = 0. \qquad (1)$$

The equation is so apparently ridiculous that we might be tempted, at first sight, to dismiss it out of hand. But wait!

Think of what an equation means. It defines the set of (in this context) (x, y) for which the statement that the equation makes is true. So for example if we write the equation

$$x^2 + y^2 = 1,$$

then if (x, y) represents a point on the unit circle, the equation is true; for any other point it is false.

Now look again at Equation (1). As long as

$$-1 \le x \le 1 \text{ and } -1 \le y \le 1 \qquad (2)$$

the equation is certainly true. However, for all other x, y the left-hand side is undefined (in real algebra). Thus Professor Amit and her co-workers claim that Equation (1) and the inequalities (2) are equivalent.

It follows that the interior (including, here and in what follows, its boundaries) of the square given by the inequalities (2) constitutes the "graph" of Equation (1), which, because of its unusual character, the authors refer to as a "contrived equation". They go on to discuss the equations that produce triangles in a similar manner, and this will here be set as a challenge to readers to do for themselves.

The convention in all this is that $\sqrt{x}$ is the positive (or zero) square root of x if $x \geq 0$, but is meaningless if $x < 0$.

If this were all there was to their article we could say of it: "This is a bit of a surprise, but so what?" However, they go on to discuss related matters – matters with rather more substance to them.

This time they consider another function $|x|$, known as the *absolute value* of x. [Some software packages call it ABS(x).] This function is defined to be x if $x \geq 0$, and $-x$ if $x < 0$. In view of our earlier remarks, we can give a rather nice alternative definition: $|x| = \sqrt{x^2}$. The three authors go on to produce "contrived equations" involving the function $|x|$.

Think first of three points on a line. Call them A, B and X. Corresponding to these three points will be co-ordinates a, b and x respectively. Then $|a-b|$ will be the distance between A and B, and similarly for the other pairs of points. Then if X lies between A and B, then

$$|x-a| + |x-b| = |a-b| \tag{3}$$

but if X lies outside this interval, this equation is false.

The three authors refer to equations like Equation (3) as also being "contrived", but here the usage is not quite the same. If we do not have $a \leq x \leq b$ or $b \leq x \leq a$, Equation (3) asserts a falsehood rather than being meaningless. However here also we can think of the interval between a and b as being the "graph" of Equation (3). The usage "contrived" describing such equations now refers not just to the apparent tautologies of equations like Equation (1) but also to other situations where an equation defines a region.

And now we have another way to represent the interiors of rectangles. If we take as well as the points A, B and X on the x-axis, points C, D and Y on the y-axis, then we can plot the y-values in the interval between c and d (the y-co-ordinates of C and D respectively). This results in another equation

$$|y-c| + |y-d| = |c-d|. \tag{4}$$

Equations (3) and (4) together give the set of points that form the interior and boundaries of a rectangle contained between the lines $x = a$, $x = b$, $y = c$ and $y = d$.

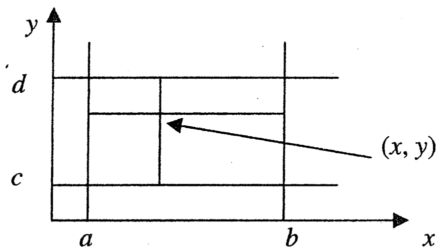
But now we can combine the two separate equations (3) and (4) into one. It is most instructive to multiply them. This gives:

$$\{|x-a|+|x-b|\}.\{|y-c|+|y-d|\}=|a-b|.|c-d|.$$

Now expand the left-hand side. We find that

$$|x-a|.|y-c|+|x-b|.|y-c|+|x-a|.|y-d|+|x-b|.|y-d|=|a-b|.|c-d| \quad (5)$$

But now this equation possesses a ready interpretation. Look at the diagram below



The rectangle we have been graphing is split up into four smaller rectangles by the horizontal and vertical lines through the interior point (x, y). What Equation (5) tells us is that the total area of the big rectangle is the sum of the areas of the four small ones. While this is not exactly news, it is surprising to see it turn up here in this context. The equation for the rectangle is a statement of one of its properties. Equation (5) is true if (x, y) lies inside the rectangle, false if not.

The three authors next go on to look again at triangles. Here they make use of a formula that we saw in *Function* last June. It goes like this. Let the vertices of a triangle be (x_1, y_1), (x_2, y_2) and (x_3, y_3). Then the area of the triangle is given by the value of the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

with one small complication that we glossed over then, but which is important in the present context.

This complication is that the value of the determinant might turn out to be negative. In some contexts, this is quite unimportant (it provides information on the orientation of the orientation of the triangle). Here however it is an unwanted complication. We can get rid of it by using the absolute value function in this context also.

To make the notation somewhat easier to follow, write

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = D(x_1, y_1; x_2, y_2; x_3, y_3).$$

The area will then be written as $|D(x_1, y_1; x_2, y_2; x_3, y_3)|$. This way there will be no complication arising from the determinant giving a negative area.

Now take another point (x, y). If this new point lies inside the triangle, we may divide the original triangle into three smaller ones whose total area is the same as that of the large one. This tells us that

$$\begin{aligned} |D(x, y; x_1, y_1; x_2, y_2)| + |D(x, y; x_2, y_2; x_3, y_3)| + |D(x, y; x_3, y_3; x_1, y_1)| \\ = |D(x_1, y_1; x_2, y_2; x_3, y_3)| \end{aligned} \qquad (6)$$

Once again, this equation will hold if (x, y) is an interior point of the triangle; otherwise not. So this statement of area conservation is the equation for the interior of the triangle.

Readers might like to explore matters further. For example, if Equations (3) and (4) had been added rather than multiplied, the new equation would still describe the same region as is given by Equation (5). Is it possible to provide a geometric interpretation?

More on the Twin Primes Conjecture

Some of the entries in the list of prime numbers form pairs separated by a common difference of 2: (3, 5), (5, 7), (11, 13), etc. It has long been conjectured that there are infinitely many such pairs, but so far no one has managed to prove this. This was one of the outstanding conjectures in Number Theory that formed the subject matter for the History column in April 2001. In the years since, there has been a flurry of interest in the conjecture, and we have reported aspects of this in several news items in recent issues of *Function*.

Recently there has been another attack on the problem. Last May, a proof (by R F Arenstorf, an American number theorist) was announced: in fact a proof of an even stronger result, i.e. a result that entailed the correctness of the twin prime conjecture and proved further matters as well.

Sadly, the proof is now known to be defective. A mistake in the demonstration of one of the subsidiary results used along the way has been shown (by the French mathematician G Tenenbaum) to be in error. Both the proposed proof and the revelation of the error are posted on the web. The first is at:

http://arXiv.org/abs/math.NT/0405509

and the second at:

http://listserv.nodak.edu/scripts/wa.exe?A2=ind0406&L=nmbrthry&F=S+&P=1119

Opinions are divided on the seriousness of the error. Remember that Wiles' initial proof of Fermat's Last Theorem needed patching up in one respect. In that instance, the defect was remedied in fairly short order. The present case may be somewhat similar as many believe and hope. However, Tenenbaum believes that the error may well indicate serious problems with the whole approach of Arensdorf's method.

For more detail, see:

http://mathworls.wolfram.com/news/2004-06-09/twinprimes/

This website (unlike those given earlier) should be quite understandable to all readers of *Function*.

"Boy bitten by a Lizard"

The recent exhibition at the National Gallery of Victoria of works by Caravaggio and his followers included two items both called *Boy bitten by a Lizard*. In the catalogue they are reproduced as Plates 4 & 5. At first glance, the two paintings seem identical, but if they are examined in more detail, differences become apparent. The first of the pair is held by the National Gallery in London and the other by the Longhi Gallery in Florence.

Between their being painted and their reaching their current places in art museums (early and late C20), there are long gaps in the "provenance" of both. That is to say, we do not know their history for much of the intervening 300+ year period. Neither has survived intact. The British version is described in the catalogue as "very damaged", while the Italian one has been covered with a "thick varnish". This means that differences in colour (which are the most obvious of the variations between them) are not truly representative of the original state of either. It may also be that very slight differences in the size are also the result of different treatments over the centuries. (The Italian version is 2mm shorter and 28mm wider.)

The catalogue discusses in some detail the debate over the authenticity of these paintings. Quite why Caravaggio would want to paint the same subject twice is the question that naturally arises. For many years it was thought that one or other was a copy by someone other than Caravaggio. Modern scholarly opinion, however, accepts both works as being from the master's own hand.

The matter is discussed in some detail by David Jaffé and John T Spike, the authors of the relevant pages of the catalogue. They believe that the London picture was executed first (perhaps somewhere between 1595 and 1600), and that the Florentine one came later (perhaps around 1600). However, they note that other authorities take a different view and put the Italian one first. The idea is, however, that the original (whichever it was) became very popular, and that Caravaggio decided to cash in on this by producing a second version for a wealthy patron.

They seem not to have considered a different possibility: that the two versions were meant to be viewed together.

Last century, the Spanish surrealist Salvador Dali painted several such pairs, of which possibly the best-known is *Le Christ de Gala*. The idea is to view the paintings together using either special glasses or with a cross-eyed vision that can be learned with practice. The result is that the two images merge into a single 3-dimensional one.

Several exhibitions of Dali's work have displayed such pairs and so placed them that a special optical device, also carefully positioned, allowed the viewer to gain the full effect. This aspect of art was taken up by Dali in 1976 and continued to interest him for the rest of his life. Readers can learn more about this from several websites, of which possibly the best is

http://www.3d-dali.com

Dali's interest in the production of 3-dimensional images sprung from his study of the Dutch painter Gerrit [Gerard] Dou (1613-1675). For an account of this, see Robert Descharnes' book *Salvador Dali* (p 168 in Eleanor Morse's English translation). This mentions that Dali had to work very hard to perfect the technique of painting such pairs well enough to achieve a satisfactory 3-D effect. Quite how successful Dou was seems not to have been discussed.

Dou, however, was influenced (via Rembrandt, and thus at some remove) by Caravaggio, and so it is not altogether fanciful to think that he was motivated by Caravaggio's having himself made such pairs, of which perhaps *Boy bitten by a Lizard* is one.

The art of making such "stereo pairs", as they are called, depends on having two almost identical, but subtly different, images. In the early years of the 20th Century, photographers would produce such pairs by taking a shot of a scene (say) as viewed by the left eye, and another of the same scene as viewed by the right. The two were then mounted side by side onto a postcard, designed to fit into a special viewing device called a *stereoscope*. When this was held up to the eyes, a striking 3-D image was seen.

It is a little difficult to know whether Caravaggio's work has this property. Clearly a stereo pair can be constructed from the two images. However, because of the damage to the originals, it is not really possible to know if it was like this back in 1600 or so!

The Cassini Project Encounter with Phoebe

On June 23 2004, the Cassini Project (a NASA exploration of our solar system) announced the discovery of large-scale deposits of carbon dioxide on the surface of Saturn's outermost orbiting major satellite Phoebe and gave a mean density for this ancient, battered moon of 1.6 grams per cubic centimeter (g/cc). Prior to these discoveries, Monash University mathematician Andrew Prentice had predicted three different possible bulk chemical compositions and mean densities for Phoebe using his controversial theory of Solar system origin (described in Function back in April 1978; see also http://www.cspa.monash.edu.au/news.html).

Three possible compositional models (Options 1, 2 and 3) for Phoebe were considered, as no one knew exactly where this moon had originated. Unlike Saturn's other main satellites, which all revolve in circular orbits close to the planet and in the same common direction as the planet's own spin, Phoebe's orbit is highly eccentric (elongated); furthermore this moon goes around the planet in the opposite direction. All this suggests that Phoebe is a captured, rather than a native, moon of Saturn.

Prentice's first model assumed that Phoebe condensed at Saturn's distance from the sun from a gas ring that was shed some 4 billion years ago by the same primitive cloud of gas that went on to form the Sun itself. The condensate from this model has a mean density of 1.33 grams per cubic centimeter. This model, however, is now clearly ruled out by the Cassini data, not only because the density is too low but also because it cannot explain the ubiquitous presence of carbon dioxide found by the Cassini spacecraft. But because the capture of a moon is much more readily explained if it starts off on the same circular orbit as Saturn, Prentice initially preferred this model prior to the Cassini encounter (http://www.aas.org/publications/baas/v36n2/aas204/887.htm).

The other options both considered the possibility that Phoebe had originally condensed much further out in the Solar system than where it is today. Somehow (not so far explained!) it then got relocated to Saturn's orbit prior to capture. Option 2 assumes that Phoebe is a left-over planetesimal from Neptune's orbit; Option 3 has it that Phoebe is a 'first

Continued on p 124.

HISTORY OF MATHEMATICS

Joseph Bertrand and his Legacy of Paradox

Joseph Bertrand (1822-1900) was a French mathematician, who contributed to Mathematics on several fronts. In the field of Number Theory, he is remembered for *Bertrand's Conjecture*: between every positive integer n (> 1) and its double, there is at least one prime. (This was later proved by the Russian mathematician Chebychev.) But he is also the source of many of the so-called paradoxes of Probability Theory. Almost all of these derive in some way from his text *Calcul des probabilités*, first published in 1889.

Several of these have already appeared in one guise or another in *Function*. The one most commonly associated with his name formed the basis of the April Fools' Day column in 1996. It bears repeating here. Consider a circle of unit radius. In it draw a chord at random. What is the probability that the length of this chord is longer than the side of an equilateral triangle inscribed in the circle, i.e. $\sqrt{3}$? There are three different answers traditionally given to this question, although others are perhaps possible as well. The traditional ones all derive from Bertrand's discussion (pp 4, 5 of his text).

The first takes the chord to be defined by one point on the circumference of the circle (A in Figure 1 on p 113) and a second one somewhere else on the circle. If that second point lies in the arc BC, then the condition is met, and the angle BAC defining the successful outcomes is precisely one third of the total angle DAE. This conclusion holds whatever initial point is chosen. Thus the required probability is ⅓.

The second possible answer imagines the chord to be defined by a direction (BC in Figure 2, p 113) and a co-ordinate in the direction perpendicular to that (AK in the figure). Whatever initial direction we choose, the chords lying in the shaded area of the figure satisfy the requirement, and these have a perpendicular co-ordinate lying in the interval MN whose length is half that of the total AK. Thus the required probability is ½.

The third takes the chord to be defined by its centre point and also by a direction. (See Figure 3, p 113.) If the centre lies in the shaded region, as with the chord FG in the figure, then the requirement is met; if not (chord *DE* in the figure), then it is not. The area of the shaded region is one quarter of the whole, and so the required probability is ¼.

The usual approach to this paradox is to deny that the problem was "well-posed" in the first place. The answer depends on quite what interpretation we place on the words "draw a chord at random". Because three different interpretations are possible (along with perhaps others), we have demonstrated that the original task was ambiguous. This is what is meant by "not well-posed". Most authors who describe the paradox take this view, and, to fill you in, this is also my own view.

I'll get back to this paradox before I finish, but let me now introduce you to others of Bertrand's paradoxes. One is so well-known as to be notorious: the so-called "box paradox". It made a brief appearance in *Function* many many years ago, but here it is again.

Three boxes are presented to us and we are told that each contains 2 coins: one contains 2 gold coins, another 2 silver coins, while the third has one of each kind. One of the boxes is opened at random and a single coin taken out and examined. It is gold. What is the probability that the other coin in the same box is also gold? A careful analysis of the various possibilities involved results in the conclusion that the probability is ⅔, but the following argument carries a certain specious authority. "Clearly the gold coin must have come from either the first or the third box; the chance that it came from the first box is ½, and thus this is the chance that the second coin is also gold." I leave readers to detect the error in this argument.

This second "paradox" is merely an example of how easily we may be led astray by failure to analyse the problem in sufficient detail. However, a further set of paradoxes arise from much more fundamental considerations. The points at issue are essentially those discussed in my columns for June and August 2000. They relate to the application of probability notions to "real life". If there is no-one to tell us with the voice of authority that a coin is fair or a die unloaded, how are probabilities to be assigned?

One possible way is to use what has been called "the Principle of Indifference": *until proved otherwise, all possible outcomes of an experiment are to be regarded as equally likely*. One strong proponent of this view, G N Schlesinger (*The Sweep of Probability*), has argued for it even in what might at first strike us as extreme circumstances.

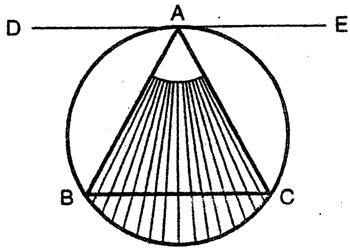


Figure 1

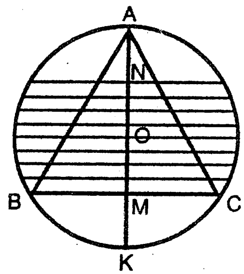


Figure 2

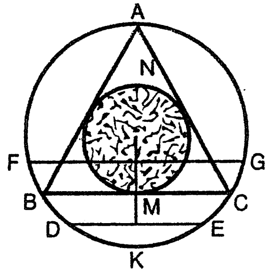


Figure 3

Suppose that a die is to be rolled and that we are warned in advance that it has been tampered with and that one face is loaded in such a way that it turns up much more often than it should, but that we have no information as to which face this is! Then if we bet on (say) a 2 turning up when the die is rolled, then we could be lucky if it so happens that the shyster has loaded the 2, or else unlucky in all other events. The same for every other possibility. We can do no better than regard the probability of a 2 as 1/6, even though we *know* that this is not the correct value!

[Here a distinction can be made and is important. If the probability is regarded as an intrinsic property of the die, then the answer 1/6 is of course quite wrong. However if we are measuring the state of our knowledge on the matter, then 1/6 is a perfectly reasonable answer. There are two different concepts here: the fact that they lead to different values being assigned to what sounds like the same probability merely reflects this. See again my columns for June and August 2000.]

But the Principle of Indifference leads to other and more troubling problems than this one. One of the simplest is the "Life on Mars Paradox" that appeared in *Function* in October 1996. We took this from Eugene P Northrop's *Riddles in Mathematics*, a work that owes a large debt to Bertrand. The Life on Mars Paradox derives form the "Weather Prediction Paradox" in Bertrand, but is considerably more telling.

An adherent of the Principle of Indifference is led to the reluctant conclusion that the probability that there are no horses on Mars is ½. Ditto for the probability of no cows, and so on for no dogs, no cats, no sheep, no goats, etc, etc, etc. The probability therefore that none of these forms of life exist on Mars is therefore $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\ldots$, a number we can make arbitrarily close to 0, thus proving almost to a certainty that life as we know it must exist on Mars!

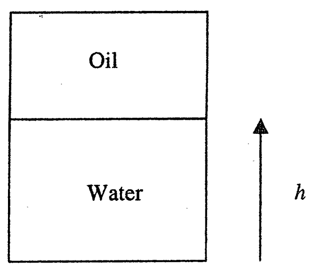
This paradox is rather easily dealt with. The events are not independent: the point Bertrand was concerned to make in his original form of the paradox. [Besides which, the Principle of Indifference is stretched to its limit when we give the probability of ½ to the probability of any *familiar* form of life living on Mars!]

Another paradox that has caused much more discussion is another variant on a Bertrand paradox. This is the *Wine-Water Paradox* that was put forward in its present form by the German probability theorist Richard von Mises. There is an account of it in his *Probability, Statistics and Truth,* first published in German in 1928. Many subsequent authors have analysed it, and the precise form has varied a little from one discussion to another. The one I use comes from a recent attempt at a resolution by Jeffrey M Mikkelson in the *British Journal for the Philosophy of Science* (March 2004).

Suppose we are presented with a container full of liquid, of which it is known that it consists of a mixture of wine and water in proportions that lie somewhere in the range $\frac{1}{3} \leq \frac{\text{wine}}{\text{water}} \leq \frac{3}{1}$. Calculate the probability that $\frac{\text{wine}}{\text{water}} \geq \frac{2}{1}$. Put $x = \frac{\text{wine}}{\text{water}}$ and consider the possible values of x. As x could lie anywhere in the range $\frac{1}{3} \leq x \leq 3$, and the favourable outcomes are those for which $2 \leq x \leq 3$, simple arithmetic seems to tell us that the probability of a favourable outcome is 3/8.

But now consider the ratio $y = \frac{\text{water}}{\text{wine}}$. Clearly $\frac{1}{3} \leq y \leq 3$. The favourable outcomes are those for which $\frac{1}{3} \leq y \leq \frac{1}{2}$, and the probability of this is 1/16.

But why should we have different answers? Surely the questions are equivalent! And as if this were not enough to have us thoroughly confused, a third answer has also been proposed. This one is mentioned by Schlesinger and favored by Mikkelson. It goes like this. Suppose the container held not water and wine, but water and oil. Then a simple line of demarcation would be visible and we could measure where it was to determine the proportions. Here is the situation.



The full height of the container is taken to be 1. Then the height h of the interface can have any value between ¼ and ¾ , as you can readily check. In the favorable cases, it must lie above ⅔, and the probability of this is 5/6. This is the answer they give. It has the virtue of not depending on which of the two possible ratios we use to reach our answer, because we are not using ratios at all.

The Principle of Indifference is much beloved of the Bayesian school of thought (see *Function*, July and August 2000). It has taken quite a battering in recent years, and von Mises, for example, regards the Wine/Water Paradox as constituting a *reductio ad absurdum*. Diehard Bayesians like Schlesinger, Mikkelson and Jaynes (of whom we will hear more later) defend the principle. Clearly there are cases in which we would instinctively use it. These have been explored in some detail by John Maynard Keynes in his *Treatise on Probability*. (Keynes is best remembered today as an economist, but he was also a formidable mathematician.) Chapter IV of this work is devoted to the principle, which he argues against (despite his having some affinities with the Bayesians).

Keynes prefers the use of a "Principle of Irrelevance": *if some piece of data is judged to be irrelevant to the problem, then we can ignore it*. One of his examples concerns two playing cards, chosen at random and each from a different pack, placed face down on a table. One card is turned over and found to be from a black suit. What is the probability that the second card is also black? Here clearly the result of the observation on the first card is quite irrelevant to the colour of the second, and hence the required probability is ½.

Or suppose that a barrel contains a collection of black and white marbles. Then we agree that the probability of drawing out (say) a black marble is determined by the numbers of marbles in the barrel, and by nothing else. The actual colour of the marble is beside the point. For most purposes the terms "black" and "white" could be interchanged. However, if we learn that the black marbles are made of iron and the white ones of tin and that furthermore the marbles are taken out using a magnet, then this changes everything! *This information is relevant.*

When we come to the Wine/Water paradox, I would say that we really have not enough information to allow an answer to the problem. The number x has a probability distribution $p(x)$, of which all that is known is that $p(x) \geq 0$ on $\frac{1}{3} \leq x \leq 3$, and $\int_{1/3}^{3} p(x)dx = 1$. The answer to the problem posed could be any number between 0 and 1. The Principle of Indifference is an attempt to manufacture data where none actually exists. The Oil/Water restatement constitutes a subtle attempt to supply further relevant data, but it does so by further elaborating the original statement.

Bayesians tend to say that the meaning of the word "probability" is that it measures *the strength of our belief* rather than some inherent property of the system under study. But if we take this idea to its logical conclusion, then there is no paradox at all. The person arguing from the x-value and the person arguing from the y-value simply have different subjective assessments, just as two bookmakers (say) might initially quote different odds on the result of a horse-race.

[Although I am critical of the Principle of Indifference, it does have a place, perhaps in Keynes' restated form. There are situations arising in actual practice (medical diagnosis, oil exploration, insurance, etc, as well as gambling) where adequate data is simply not available. Yet we need to make decisions even so. This is the territory claimed by the Bayesians. Schlesinger's account of the paradoxes contains a lengthy and closely reasoned study of such cases. For more detail, see also my column for August 2000.]

But now back to the original Bertrand Paradox: the "random chord in the circle". This received a very thorough treatment in an article by E T Jaynes in the journal *Foundations of Physics* (1973). (This article also has a useful appendix summarising the views of previous authors who also dealt with the paradox.) Jaynes replaces the original statement with a supposed experiment in which a circle is placed on the floor and long straws are dropped onto it. Because the straws are "long" the possibility of a straw's having an end inside the circle is remote, and such events are omitted from the count. Otherwise the straw overlying the circle defines a chord in it and this can be measured to see how long it is.

He shows that, of the three proposed solutions, only the second is insensitive to small changes in the conditions of the experiment: a small alteration to the size of the circle and a small sideways shift in its position. Not surprisingly he is able to report that when the experiment was actually performed, this was the result he found. His restatement of the problem seems to me to introduce a subtle shift in our understanding of it. No wonder it rejects the first solution as being incompatible with a change in the radius of the circle; the straws are not also scaled. (Jaynes does however agree that the Wine/Water Paradox cannot have a sensible solution.)

Keynes offers a different analysis of the chord paradox. He sees Solution 1 as regarding the chord as a degenerate form of a long thin triangle, Solution 2 as seeing it as a limiting form of a long thin quadrilateral, and Solution 3 as the result of taking a more general (symmetric) shape and letting its thickness tend toward zero. Again this seems to be an attempt to provide data that are not really there, but it does serve to illustrate that the three different answers correspond to different geometric situations. Perhaps a little surprisingly, Keynes's resolution of the paradox is not listed in Jaynes's summary of other opinions.

Of the authors whose opinions *are* summarised, almost all agree with the view I have here espoused. Bertrand himself wrote: "None of the three is wrong, none is right, the question is ill-posed." Seven other authors (including Northrop) give similar verdicts. von Mises also agrees but goes rather further: "Which one of these or many other assumptions should be made is a question of fact and depends on how the needles are thrown. It is not a problem of probability calculus to decide which distribution prevails …". Jaynes is concerned to dispute this somewhat narrow understanding of what constitutes probability theory, although many mathematicians would espouse it.

Borel (who appeared in *Function* in April 2002) is seen by Jaynes as the only author who agrees with his overall view: " … it is a simple matter to see that the majority of natural interpretations lead us to [Solution 2]". Borel give no further details as to how he arrived at this conclusion. But however he did, he must have made some assumption as to what constituted a "natural interpretation", and thus given grounds for disagreeing with his judgement.

Final Note:

There is a current project to put Bertrand's book up on the web. See

http://www.york.ac.uk/depts/maths/histstat/lifework.htm

However, as I write, only a small amount is actually available. The project is very much a work in progress.

COMPUTERS AND COMPUTING

Computer Origami

This month's column is based on an article in *The New York Times* on June 22, 2004. The author was the Australian expatriate Margaret Wertheim, who has developed a formidable reputation as a science writer. The title is "Curves, Cones, Shells, Towers: He made Paper Jump to Life", and it concerns the work of the late David Huffman.

The article can be read online by going to

http://www.nytimes.com/2004/06/22/Science/22orig.html

and following the prompts from there. However, we here give a summary.

Whereas in traditional origami (the Japanese art of paper-folding) all the folds are straight, computer simulations are exempt from this restriction, and Dr Huffman moved beyond the strictly traditional to develop a more general artform based on curved folds, and giving rise to surprisingly lifelike structures.

In a related exploration, he looked at 3-dimensional analogues of tiling patterns, such as those described in our feature article for this issue. Huffman and those who continue his work have greatly enlarged the origami repertoire.

They have also been exploring the Mathematics behind the art. Last June, a conference on origami was held in New York, and included sessions on computational origami (*origami sekkei* in Japanese). The underlying Mathematics involves among other specialist areas computational geometry, number theory, coding theory and linear algebra. At the conference Dr. Robert Lang gave a talk on Mathematics and its application to origami design. In it he included applications to such real-world problems as folding airbags and space-based telescopes.

Dr Lang until recently was a laser physicist but he gave up that career to become a "full-time folder". He is the author of a recent book on technical folding: *Origami Design Secrets: Mathematical Methods for an Ancient Art.*

As well as the aesthetics involved in the art, there are surprisingly many applications. Besides the astronomical and engineering ones just mentioned, there is the possibility that there may be biological ramifications for the theory of how proteins fold (and thus become biochemically active).

Lang has been studying the models and research notes left behind with Huffman's death in 1999. Huffman pursued his interest as a hobby and was known for this aspect of his life only within the narrow world of origami sekkei. During his life he published only one paper on the subject.

One of his concerns was the precise calculation of what happened at the joins of folds. This is where we expect strain on the paper. It is important not to let the paper stretch or tear.

One of his discoveries was the so-called *pi condition* which says that if you have a point surrounded by four creases and you want the form to fold flat, then opposite angles around the vertex must sum to 180°, i.e. π radians, whence the name. The condition was rediscovered by others and has since been generalised to the case of more than four creases. The generalisation states that whatever the number of creases, all alternate angles must sum to π. The conditions under which things can fold flat is a major concern in computational origami.

Another mathematical link is to the theory of minimal surfaces: these are naturally adopted by such things as soap bubbles and certain biological structures such as the shells of sea urchins. They are of interest to engineers because of their saving on material and their strength. A typical application is to the design of pressed-metal car bodies.

Visit the website given above to see some pictures of what can be achieved in this unusual but fascinating and surprisingly important area.

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OLYMPIAD NEWS

Hans Lausch, Monash University

The 2004 Australian Mathematical Olympiad

The Australian Mathematical Olympiad (AMO) for 2004 was held in Australian schools on February 10 and 11. On both days, 104 students in years 9 to 12 sat a paper consisting of four problems, for which they were given four hours. These are the two papers:

First Day

1. Determine all pairs (a,b) of real numbers for which the equation

$$x^3+3x^2+ax+b=0$$

has three different real solutions that can be arranged in arithmetic progression (that is, the third minus the second is equal to the second minus the first).

2. Suppose $0\le x\le a\le y\le b\le z$ and $a+b+x+y+z=2004$.

Determine, with proof, the minimum possible value of $x+y+z$.
Determine, with proof, the maximum possible value of $x+y+z$.

3. Determine the number of sequences $a_1, a_2, \ldots, a_{2004}$ which are the numbers 1, 2, …, 2004 in some order and satisfy

$$|a_1-1|=|a_2-2|=\ldots=|a_{2004}-2004|>0..$$

4. Let ABC be an equilateral triangle, and let D be a point on AB between A and B. Next, let E be a point on AC with DE parallel to BC. Further, let F be the midpoint of CD and G the circumcentre of triangle ADE. Determine the angles of triangle BFG.

Second Day

5. Determine all non-negative integers m and n for which $6^m + 2^n + 2$ is a perfect square.

6. Decide whether or not there is a function f defined for all positive integers and taking positive integers as values such that

$$f(f(1)) = 5,\ f(f(2)) = 6,\ f(f(3)) = 4,\ f(f(4)) = 3,$$

$$f(f(n)) = n + 2 \text{ for } n \geq 5$$

7 A necklace is made from an even number, $n \geq 4$, of beads, each of which is coloured red, blue or green. There is an equal number of blue beads and green beads on the necklace. It is impossible to cut the necklace into two separate strings each of which contains a positive even number of beads and each of which contains the same number of blue and green beads.

Find all the possibilities for the number of red beads on the necklace.

8. Let $ABCD$ be a parallelogram. Suppose there exists a point P in the interior of $ABCD$ such that

$$\angle ABP = 2\angle ADP \text{ and } \angle DCP = 2\angle DAP.$$

Prove that $AB = BP = CP$.

The Sixteenth Asian Pacific Mathematics Olympiad

The Asian Pacific Mathematics Olympiad (APMO) was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the APMO has grown into a major international competition for students from about twenty countries on the Pacific Rim as well as from Argentina, Kazakhstan, South Africa and Trinidad & Tobago. It was held on March 15/16, Australian participants numbering 30. Here is the contest paper.

1. Find all non-empty finite sets S of positive integers such that if $m,n \in S$ then $\dfrac{m+n}{\gcd(m,n)} \in S$.

2. ABC is an acute-angled triangle with circumcenter O and orthocenter H. Show that one of area AOH, area BOH, area COH is the sum of the other two.

3. 2004 points are in the plane, no three collinear. S is the set of lines through any two of the points. Show that the points can be colored with two colors so that any two of the points have the same color iff there are an odd number of lines in S which separate them (a line separates them if they are on opposite sides of it).

4. Show that $\left\lfloor \dfrac{(n-1)!}{n(n+1)} \right\rfloor$ is even for every positive integer n.

5. Show that $(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$ for all positive real numbers a, b, c.

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This year's International Mathematical Olympiad (IMO), attended by teams from more than 80 countries, was held at Athens, Greece, in July. Australia has participated in this competition since 1979 and this year was represented by six secondary-school students. At the 2001 IMO in Washington, USA, an Australian team finished in 25th position, at the 2002 IMO in Glasgow, UK, as well as at the 2003 IMO in Tokyo, Japan, our team came 26th, while at the 1997 IMO in Mar del Plata, Argentina, our team took 9th position. Our 2004 results will be reported in the next issue.

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(continued from p 110)

cousin' of the recently discovered object Quaoar. This body orbits the sun at an even greater distance, 1.44 times further from the Sun than Neptune.

The expected bulk chemical compositions for Options 2 and 3 were both computed on 6 July 2003 and are archived for that date on the computer file VC$3:[AJRP.GRAVI]SUN7Q.OUT;299 on the Monash University ITS Vax Cluster. The Option 2 model for Phoebe gives a density of 1.50 g/cc. Option 3 predicts a density of 1.51 g/cc. Options 2 and 3 both account for the carbon dioxide announced by the Cassini Project. And both models just meet the lower limit on the observed density, which has a formal error of 0.1 g/cc.

But when Prentice posted Option 3, he noted that the relocation of Phoebe from the frigid world of Quaoar to the relatively warm environment of Saturn would cause the subsequent loss of all the frozen methane from its surface. Since frozen methane is a very light substance (density of only 0.5 g/cc), the loss of all such ice from Phoebe's chemical inventory would cause the mean density to rise from 1.5 to 1.6 g/cc. This is the very value found by Cassini!

Prentice now regrets choosing Option 1 as his preferred pre-Cassini model. But his Option 3 model, whose possibility was anticipated by his modern Laplacian model of Solar system origin one year prior to the arrival of the spacecraft at Phoebe, accounts precisely for the two main scientific discoveries that were announced by the Cassini Project.

PROBLEMS AND SOLUTIONS

We begin with the solutions to the problems posed last February in *Volume* **28**, *Part* 1.

Problem 28.1.1 (submitted by Julius Guest) read:

> Prove that $4\times 6^n + 5^{n+1} - 9$ is divisible by 20 for all positive integers n.

The solution below is by Anson Huang (Year 11, The Gap High School, Queensland). [His technique of proof is known as "mathematical induction". See *Function* for June 1998. Eds] Other solutions were received from Šefket Arslangić (Bosnia), John Barton, Derek Garson and the proposer.

Let $P_n = 4\times 6^n + 5^{n+1} - 9$. Now suppose that for some particular value of n, k say, the statement is true. I.e. $P_k = 20P$, for some integer P. Now consider P_{k+1}.

$$\begin{aligned}P_{k+1} &= 4\times 6^{k+1} + 5^{k+2} - 9\\ &= 4\times 6\times 6^k + 5\times 5^{k+1} - 9\\ &= 6\left[4\times 6^k + 5^{k+1} - 9\right] - 5^{k+1} + 5\times 9\\ &= 6P_k - 5^{k+1} + 5\times 9\\ &= 6\times 20P - 5^{k+1} + 5\times 9.\end{aligned}$$

P_{k+1} will therefore be divisible by 20 if $5^{k+1} - 5\times 9$ is divisible by 20. Clearly this number is divisible by 5. The quotient when this division is performed is $5^k - 9$. Write this as

$$5^k - 5 - 4 = 5\times(5^{k-1} - 1) - 4.$$

But now $5^{k-1} - 1$ is divisible by $5-1$, i.e. 4. Thus $5^k - 9$ is divisible by 4, and so $5^{k+1} - 5\times 9$ is divisible by both 5 and 4. It is thus divisible by 20. The net result of all this is that if P_k is divisible by 20 then so is P_{k+1}. But $P_1 = 40$, which is divisible by 20, and so P_2, P_3, P_4, etc are all divisible by 20.

Problem 28.1.2 (submitted by Šefket Arslangić, Bosnia) read:

Let *ABC* be a triangle with sides *a*, *b*, *c*. Let

$$p = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \text{ and } q = \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

Prove that $|p - q| < 1$.

Solutions were received from John Barton, Julius Guest and the proposer. Here is Barton's.

$$p = \frac{ab^2 + bc^2 + ca^2}{abc} \qquad \text{and} \qquad q = \frac{a^2b + b^2c + c^2a}{abc}$$

and so

$$p - q = \frac{a^2(c-b) + b^2(a-c) + c^2(b-a)}{abc}$$

But now if $a = b$, then $p - q = 0$. Thus the numerator has a factor $a - b$, and by symmetry also factors of $b - c$ and $c - a$. It may readily be deduced that

$$p - q = \frac{(a-b)(b-c)(c-a)}{abc} = \frac{a-b}{c}\,\frac{b-c}{a}\,\frac{c-a}{b}$$

Then

$$|p - q| = \frac{|a-b|}{c}\,\frac{|b-c|}{a}\,\frac{|c-a|}{b}$$

But now by the triangle inequality, each of the factors on the right is less than 1, and so the result follows.

Problem 28.1.3 (from *School Science and Mathematics*) read:

> Show that for all natural numbers n, $n^9 - 6n^7 + 9n^5 - 4n^3$ is divisible by 8640.

Solutions were received from Šefket Arslangić (Bosnia), John Barton, Derek Garson and Julius Guest.

What follows is a composite, also making use of the solution published in *School Science and Mathematics*, and sent to them by Vicki Schell.

The given polynomial may be written as

$$n^3(n-1)^2(n-2)(n+1)^2(n+2).$$

This product involves five consecutive numbers and so must contain a multiple of 5. It must also contain three of its nine factors that are divisible by 3, and at least four that are divisible by 2 and one that is divisible by 4. Hence the product is divisible by $2^6 \times 3^3 \times 5$, that is to say, by 8640.

Several solvers also submitted an inductive proof.

Problem 28.1.4 (also from *School Science and Mathematics*) read:

> A fair coin is tossed n times. What is the probability that the outcome sequence does not contain two successive heads?

Solutions were received from Bernard Anderson, John Barton, and Derek Garson. Here is Anderson's.

Call a sequence that meets the requirement an "allowable sequence". Let S_n be the number of allowable sequences of length n. Let H_n be the number of these ending with a head, and let T_n be the number ending with a tail. Note that $H_2 = 1$, $T_2 = 2$, $H_3 = 2$, $T_3 = 3$.

Furthermore $H_{n+1} = T_n$, and, because a final tail may follow either an allowable sequence ending in either a head or a tail, $T_{n+1} = H_n + T_n$. But $T_{n+1} = H_{n+2}$ by the first result in the previous sentence, and so we have $H_{n+2} = H_{n+1} + H_n$, which is the defining relation for the Fibonacci sequence. Thus $H_n = F_n$ where F_n is the nth Fibonacci number. Then $T_n = F_{n+1}$, and so $S_n = H_n + T_n = F_n + F_{n+1} = F_{n+2}$. This is the total number of allowable sequences out of a total of 2^n possible sequences.

The required probability is thus $F_{n+2}/2^n$.

Corrections: In the second line of the solution to Problem 27.5.1, sin 20° should have read $\sin^2 20^\circ$ throughout. In Problem 28.3.2, sin x should have read $\sin\frac{1}{x}$ and $4 + x^2$ should have read $1 + 4x^2$. Apologies!

We close with a new set of problems.

Problem 28.4.1 (from *Australian Senior Mathematics Journal*)

Find all solutions of the equation

$$\left(x^2 - 5x + 5\right)^{x^2-9x+20} = 1.$$

Problem 28.4.2 (from *Mathematical Bafflers*, Ed Angela Dunn)

Find all pairs of rational numbers (x, y) such that $x^y = y^x$.

Problem 28.4.3 (from *Exploring, Investigating and Discovering Mathematics*, by Vasile Berinde, and following a news item in our previous issue)

Find all prime numbers n such that $n + 4$ and $n + 8$ are both also prime.

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Published by the School of Mathematical Sciences, Monash University